ISOSPECTRAL FINITE SIMPLE GROUPS

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Abstract. The spectrum of a finite group is the set of its element orders. Two groups are called isospectral if their spectra coincide. It is known that $PSp_6(2)$ is isospectral to $PO_8^+(2)$ and $O_7(3)$ is isospectral to $PO_8^+(3)$. In the present paper we prove that there are no other pairs of non-isomorphic isospectral finite simple groups. In particular, we prove that there are no three finite simple groups with the same spectrum.

Keywords: finite group, simple group, spectrum of a group, isospectral simple groups.

The spectrum $\omega(G)$ of a finite group $G$ is the set of its element orders. Two groups are called isospectral if their spectra coincide. Given a finite group $G$, we write $h(G)$ for the number of isomorphism classes of finite groups isomorphic to $G$.

A finite group $G$ is called recognizable by spectrum if $h(G) = 1$.

Let $\{G, H\}$ be one of the pairs $\{PSp_6(2), PO_8^+(2)\}$ and $\{O_7(3), PO_8^+(3)\}$. Then $\omega(G) = \omega(H)$. Moreover, it is proved in [1, 2] that $h(G) = 2$, i.e., if $K$ is a finite group such that $\omega(K) = \omega(G)$, then either $K \cong G$ or $K \cong H$. In this context the following question naturally arises ([3, Question 16.25]):

Do there exist three pairwise non-isomorphic finite nonabelian simple groups with the same spectrum?

In the present paper we prove that there are no pairs of non-isomorphic isospectral finite simple groups except for the listed above. In particular, we give a negative answer to Question 16.25.
The set $\omega(G)$ is closed under taking divisors, i.e., if $n \in \omega(G)$ and $d$ divides $n$ then $d \in \omega(G)$. Therefore it is uniquely determined by its subset $\mu(G)$ of elements that are maximal under the divisibility relation. Thus the spectra of finite groups $G$ and $H$ coincide if and only if $\mu(G) = \mu(H)$.

Given a finite group $G$, denote by $\pi(G)$ the set of all prime divisors of its order and by $\exp(G)$ the period of $G$. If $G$ is a finite group of Lie type over a field of characteristic $p$, then $\omega_p(G)$ denotes the subset of $\omega(G)$ consisting of all elements that are not divisible by $p$.

**Theorem 1.** Let $G$ and $H$ be finite simple groups such that $\omega(H) = \omega(G)$ and $H \not\cong G$. Then either \( \{ G, H \} = \{ PSp_6(2), P\Omega_8^+(2) \} \) or \( \{ G, H \} = \{ \Omega_7(3), P\Omega_8^+(3) \} \). In particular, there are no three pairwise non-isomorphic isospectral finite simple groups.

**Proof.** Let $G$ be a finite simple group such that there exists a simple group $H$ non-isomorphic to $G$ and having the same spectrum as $G$. Obviously, $G$ is nonabelian.

First, we show that $G$ is not a sporadic group. Indeed, all sporadic groups except for the group $J_2$ are recognizable by spectrum [4]. Suppose that $G = J_2$. Then $\mu(G) \subseteq \{ 15, 12, 10, 8, 7 \}$ and $\pi(G) = \{ 2, 3, 5, 7 \}$ (see [5]). The list of simple groups $K$ with $\pi(K) = \{ 2, 3, 5, 7 \}$ can be found in [6, Table 1]. These groups are $Alt_7$, $Alt_8$, $Alt_{10}$, $PSL_2(49)$, $PSU_3(5)$, $PSL_3(4)$, $PSU_4(3)$, $PSp_4(7)$, $PSp_6(2)$, and $P\Omega_8^+(2)$. The spectra of $Alt_7$, $Alt_8$, $Alt_{10}$, and $PSL_3(4)$ do not contain 8, hence $H$ is not one of these groups. The spectra of $Alt_{10}$, $PSU_3(3)$, $PSp_6(2)$, and $P\Omega_8^+(2)$ contain 9 (see [5] and [7, Corollaries 3 and 4]). By [7, Corollary 2] the spectrum of $PSp_4(7)$ contains $25 = (7^2 + 1)/2$. The spectrum of $PSL_2(49)$ also contains 25. Finally, $\mu(PSU_3(5)) = \{ 10, 8, 7, 6 \}$ (see [8, Corollary 3]). Thus, $H \neq J_2$.

Suppose that $G$ is an alternating group. We show that $H$ is not an alternating group in this case. By [9, Lemma 2.1] if $n \geq 7$, then $\omega(Alt_n) \not\subseteq \omega(Sym_{m-1})$. Hence if $6 \leq n < m$ then $\omega(Alt_n)$ is a proper subset of $\omega(Alt_m)$. Obviously, 4 $\in \omega(Alt_6) \setminus \omega(Alt_5)$. Thus $H$ is a group of Lie type. It follows from [10, 11] that for every $p \in \pi(H)$ there exists $q \in \pi(H)$ such that $pq \not\in \omega(H)$. Since $\omega(H) = \omega(G)$, this property also holds for $G$. Let $p = 3$ and $q \in \pi(G)$ is such that $3q \not\in \omega(G)$. Then $G$ is one of the groups $Alt_7$, $Alt_{q+1}$, and $Alt_{q+2}$. The alternating groups of such degrees, except for the group $Alt_6$, are recognizable by spectrum [9, 12]. Suppose that $G = Alt_6$. Then $\pi(G) = \{ 2, 3, 5 \}$. Using [6, Table 1] we infer that the only possibility for $H$ is $PSU_3(3)$. But $12 \in \omega(PSU_3(3))$. Thus $G$ is not an alternating group.

It remains to consider the case where $G$ and $H$ are groups of Lie type. The following result is a consequence of [13, Theorem 1.1].

**Lemma 1.** Let $G$ and $H$ be finite simple groups of Lie type such that $\omega(G) = \omega(H)$ and $G \not\cong H$. Then one of the following holds:

(a) $\{ G, H \} = \{ PSp_{2n}(q), \Omega_{2n+1}(q) \}$ for $n \geq 5$ and odd $q$;
(b) $\{ G, H \} = \{ PSp_4(q), PSL_2(q^2) \}$;
(c) $\{ G, H \} \subseteq \{ PSp_6(q), P\Omega_8^q(q), \Omega_7(q) \}$;
(d) $\{ G, H \} \subseteq \{ PSp_8(q), P\Omega_8^q(q), \Omega_9(q) \}$;
(e) $\{ G, H \} = \{ PSL_3(2), PSU_3(3) \}$.

Since $PSL_3(2)$ is recognizable, (e) is impossible. If $q$ is a power of a prime $p$ then $p(q+1) \in \omega(PSp_4(q)) \setminus \omega(PSL_2(q^2))$ by [7, Corollary 2] and [8, Corollary 3]. Hence (b) is not possible either. By [14, 15] or [7, Theorems 2 and 6], $\omega(PSp_{2n}(q)) \neq \omega(PSp_{2n}(q))$. Therefore $\mu(G) = \mu(H)$, and the theorem holds.
\( \omega(\Omega_{n+1}(q)) \) for \( n > 2 \) and odd \( q \). Therefore (a) is impossible. Moreover, the set \( \{G, H\} \) contains \( P\Omega_4^-(q) \) in (c) and \( P\Omega_6^-(q) \) in (d).

Let \( G = P\Omega_4^+(q) \) and \( H = PSp_6(q) \). By [16, Theorem 5], \( G \) contains an element of order

\[
\frac{q^4 - 1}{(4, q^4 - 1)} = \frac{1}{(4, q^4 - 1)}(q^2 + 1)(q + 1)(q - 1).
\]

Let \( T \) be a maximal torus of \( H \) such that \( q^2 + 1 \) divides the order of \( T \). By [16, Theorems 3 and 7], the period of \( T \) is equal to \( [q^2 + 1, q \pm 1] = (q^2 + 1)(q \pm 1)/(q^2 + 1, q \pm 1) \). If \( q > 5 \) or \( q = 4 \)

\[
\exp(T) \leq (q^2 + 1)(q + 1) < (q^2 + 1)(q + 1) - \frac{q - 1}{(4, q^4 - 1)}.
\]

hereinafter \( \exp(K) \) denotes the period of a group \( K \). Let \( q = 5 \). Then \( (q^2 + 1, q \pm 1) = 2 \). So

\[
\exp(T) \leq (q^2 + 1)(q + 1)/2 < (q^2 + 1)(q + 1) = \frac{1}{4}(q^2 + 1)(q + 1)(q - 1).
\]

Thus the spectra of groups \( G \) and \( H \) are different for \( q > 3 \). If \( q = 3 \) then \( \omega(H) \setminus \omega(G) \) contains 30 = 3(3^2 + 1) by [7, Theorems 2 and 9]. If \( q = 2 \) then \( \{G, H\} = \{PSp_6(2), P\Omega_8^+(2)\} \).

Let \( G = P\Omega_6^-(q) \) and \( H = \Omega_7(q) \). Since \( \omega_p'(PSp_6(q)) = \omega_p'(\Omega_7(q)) \) by [16, Theorems 3 and 4], a word-by-word repetition of the previous argument yields that for \( q > 3 \) the spectra of groups \( G \) and \( H \) are different. If \( q = 2 \) then \( \Omega_7(2) \simeq PSp_6(2) \). If \( q = 3 \) then \( \{G, H\} = \{\Omega_2(3), P\Omega_8^+(3)\} \).

Let \( G = P\Omega_4^+(q) \) and \( H \in \{PSp_6(q), \Omega_3(q)\} \) and \( q \) is a power of a prime \( p \). By [7, Theorems 2–4, 6 and 8] we have \( p(q^4 + 1)/(2, q - 1) \in \omega(H) \setminus \omega(G) \). The theorem is proved.

References


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A.A. Buturlakin and M.A. Grechkoseeva, The cyclic structure of maximal tori of the finite classical groups, Algebra and Logic, 46: 2 (2007), 73–89.

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