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EXTENDING PAIRINGS TO HAMILTONIAN CYCLES

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Abstract. Recently J.Fink proved that every 1-factor of the complete graph on the vertex set of the hypercube Q_n can be extended to a cycle by adding some edges of this hypercube. We prove that, for $n \geq 4$, one can remove some edges of Q_n so that the resulting graph still has this property. Also we give upper and lower bounds on the minimum number of edges of a $2n$-vertex graph having this property.

Keywords: 1-factor, Hamiltonian cycle, Kreweras Conjecture, hypercube

Let $G = (V, E)$ be a simple graph on $2n$ vertices. By *a pairing* on G we mean any partition of the vertex set $V(G)$ into 2-element sets. It will be convenient to define a pairing by a fixed-point-free permutation p of V of order 2, the sets being of the form $\{x, p(x)\}$.

Definition 1. *A pairing p of a graph $G = (V, E)$ on $2n$ vertices is called* **extendable** *if there exists a cyclic ordering $(v_0, v_1, \ldots, v_{2n-1})$ of $V(G)$ (indices modulo $2n$) such that for every $i = 0, \ldots, n-1$ we have $v_{2i+1} = p(v_{2i})$, and $v_{2i+1}v_{2i+2}$ is an edge of G.*

If every pairing is extendable, we will call G **a Fink graph**.

In particular, the single edge ($n = 1$) is trivially a Fink graph.

Recently J.Fink [1] proved that the hypercubes Q_n are Fink graphs, thus proving in particular that every 1-factor of Q_n is contained in a Hamiltonian cycle (the Kreweras Conjecture). We will give here a different exposition of Fink's proof, based on a simple inductive lemma, and then will further exploit this lemma to obtain some new results on Fink graphs. Finally, we shall show that, for $n \geq 4$, one can remove some edges of Q_n so that the graph remains Fink.

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Definition 2. *By a* **Fink partition** *of a graph $G = (V, E)$ we mean a partition $V = X \cup Y$, $X \cap Y = \emptyset$, such that the subgraphs induced on X and Y are both Fink graphs.*

Lemma 1. (i) *Let (X, Y) be a Fink partition of a graph G. Then every pairing p such that $p(x) \notin X$ for some $x \in X$, is extendable.*

(ii) *If $(X_1, Y_1), \ldots, (X_k, Y_k)$ are Fink partitions of a graph G such that $|X_1 \cap \ldots \cap X_k|$ is odd then G is a Fink graph.*

Proof. **(i)** Let $A = \{x \in X \mid p(x) \notin X\}$ and $B = p(A)$. So, $A \subseteq X$, $B \subseteq Y$, and $|A| = |B| = 2k$ is even, since $|X|$ and $|Y|$ are even.

Take any pairing q on X that coincides with p on $X \setminus A$. Let

$$(a_0, a_1, X_1, a_2, a_3, X_2, \ldots, a_{2k-2}, a_{2k-1}, X_k)$$

be its extension. Here a_i are all elements of A, and $X_1, \ldots, X_k$ are sequences of vertices from $X \setminus A$ (possibly, empty).

For $i = 0, \ldots, 2k-1$ let $b_i = p(a_i)$. Now we define a pairing r on Y as follows: it coincides with p on $Y \setminus B$, and the pairs on B are

$$\{b_1, b_2\}, \{b_3, b_4\}, \ldots, \{b_{2k-1}, b_0\};$$

that is, $r(b_{2i}) = b_{2i-1}$ ($i = 1, \ldots, k$, indices modulo $2k$).

Finally, we take any extension of r on Y, and for every $i = 1, \ldots, k$ insert into it between b_{2i-1} and b_{2i} the sequence a_{2i-1}, X_i, a_{2i}. The resulting cyclic ordering of vertices of G will be an extension of the pairing p, which proves the statement.

(ii) Let $X = X_1 \cap \ldots \cap X_k$. Take an arbitrary pairing p of G. There is a vertex $x \in X$ with $p(x) \notin X$, since $|X|$ is odd. Therefore, for some i we have $p(x) \notin X_i$. But $x \in X_i$, so by **(i)** applied to the partition (X_i, Y_i) we conclude that p is extendable, and the lemma is proved. □

The result that Q_n is a Fink graph immediately follows by induction. Indeed, the n-dimensional cube admits n partitions (X_i, Y_i), each into two $(n-1)$-dimensional subcubes, and $|X_1 \cap \ldots \cap X_n| = 1$.

The simplest way to produce a new Fink graph from two smaller ones is the following. Let X, Y be two disjoint Fink graphs. Choose arbitrary vertices $x \in X$, $y \in Y$, and join x to all neighbours of y in Y, and y to all neighbours of x in X. The resulting graph (we shall denote it by $X +_{(x,y)} Y$) is Fink, because it has two Fink partitions, (X, Y) and $(X \setminus \{x\} \cup \{y\}, Y \setminus \{y\} \cup \{x\})$, which satisfy the condition of Lemma 1(ii).

Take $n > 1$ disjoint one-edge graphs $X_i = \{a_i, b_i\}$, $i = 1, \ldots, n$. The graph

$$G_n = X_1 +_{(a_1,b_2)} X_2 +_{(a_2,b_3)} X_3 +_{(a_3,b_4)} \ldots +_{(a_{n-1},b_n)} X_n$$

(order of the operations from left to right) is a Fink graph on $2n$ vertices with $4n-4$ edges.

Now we shall study the number of edges in Fink graphs. The following lemma will be of use.

Lemma 2. *Let u be a vertex of degree 2 in a graph G, let v, w be its neighbours. The graph G is Fink if and only if both $G \setminus \{u, v\}$ and $G \setminus \{u, w\}$ are Fink graphs.*

Proof. The "if"part directly follows from Lemma 1(ii). For the "only if"part, suppose that $H = G \setminus \{u, v\}$ is not Fink, and p is a pairing of H which is not extendable. Let $p(w) = x$. Replace the pair $\{w, x\}$ by two pairs $\{w, u\}$, $\{v, x\}$. The resulting pairing p' of G is not extendable. Indeed, suppose that $(v_0 = w, v_1, \ldots, v_{2n-1})$ is an extension of p' in G. Then it follows that $v_1 = u$, $v_2 = v$, and $v_3 = x$. But then $(v_0 = w, v_3 = x, v_4, \ldots, v_{2n-1})$ is an extension in H of the pairing p, a contradiction. □

Let $E(n)$ denote the minimum number of edges in a Fink graph on $2n$ vertices.

Theorem 1. *$E(1) = 1$, $E(2) = 4$, $E(3) = 8$, $E(4) = 12$.*
For $n \geq 4$, $3n \leq E(n) \leq 4n - 4$.

Proof. The first equality is trivial. Now, notice that the only Fink graph with a vertex of degree 1 is a single edge, which implies the second equality.

Next, if a Fink graph has two adjacent vertices of degree 2 then it is the 4-cycle. Indeed, remove one of them together with its second neighbour: the remaining graph has a vertex of degree 1, and is Fink by Lemma 2.

Therefore, if a Fink graph on $2n > 4$ vertices with e edges has a vertex of degree 2 then removing it and one of its neighbours produces a Fink graph on $2n-2$ verices with at most $e - 4$ edges. This implies, consequently, the inequalities $E(3) \geq 8$ and $E(4) \geq 12$. These bounds are exact, as is shown by the Fink graphs G_3 and G_4 constructed above.

The inequality $E(n) \geq 3n$ for $n \geq 4$ can now be proved by induction via precisely the same argument, since the inequality $|E(G)| < 3n$ implies that G has a vertex of degree at most 2; the case $E(4) = 12$ serving as the induction base. The inequality $E(n) \leq 4n - 4$ is demonstrated by the graphs G_n. □

We conjecture that the true value of $E(n)$ is $4n - 4$.

Finally, we shall exploit the ideas of Lemma 1 to show that, for $n \geq 4$, some edges of the hypercube Q_n can be removed so that the resulting graph is still Fink.

Lemma 3. *The three-dimensional cube with one deleted edge has exactly two pairings which are not extendable.*

Proof. Q_3 has three partitions into two four-cycles. Suppose that the deleted edge e joined the 4-cycles X and Y. Then every pairing joining X to Y is extendable, by Lemma 1(i).

To examine the few remaining possibilities is easy. □

Lemma 4. *The four-dimensional cube with one deleted edge is a Fink graph.*

Proof. For the contrary, suppose that p is a pairing on $G = Q_4 - \{e\}$ which is not extendable.

Q_4 has four partitions into two subcubes Q_3; denote them, as before, by $X_i \cup Y_i$, $i = 1, \ldots, 4$. Suppose that the deleted edge e joined X_1 to Y_1. Then, after deleting the edge, X_1 and Y_1 remain isomorphic to Q_3, and one graph from each of the other pairs, say X_i, becomes isomorphic to $Q_3 \setminus \{e\}$.

Thus, X_1 and Y_1 are both Fink, and so no pair of p joins them, by Lemma 1(i). It follows that at least one of the other three partitions is joined by at least 4 pairs of p; let it be $X_4 \cup Y_4$, where $X_4 = Q_3 \setminus \{e\}$, and $Y_4 = Q_3$.

Now we can repeat the argument of Lemma 1(i). The pairing on X_4 can be chosen in at least three ways. By Lemma 3, at least one of them will be extendable in X_4. Since Y_4 is Fink, we can continue the argument, and find an extension for the pairing p. The lemma is proved. □

Now, as before, a direct inductive argument, using Lemma 4 as the induction base, immediately gives us

Theorem 2. *If some edges of the hypercube Q_n, $n \geq 4$, are removed, so that from every 4-face is removed at most one edge, then the resulting graph is Fink.*

We did not try to decide whether Q_4 with two deleted edges is Fink. If, at least for some choices of the pair of edges, this turned out to be so, then this would give a strengthening of the theorem.

References

[1] J. Fink, *Perfect matchings extend to Hamilton cycles in hypercubes*, J. Combin. Theory Ser. B 97 (2007), 1074–1076.

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