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ON NEW ESTIMATES FOR DISTANCES IN ANALYTIC FUNCTION SPACES IN HIGHER DIMENSION

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Abstract. We provide various new sharp estimates for distances of fixed analytic functions to certain subspaces of analytic Besov classes in the unit ball and unit polydisk.

Keywords: holomorphic function, distances, Bloch-type spaces, Bergman type classes, polydisk, unit ball.

1. Introduction

To formulate our results we will need some standard definitions (see [1], [9]). We denote the open unit ball in $\mathbb{C}^n$ by $\mathbf{B} = \{z \in \mathbb{C}^n : |z| < 1\}$. The boundary of $\mathbf{B}$ will be denoted by $\mathbf{S}$, $\mathbf{S} = \{z \in \mathbb{C}^n : |z| = 1\}$. By dv we denote the volume measure on $\mathbf{B}$, normalized so that $v(\mathbf{B}) = 1$, and by $d\sigma$ we denote the surface measure on $\mathbf{S}$ normalized so that $\sigma(\mathbf{S}) = 1$. As usual, we denote by $H(\mathbf{B})$ the class of all holomorphic functions on $\mathbf{B}$.

We denote the unit polydisk by $\mathbf{D}^n = \{z \in \mathbb{C}^n : |z_k| < 1,\ 1 \le k \le n\}$ and the distinguished boundary of $\mathbf{D}^n$ by $\mathbf{T}^n = \{z \in \mathbb{C}^n : |z_k| = 1,\ 1 \le k \le n\}$. By dA_{2n} we denote the volume measure on $\mathbf{D}^n$ and by dm_n we denote the normalized Lebesgue measure on $\mathbf{T}^n$. Let $H(\mathbf{D}^n)$ be the space of all holomorphic functions on $\mathbf{D}^n$. For every function $f \in H(\mathbf{D^n})$ and $f(z_1, \dots, z_n) = \sum_{k_1,\dots k_n} a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n}$, we define the operator of fractional differentiation by
$\mathcal{D}^\alpha f(z_1, \dots, z_n) = \sum_{k_1=0}^\infty \cdots \sum_{k_1=0}^\infty \prod_{j=1}^n (k_j+1)^\alpha a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n},\ \alpha \in \mathbb{R}$.

We will write $Df(z)$ if $\alpha = 1$. For any α, $\mathcal{D}^\alpha$ is an operator acting from $H(\mathbf{D^n})$ to $H(\mathbf{D^n})$ (see [1]).

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We denote by $BMOA(\mathbf{D})$ and $B(\mathbf{D})$, as usual, the well known subspace of BMO and the Bloch class (see for example [1] and the reference there).

Let further $\Omega^k_{\alpha,\varepsilon} = \{z \in \mathbf{D} : |D^k f(z)|(1-|z|^2)^\alpha \geq \varepsilon\}$, $\alpha \geq 0$, $\varepsilon > 0$, $\Omega^0_{\alpha,\varepsilon} = \Omega_{\alpha,\varepsilon}$.

Applying famous Fefferman duality theorem, P. Jones proved the following

Theorem A. *([2], [8]) Let $f \in \mathcal{B}$. Then the following are equivalent:*

(*a*) $d_1 = dist_{\mathcal{B}}(f, BMOA)$;

(*b*) $d_2 = \inf\{\varepsilon > 0 : \chi_{\Omega^1_{1,\varepsilon}(f)}(z)\frac{dA(z)}{1-|z|^2}$ *is a Carleson measure*$\}$,

where χ denotes characteristic function of the mentioned set.

Let X, Y be subspaces of $H(\mathbf{D}^n)$ (or $H(\mathbf{B})$). In this paper we will provide direct proofs for estimation of $dist_Y(f, X) = \inf_{g\in X} \|f - g\|_Y$, $X \subset Y$, $X, Y \subset H(\mathbf{D}^n)(orH(\mathbf{B}))$, $f \in Y$, in the unit polydisk (or ball). In assertions we formulated below we provide only direct estimates for Dist function and we do not discuss inclusions $X \subset Y$ since they are well known or they can be easily proved. We briefly mention some results obtained recently on this extremal problem.

Recently, R. Zhao (see [8]) and W. Xu (see [7]), repeating arguments of R. Zhao in unit ball, obtained results on distances from Bloch functions to some Möbius invariant function spaces in one and higher dimension. The main intention of this note is to develop further their ideas and present new sharp theorems in the unit polydisk and unit ball.

2. Main results

The goal of this section to present new sharp theorems for Dist function in holomorphic Besov classes in unit ball and in unit polydisk.

For $0 < p < \infty$ and $\alpha > 0$, we define analytic Besov spaces in the unit ball in a standard way as follows (see [3])

$B^{\infty,1}_{-\alpha}(\mathbf{B}) = \left\{f \in H(\mathbf{B}) : \sup_{r<1}\left(\int_{\mathbf{S}} |f(r\xi)||d\sigma(\xi)|\right)(1-r)^\alpha < \infty\right\}$,

$B^{p,1}_{-\alpha}(\mathbf{B}) = \left\{f \in H(\mathbf{B}) : \int_0^1 \left(\int_{\mathbf{S}} |f(r\xi)||d\sigma(\xi)|\right)^p (1-r)^{\alpha p-1}dr < \infty\right\}$.

It is easy to prove that $B^{\infty,1}_{-\alpha}(\mathbf{B})$ and $B^{p,1}_{-\alpha}(\mathbf{B})$ classes are Banach spaces for $p > 1$ and quazinormed spaces for $p \leq 1$.

We now define a new set on unit interval and then using it is characteristic function we will give a new sharp assertion concerning distance function.

For $\varepsilon > 0$, $f \in H(\mathbf{B})$, let $L_{\varepsilon,\alpha}(f) = \{r \in (0,1) : (1-r)^\alpha \int_{\mathbf{S}} |f(r\xi)||d\sigma(\xi)| \geq \varepsilon\}$.

Theorem 1. *Let $f \in B^{\infty,1}_{-\alpha}(\mathbf{B})$, $\alpha > 0$, $1 \leq p < \infty$. Then the following are equivalent:*

(*a*) $\widehat{s}_1 = dist_{B^{\infty,1}_{-\alpha}(\mathbf{B})}(f, B^{p,1}_{-\alpha}(\mathbf{B}))$;

(*b*) $\widehat{s}_2 = \inf\{\varepsilon > 0 : \int_0^1 (1-r)^{-1}\chi_{L_{\varepsilon,\alpha}(f)}(r)dr < \infty\}$.

Let $I = (0,1)$. We denote by $r\xi = (r\xi_1, \ldots, r\xi_n)$ where $r \in I$, $\xi_j \in \mathbf{T}$, $j = 1, \ldots, n$, $\xi = (\xi_1, \ldots, \xi_n)$ and also $\overrightarrow{r}\xi = (r_1\xi_1, \ldots, r_n\xi_n)$, where $\overrightarrow{r} \in I^n$, $\overrightarrow{r} = (r_1, \ldots, r_n)$, $r_j \in I$, $\xi_j \in \mathbf{T}$, $j = 1, \ldots, n$.

For $0 < p < \infty$ and $\alpha > 0$, we define analytic Besov spaces $B^{\infty,1}_{-\alpha}(\mathbf{D^n})$ and $B^{p,1}_{-\alpha}(\mathbf{D^n})$ in polydisk as the sets

$\left\{f \in H(\mathbf{D^n}) : \sup_{r_1<1,\ldots,r_n<1}\left(\int_{\mathbf{T^n}} |f(\overrightarrow{r}\xi)||dm_n(\xi)|\right)\prod_{k=1}^n (1-r_k)^\alpha < \infty\right\}$ and

$\left\{f \in H(\mathbf{D^n}) : \int_0^1 \cdots \int_0^1 \left(\int_{\mathbf{T^n}} |f(\overrightarrow{r}\xi)||dm_n(\xi)|\right)^p \prod_{k=1}^n (1-r_k)^{\alpha p-1}dr < \infty\right\}$,

respectively (see [1]).

It is easy to prove that $B^{\infty,1}_{-\alpha}(\mathbf{D^n})$ and $B^{p,1}_{-\alpha}(\mathbf{D^n})$ classes are Banach spaces for $p > 1$ and quazinormed spaces for $p \le 1$.

For $\varepsilon > 0,\ f \in H(\mathbf{D^n})$, let

$L_{\varepsilon,\alpha}(f) = \{\overrightarrow{r} = (r_1,\dots r_n) \in I^n : \prod_{k=1}^n (1-r_k)^\alpha \int_{\mathbf{T^n}} |f(\overrightarrow{r}\xi)||dm_n(\xi)| \ge \varepsilon\}$.

In our next theorems we present complete analogues of Theorem 1 and Theorem A in case of polydisk and subframe. Note various properties of analytic classes in polydisk and subframe were studied before by many authors (see for example [1], [4], [5], [6] and the references there).

Theorem 2. *Let $f \in B^{\infty,1}_{-\alpha}(\mathbf{D^n}),\ \alpha > 0,\ 1 \le p < \infty$. Then the following are equivalent:*

(*a*) $\widehat{\widehat{s}}_1 = dist_{B^{\infty,1}_{-\alpha}(\mathbf{D^n})}(f, B^{p,1}_{-\alpha}(\mathbf{D^n}))$;

(*b*) $\widehat{\widehat{s}}_2 = \inf\{\varepsilon > 0 : \int_0^1 \cdots \int_0^1 \prod_{k=1}^n (1-r_k)^{-1} \chi_{L_{\varepsilon,\alpha}(f)}(r_1,\dots r_n) dr_1 \cdots dr_n < \infty\}$.

Let $\widetilde{\mathcal{B}}^\alpha(\mathbf{D^n}),\ \alpha > 0$, be the collection of Bloch type spaces of analytic functions on the polydisk satisfying $\|f\|_{\widetilde{\mathcal{B}}^\alpha(\mathbf{D^n})} = \sup_{z_1\in\mathbf{D},\dots,z_n\in\mathbf{D}} |f(z_1,\dots,z_n)| \prod_{k=1}^n (1 - |z_k|^2)^\alpha < \infty$. $\widetilde{\mathcal{B}}^\alpha(\mathbf{D^n})$ is a Banach space with the norm $\|f\|_{\widetilde{\mathcal{B}}^\alpha(\mathbf{D^n})}$.

For $k > s,\ 0 < p, q \le \infty$, $\mathcal{B}^{q,p}_s(\mathbf{D^n})$ let be the class of analytic functions on polydisk satisfying (see [3])

$\|f\|^q_{\mathcal{B}^{q,p}_s(\mathbf{D^n})} = \int_0^1 \cdots \int_0^1 \left(\int_{\mathbf{T^n}} |D^k f(\overrightarrow{r}\xi)|^p |dm_n(\xi)|\right)^{\frac{q}{p}} \prod_{k=1}^n (1-r_k)^{(k-s)q-1} dr < \infty.$

Theorem 3. *Let $0 < q \le 1,\ s < 0,\ t \le s - \frac{1}{q},\ \beta > \frac{1-sq}{q}$ and $\beta > -1-t$. Let $f \in \widetilde{\mathcal{B}}^{-t}(\mathbf{D^n})$. Then the following are equivalent:*

(*a*) $l_1 = dist_{\widetilde{\mathcal{B}}^{-t}(\mathbf{D^n})}(f, \mathcal{B}^{q,q}_s(\mathbf{D^n}))$;

(*b*) $l_2 = \inf\{\varepsilon > 0 : \int_{\mathbf{D^n}} \left(\int_{\Omega_{\varepsilon,-t}(f)} \frac{\prod_{k=1}^n (1-|w_k|)^{\beta+t} dA_{2n}(w_1,\dots,w_n)}{\prod_{k=1}^n |1-\overline{z_k}w_k|^{2+\beta}}\right)^q \times$
$\times \prod_{k=1}^n (1-|z_k|)^{-sq-1} dA_{2n}(z_1,\dots,z_n) < \infty\}$.

We will formulate now a sharp theorem for analytic classes on subframe.

For every function $f \in H(\mathbf{D^n})$ and $f(z_1,\dots,z_n) = \sum_{k_1,\dots k_n} a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n}$, we define the operator $\mathcal{R}^s$ by

$\mathcal{R}^s f(z) = \sum_{k_1,\dots k_n \ge 0} (k_1 + \cdots + k_n + 1)^s a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n}$ and $\widetilde{\mathbf{D}}^n = (0,1] \times \mathbf{T}^n$.

It is easy to note that $\|f\|_{B^{\infty,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n)} = \sup_{r<1} (1-r)^\alpha \int_{\mathbf{T^n}} |\mathcal{R}^s f(r\xi)| dm_n(\xi)$

$\le C \left(\int_0^1 \left(\int_{\mathbf{T^n}} |\mathcal{R}^s f(r\xi)| dm_n(\xi)\right)^p (1-r)^{\alpha p-1} dr\right)^{\frac{1}{p}} = \|f\|_{B^{p,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n)}$, where $s \in \mathbb{R}$, $\alpha > 0,\ 0 < p < \infty$.

For $\varepsilon > 0$ and $f \in H(\mathbf{D}^n)$, let $K_{\varepsilon,\alpha,s} = \{r \in I : (1-r)^\alpha \int_{\mathbf{T}^n} |\mathcal{R}^s f(r\xi)| dm_n(\xi) \ge \varepsilon\}$.

Theorem 4. *Let $f \in B^{\infty,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n),\ \alpha > 0,\ 1 \le p < \infty,\ s \in \mathbb{R}$. Then the following are equivalent:*

(*a*) $\nu_1 = dist_{B^{\infty,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n)}(f, B^{p,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n))$;

(*b*) $\nu_2 = \inf\{\varepsilon > 0 : \int_0^1 (1-r)^{-1} \chi_{K_{\varepsilon,\alpha,s}(f)}(r) dr < \infty\}$.

Proofs of all mentioned assertions are partially based on approaches that appeared recently in Zhao's paper [8] and they will be presented by us elsewhere.

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