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VIRTUAL 3-MANIFOLDS

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Abstract. We generalize the class if all compact 3-manifolds to a class of new objects called *virtual 3-manifolds*. Each virtual 3-manifold determines a 3-manifold with singularities of the type $\mathrm{Con}(RP^2)$ and may be presented by a triangulation as well as by a special spine. Many properties and invariants of 3-manifolds can be extended to the virtual ones. We restrict ourselves to mentioning Turaev-Viro invariants and two-sheeted branched coverings of virtual 3-manifolds.
Keywords: 3-manifold, special spine, virtual 3-manifold.

1. Basic facts of the special spine theory

Let P be a subpolyhedron of a 3-dimensional polyhedron M. We say that P is a *spine* of M, if for some triangulation (T_M, T_P) of the pair (M, P) there is a finite sequence of elementary simplicial collapses transforming T_M to T_P. If M is a 3-manifold with boundary, then $P \subset M$ is a spine of M if $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$. It is convenient to define a spine of a closed 3-manifold M to be a spine of M with a removed open 3-ball.

Definition 1. *A compact 2-dimensional polyhedron P is called* special*, if*

(1) *The link of any point of P is homeomorphic either to a circle, or to a circle with a diameter, or to a circle with three radii. Points of the first, second, and third types are called* true vertices, triple points, *and* nonsingular points, *respectively.*
(2) *P contains at least one vertex and the set of all nonsingular points of P is a collection of disjoint open 2-cells.*

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Notice that the set of all triple points of a special polyhedron is a union of open intervals. The following is easy (see [1,2]).

Theorem 1. *Any compact 3-manifold admits a special spine. Moreover, any homeomorphism between special spines of 3-manifolds can be extended to a homeomorphism between those manifolds (assuming that both are either closed or not).* □

Let us recall the following basic transformation (T-*move*) of special polyhedra. Sometimes it is called *two-to-three move* since it replaces a fragment E with two true vertices by a fragment E' with three true vertices, see Fig. 1.

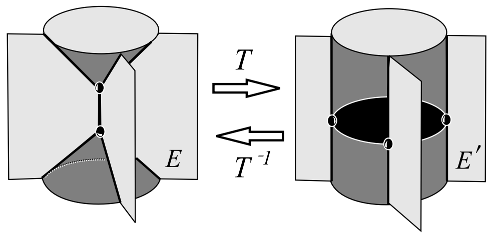

FIGURE 1. T-move

The following theorem is one of the main results of the special spine theory (see [1,2]).

Theorem 2. *Let P and Q be special polyhedra with at least two true vertices each. Then the following holds:*

(1) *If P and Q are special spines of the same 3-manifold, then one can transform P into Q by a finite sequence of moves $T^{\pm 1}$;*
(2) *If one can transform P into Q by a finite sequence of moves $T^{\pm 1}$ and one of them is a special spine of a 3-manifold, then the other is a special spine of the same 3-manifold.* □

Notice that if a special polyhedron is embedded into a 3-manifold, then it is a special spine of its regular neighborhood. However, there are special polyhedra which cannot be embedded into 3-manifolds or, equivalently, cannot be thickened to 3-manifolds. For example, projective plane RP^2 with a disc D attached along a projective line is non-embeddable because the normal bundle of ∂D in RP^2 is nontrivial. This polyhedron is not special but can be easily converted into a special polyhedron by inserting self-intersections for the attaching circle. The normal bundle obstruction does work also in this case.

2. WHAT IS A VIRTUAL 3-MANIFOLD?

We say that two special polyhedra are equivalent if one of them can be transformed to the other by a sequence of moves $T^{\pm 1}$. It follows from Theorems 1 and 2 that there is a natural bijection between the equivalence classes of *thickenable* special polyhedra with ≥ 2 vertices and compact 3-manifolds with boundary considered up to homeomorphisms. This fact (as well as an analogy with the virtual knot theory [3]) motivates the following definition.

Definition 2. *A* virtual *3-manifold is an equivalence class of special polyhedra (not necessarily thickenable).*

It may be slightly disturbing to think of a class of 2-dimensional objects as a 3-dimensional object. The following definition would help us to return to the 3-dimensional world.

Definition 3. *A compact 3-dimensional polyhedron M is called* a 3-manifold with RP^2-singularities *if the link of any point of M is either a 2-sphere, or a 2-disc, or RP^2.*

For example, the suspension (double cone) over RP^2 is a 3-manifold with two RP^2-singularities. Let us describe a construction which assigns a 3-manifold with RP^2-singularities to each special polyhedron in a natural way.

Let P be a given special polyhedron. Replacing each true vertex of P by a 3-ball and each triple edge of P by a handle of index 1, we get a not necessarily orientable handlebody H such that each 2-cell C of P intersects ∂H along one circle. A regular neighborhood N of that circle in ∂H is either an annulus or a Möbius strip. In the first case we thicken the rest of C to an index 2 handle. In the second case we replace the rest of C by $\mathrm{Con}(RP^2)$ by attaching to H a 2-cell D along ∂N and taking the cone over $N \cup D \approx RP^2$. Doing so for all 2-cells of P, we get a 3-manifold with RP^2-singularities, which is denoted by $W(P)$.

Theorem 3. *The assignment $P \mapsto W(P)$ induces a correctly defined surjection $\varphi\colon \mathcal{V} \to \mathcal{W}$ of the set $\mathcal{V}$ of all virtual 3-manifolds to the set $\mathcal{W}$ of all 3-manifolds with nonempty boundary and RP^2-singularities.*

Proof. For proving correctness of φ we have to show that if two special polyhedra P_1, P_2 are related by $T^{\pm 1}$-moves then 3-manifolds $\varphi(P_1)$ and $\varphi(P_1)$ are homeomorphic. It suffices to consider the case when P_2 is obtained from P_1 by only one T-move. We may assume that the fragment E of P_1 participating in the move is contained inside the handlebody H used for construction of $W(P_1)$. It follows that H remains unchanged and so is $W(P_1)$.

In order to see that φ is surjective, it suffices to prove that any 3-manifold with RP^2-singularities has a special spine. This can be done in three steps. First, we cut off all cones over RP^2. Then we construct a special spine of the obtained genuine 3-manifold such that it contains all copies of RP^2 corresponding to the bases of the removed cones. At last, we attach discs along projective circles in those RP^2. □

Problem. *Is φ bijective? If not, what additional moves on special polyhedra may be added to $T^{\pm 1}$ in order to get bijectivity?*

3. Turaev-Viro invariants and coverings

The Turaev-Viro invariants of 3-manifolds [4] had been originally constructed via triangulations. The dual special spine approach seems to be more convenient, see [2]. It turns out that without any modifications this approach works also for virtual 3-manifolds. Therefore, the following theorem is true.

Theorem 4. *Turaev-Viro invariants of 3-manifolds can be extended to the class of all virtual 3-manifolds.* □

Let P be a special polyhedron which cannot be embedded into a 3-manifold. Then the corresponding 3-manifold $W(P)$ has at least one RP^2-singularity and thus is nonorientable. Let $p\colon \tilde{W}(P) \to W(P)$ be the two-sheeted covering corresponding to the subgroup of orientation-preserving elements of $\pi_1(W(P))$. The following is evident.

Theorem 5. *For any special polyhedron P the covering space $\tilde{W}(P)$ constructed above is an orientable genuine 3-manifold.* □

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