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UNIQUENESS OF THE PRIME GRAPH OF $\mathrm{L}_{16}(2)$

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Abstract. We complete the proof that the simple group $\mathrm{L}_{16}(2)$ is uniquely determined by its prime graph among all finite groups thus giving the first example of a recognizable-by-graph group whose prime graph is connected. We bridge the gap in the argument from [1] which purported to establish the same result.

Keywords: finite simple groups, prime graph, recognition

1. Introduction

Recall that the *prime graph* $\Gamma(G)$ of a finite group G is the graph whose vertex set is the set $\pi(G)$ of prime divisors of the order $|G|$ in which two distinct vertices $p, q \in \pi(G)$ are joined by an edge if and only if G contains an element of order pq.

It is often the case that the structure of the prime graph carries much information about the underlying group. For example, the finite groups whose prime graph is disconnected have a very limited structure as shown by the Gruenberg-Kegel theorem [2].

Of particular interest are the finite groups that are uniquely determined by their prime graph. The group G is said to be *recognizable by graph* if, for every finite group H, the equality of vertex-labeled graphs $\Gamma(H) = \Gamma(G)$ implies the isomorphism $H \cong G$. Examples of recognizable by graph groups are the simple groups J_1, M_{22}, M_{23}, M_{24}, Co_2, $\mathrm{G}_2(7)$, J_4, ${}^2\mathrm{G}_2(q)$, $q > 3$, see [3, 4]. One observes that all the known examples of such groups G have the property that $\Gamma(G)$ is disconnected. Whether

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there exists a recognizable-by-graph group whose prime graph is connected or there does not, has been unknown so far.

It turns out that proving recognizability of G by graph may require the knowledge of subtle properties of modular representations for G such as the presence of non-trivial fixed points of large prime-order elements. In this paper, we show how one can use this information in the case of 2-modular representations for $\mathrm{L}_{16}(2)$.

A more general approach to the study of fixed points of prime-order elements of simple linear and unitary groups in their irreducible equicharacteristic modules is taken in [8].

The main result of the present paper can be stated as follows:

Theorem 1. *If G is a finite group such that $\Gamma(G) = \Gamma(\mathrm{L}_{16}(2))$ and $G/\mathrm{O}_2(G) \cong \mathrm{L}_{16}(2)$ then $\mathrm{O}_2(G) = 1$.*

This, together with [1], gives the first example of a recognizable-by-graph group whose prime graph is connected.

Corollary 2. $\mathrm{L}_{16}(2)$ *is recognizable by its prime graph.*

We remark that an attempt to give a complete proof that $\mathrm{L}_{16}(2)$ is recognizable by graph was made in [1]. However, there appears to be a flaw in the proof of Lemma 3.4 (p. 56, line 20), where Lemma 2.5 is applied to conclude that $(2^{15}-1)p$ is an element order of G. But Lemma 2.5 cannot be used in the case $p = 2$. We do not see a way to correct this argument using the methods employed in [1].

2. Preliminaries

We first recollect some facts from the representation theory of algebraic groups. Details can be found in [5].

Let $G = \mathrm{SL}_n(F)$, where F is an algebraically closed field of characteristic p. Then G is a simply connected simple algebraic group of type A_l, where $l = n - 1$. Denote by ω_0 the zero weight and by $\omega_1, \ldots, \omega_l$ the fundamental weights of G (with respect to a fixed maximal torus of G and a system of positive roots). Let Ω denote the weight lattice of G. A weight $a_1\omega_1 + \ldots + a_l\omega_l \in \Omega$ is called *p-restricted*, if $0 \leqslant a_i < p$ for $i = 1, \ldots, l$.

For an irreducible (rational, finite dimensional) G-module L, we denote by $\Omega(L)$ the set of weights of L. Such modules are parameterized by their highest weights. The irreducible G-module of highest weight λ is customarily denoted by $L(\lambda)$. The module $L(\omega_i)$, $i = 0, \ldots, l$, is known to be isomorphic to the i-th exterior power of the natural n-dimensional module, see [5, II.2.15].

Lemma 3 (See [6, Lemma 13]). *Let $G = \mathrm{SL}_{l+1}(F)$ and let L be an irreducible G-module with highest weight λ. Suppose that $\lambda = a_1\omega_1 + \ldots + a_l\omega_l$ is p-restricted and $i \in \{0, 1, \ldots, l\}$ is the uniquely defined integer such that*

$$a_1 + 2a_2 + \ldots + la_l \equiv i \pmod{l+1}. \tag{1}$$

Then $\Omega(L(\omega_i)) \subseteq \Omega(L)$.

3. Main result

Proposition 4. *Let W be a nontrivial absolutely irreducible module for $H = \mathrm{L}_{16}(2)$ over a field of characteristic 2. Then the natural semidirect product $W \leftthreetimes H$ contains an element of one of the orders $2 \cdot 151$ or $2 \cdot 257$.*

Proof. By the Steinberg theorem [7, Theorem 43], W is the restriction to H of an irreducible rational module (which we also denote by W) for the algebraic group $SL_{16}(F)$, where F is an algebraically closed field of characteristic 2. Let $\lambda = a_1\omega_1 + \ldots + a_l\omega_l$ be the highest weight of W, where $l = 15$. We observe that λ is 2-restricted. Suppose that $i \in \{0, 1, \ldots, l\}$ is such that $a_1 + 2a_2 + \ldots + la_l \equiv i \pmod{l+1}$ and let V be the natural module for $SL_{16}(F)$. By Lemma 3, we have

$$\Omega(\wedge^i V) \subseteq \Omega(W), \tag{2}$$

where $\wedge^i V$ is the ith exterior power of V.

There is an element $a \in H$ of order 257 whose set of eigenvalues on V is

$$\{\zeta, \zeta^2, \zeta^{2^2}, \ldots, \zeta^{2^{15}}\}$$

for some $\zeta \in F^\times$ of order 257. Therefore, if i is even, there is an eigenvalue

$$\prod_{k=0}^{i/2-1} \zeta^{2^k} \cdot \zeta^{2^{8+k}} = 1$$

of a on $\wedge^i V$. Form (2) it follows that a fixes a nonzero vector in W, and $W \leftthreetimes H$ has an element of order $2 \cdot 257$.

Suppose that i is odd. There is an element $b \in H$ of order 151 whose set of eigenvalues on V is

$$\{1, \eta, \eta^2, \eta^{2^2}, \ldots, \eta^{2^{14}}\} \tag{3}$$

for some $\eta \in F^\times$ of order 151. In this case, b has on $\wedge^i V$ an eigenvalue 1. Indeed, if $i = 1, 3, 5, 7$ then each of the respective values

$$1, \qquad \eta^{1+2^5+2^{10}}, \qquad \eta^{1+2^3+2^6+2^9+2^{12}}, \qquad 1 \cdot \eta^{1+2^5+2^{10}} \cdot \eta^{2+2^6+2^{11}} \tag{4}$$

equals 1, whereas if $i = 15, 13, 11, 9$ then we have $16 - i = 1, 3, 5, 7$, respectively, and the product of the elements in (3) which do not occur in (4) equals 1 as well. Again, (2) implies that b fixes a nonzero vector in W, and $W \leftthreetimes H$ has an element of order $2 \cdot 151$. □

We are now ready to prove Theorem 1 stated in the introduction.

Proof. Let $H = L_{16}(2)$ and let G be such that $\Gamma(G) = \Gamma(H)$ and $G/W \cong H$, where $W = O_2(G)$. Without loss of generality (cf. [6, Lemma 9]) we may assume that G is a split extension of W by H with H acting on W absolutely irreducibly. If $W \neq 1$ then Proposition 4 implies that G has an element of one of the orders $2 \cdot 151$ or $2 \cdot 257$. However, the prime 2 is connected in $\Gamma(H)$ with neither 151 nor 257. This contradicts the assumption $\Gamma(G) = \Gamma(H)$ and completes the proof. □

References

[1] B. Khosravi, B. Khosravi, B. Khosravi, *A characterization of the finite simple group $L_{16}(2)$ by its prime graph*, Manuscr. Math. **126**: 1 (2008), 49–58.

[2] J. S. Williams, *Prime graph components of finite groups*, J. Algebra. **69**: 2 (1981), 487–513.

[3] M. Hagie, *The prime graph of a sporadic simple group*, Commun. Algebra, **31**: 9 (2003), 4405–4424.

[4] A. V. Zavarnitsine, *On recognition of finite groups by the prime graph*, Algebra and Logic. **45**: 4 (2006), 220–231.

[5] J. C. Jantzen, *Representations of algebraic groups.* Second edition. Mathematical Surveys and Monographs, 107. American Mathematical Society, Providence, RI, 2003.

[6] A. V. Zavarnitsine, *Properties of element orders in covers for* $\mathrm{L}_n(q)$ *and* $\mathrm{U}_n(q)$, Sib. Math. J., **49**: 2 (2008), 246–256.
[7] R. Steinberg, *Lectures on Chevalley Groups*, Yale University. New Haven. 1968.
[8] A. V. Zavarnitsine, *Fixed points of large prime-order elements in equicharacteristic action of linear and unitary groups*, in preparation.

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