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ISOSPECTRAL FINITE SIMPLE GROUPS

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Abstract. The spectrum of a finite group is the set of its element orders. Two groups are called isospectral if their spectra coincide. It is known that $PSp_6(2)$ is isospectral to $P\Omega_8^+(2)$ and $\Omega_7(3)$ is isospectral to $P\Omega_8^+(3)$. In the present paper we prove that there are no other pairs of non-isomorphic isospectral finite simple groups. In particular, we prove that there are no three finite simple groups with the same spectrum.

Keywords: finite group, simple group, spectrum of a group, isospectral simple groups.

The spectrum $\omega(G)$ of a finite group G is the set of its element orders. Two groups are called isospectral if their spectra coincide. Given a finite group G, we write $h(G)$ for the number of isomorphism classes of finite groups isomorphic to G. A finite group G is called recognizable by spectrum if $h(G) = 1$.

Let $\{G, H\}$ be one of the pairs $\{PSp_6(2), P\Omega_8^+(2)\}$ and $\{\Omega_7(3), P\Omega_8^+(3)\}$. Then $\omega(G) = \omega(H)$. Moreover, it is proved in [1, 2] that $h(G) = 2$, i.e., if K is a finite group such that $\omega(K) = \omega(G)$, then either $K \simeq G$ or $K \simeq H$. In this context the following question naturally arises ([3, Question 16.25]):

Do there exist three pairwise non-isomorphic finite nonabelian simple groups with the same spectrum?

In the present paper we prove that there are no pairs of non-isomorphic isospectral finite simple groups except for the listed above. In particular, we give a negative answer to Question 16.25.

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The set $\omega(G)$ is closed under taking divisors, i.e., if $n \in \omega(G)$ and d divides n then $d \in \omega(G)$. Therefore it is uniquely determined by its subset $\mu(G)$ of elements that are maximal under the divisibility relation. Thus the spectra of finite groups G and H coincide if and only if $\mu(G) = \mu(H)$.

Given a finite group G, denote by $\pi(G)$ the set of all prime divisors of its order and by $\exp(G)$ the period of G. If G is a finite group of Lie type over a field of characteristic p, then $\omega_{p'}(G)$ denotes the subset of $\omega(G)$ consisting of all elements that are not divisible by p.

Theorem 1. *Let G and H be finite simple groups such that $\omega(H) = \omega(G)$ and $H \not\simeq G$. Then either $\{G, H\} = \{PSp_6(2), P\Omega_8^+(2)\}$ or $\{G, H\} = \{\Omega_7(3), P\Omega_8^+(3)\}$. In particular, there are no three pairwise non-isomorphic isospectral finite simple groups.*

Proof. Let G be a finite simple group such that there exists a simple group H non-isomorphic to G and having the same spectrum as G. Obviously, G is nonabelian.

First, we show that G is not a sporadic group. Indeed, all sporadic groups except for the group J_2 are recognizable by spectrum [4]. Suppose that $G = J_2$. Then $\mu(G) = \{15, 12, 10, 8, 7\}$ and $\pi(G) = \{2, 3, 5, 7\}$ (see [5]). The list of simple groups K with $\pi(K) = \{2, 3, 5, 7\}$ can be found in [6, Table 1]. These groups are Alt_7, Alt_8, Alt_9, Alt_{10}, $PSL_2(49)$, $PSU_3(5)$, $PSL_3(4)$, $PSU_4(3)$, $PSp_4(7)$, $PSp_6(2)$, and $P\Omega_8^+(2)$. The spectra of Alt_7, Alt_8, Alt_9, and $PSL_3(4)$ do not contain 8, hence H is not one of these groups. The spectra of Alt_{10}, $PSU_4(3)$, $PSp_6(2)$, and $P\Omega_8^+(2)$ contain 9 (see [5] and [7, Corollaries 3 and 4]). By [7, Corollary 2] the spectrum of $PSp_4(7)$ contains $25 = (7^2 + 1)/2$. The spectrum of $PSL_2(49)$ also contains 25. Finally, $\mu(PSU_3(5)) = \{10, 8, 7, 6\}$ (see [8, Corollary 3]). Thus, $H \neq J_2$.

Suppose that G is an alternating group. We show that H is not an alternating group in this case. By [9, Lemma 2.1] if $n \geqslant 7$, then $\omega(Alt_n) \not\subseteq \omega(Sym_{n-1})$. Hence if $6 \leqslant n < m$ then $\omega(Alt_n)$ is a proper subset of $\omega(Alt_m)$. Obviously, $4 \in \omega(Alt_6) \setminus \omega(Alt_5)$. Thus H is a group of Lie type. It follows from [10, 11] that for every $p \in \pi(H)$ there exists $q \in \pi(H)$ such that $pq \notin \omega(H)$. Since $\omega(H) = \omega(G)$, this property also holds for G. Let $p = 3$ and $q \in \pi(G)$ is such that $3q \notin \omega(G)$. Then G is one of the groups Alt_q, Alt_{q+1}, and Alt_{q+2}. The alternating groups of such degrees, except for the group Alt_6, are recognizable by spectrum [9, 12]. Suppose that $G = Alt_6$. Then $\pi(G) = \{2, 3, 5\}$. Using [6, Table 1] we infer that the only possibility for H is $PSU_3(3)$. But $12 \in \omega(PSU_3(3))$. Thus G is not an alternating group.

It remains to consider the case where G and H are groups of Lie type. The following result is a consequence of [13, Theorem 1.1].

Lemma 1. *Let G and H be finite simple groups of Lie type such that $\omega(G) = \omega(H)$ and $G \not\simeq H$. Then one of the following holds:*

(a) $\{G, H\} = \{PSp_{2n}(q), \Omega_{2n+1}(q)\}$ *for $n \geqslant 5$ and odd q;*
(b) $\{G, H\} = \{PSp_4(q), PSL_2(q^2)\}$*;*
(c) $\{G, H\} \subseteq \{PSp_6(q), P\Omega_8^+(q), \Omega_7(q)\}$*;*
(d) $\{G, H\} \subseteq \{PSp_8(q), P\Omega_8^-(q), \Omega_9(q)\}$*;*
(e) $\{G, H\} = \{PSL_3(2), PSU_3(3)\}$.

Since $PSL_3(2)$ is recognizable, (e) is impossible. If q is a power of a prime p then $p(q+1) \in \omega(PSp_4(q)) \setminus \omega(PSL_2(q^2))$ by [7, Corollary 2] and [8, Corollary 3]. Hence (b) is not possible either. By [14, 15] or [7, Theorems 2 and 6], $\omega(PSp_{2n}(q)) \neq$

$\omega(\Omega_{2n+1}(q))$ for $n > 2$ and odd q. Therefore (a) is impossible. Moreover, the set $\{G, H\}$ contains $P\Omega_8^+(q)$ in (c) and $P\Omega_8^-(q)$ in (d).

Let $G = P\Omega_8^+(q)$ and $H = PSp_6(q)$. By [16, Theorem 5], G contains an element of order

$$\frac{q^4-1}{(4,q^4-1)} = \frac{1}{(4,q^4-1)}(q^2+1)(q+1)(q-1).$$

Let T be a maximal torus of H such that q^2+1 divides the order of T. By [16, Theorems 3 and 7], the period of T is equal to $[q^2+1, q\pm1] = (q^2+1)(q\pm1)/(q^2+1, q\pm1)$. If $q > 5$ or $q = 4$ then

$$\exp(T) \leqslant (q^2+1)(q+1) < (q^2+1)(q+1)\frac{q-1}{(4,q^4-1)},$$

hereinafter $\exp(K)$ denotes the period of a group K. Let $q = 5$. Then $(q^2+1, q\pm1) = 2$. So

$$\exp(T) \leqslant (q^2+1)(q+1)/2 < (q^2+1)(q+1) = \frac{1}{4}(q^2+1)(q+1)(q-1).$$

Thus the spectra of groups G and H are different for $q > 3$. If $q = 3$ then $\omega(H) \setminus \omega(G)$ contains $30 = 3(3^2+1)$ by [7, Theorems 2 and 9]. If $q = 2$ then $\{G, H\} = \{PSp_6(2), P\Omega_8^+(2)\}$.

Let $G = P\Omega_8^+(q)$ and $H = \Omega_7(q)$. Since $\omega_{p'}(PSp_6(q)) = \omega_{p'}(\Omega_7(q))$ by [16, Theorems 3 and 4], a word-by-word repetition of the previous argument yields that for $q > 3$ the spectra of groups G and H are different. If $q = 2$ then $\Omega_7(2) \simeq PSp_6(2)$. If $q = 3$ then $\{G, H\} = \{\Omega_7(3), P\Omega_8^+(3)\}$.

Let $G = P\Omega_8^-(q)$ and $H \in \{PSp_8(q), \Omega_9(q)\}$ and q is a power of a prime p. By [7, Theorems 2–4, 6 and 8] we have $p(q^3+1)/(2, q-1) \in \omega(H) \setminus \omega(G)$. The theorem is proved.

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