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COMBINING INTUITIONISTIC CONNECTIVES AND ROUTLEY NEGATION

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Abstract. Logic N^* was defined as a logical framework for studying deductive bases of the well founded semantics (WFS) of logics programs with negation. Its semantical definition combines Kripke frames for intuitionistic logic with Routley's $*$-operator, which is used to interpret the negation operation. In this paper we develop algebraic semantics for N^*, describe its subdirectly irreducible algebraic models, describe completely the lattice of normal HT^2-extensions. The logic HT^2 is a finite valued extension of N^*, which is a deductive base of WFS. The last result can be used to check the maximality of this deductive base.

Keywords: Routley semantics, negation as modality, negation in logic programming, algebraic semantics, Heyting-Ockham algebra.

1. Introduction

The main object of investigations in this paper is the logic N^*, introduced in [5] as a logical framework for investigation of the well founded semantics (WFS) of logic programs with negation. Main features of the semantical Kripke style definition of this logic are reflected in the title of this paper. Positive connectives are defined as in intuitionistic logic whereas the negation is interpreted via Routley's $*$-operator [13]. This kind of semantics for negation is very popular in relevant logic. So, Kripke frames for N^* have the form $\langle W, \leq, *\rangle$, where $\langle W, \leq\rangle$ is a Kripke frame for intuitionistic logic and the antimonotonic wrt $\leq$ function $* : W \to W$ is used to interpret negation. In [14], R. Routley used frames of the form $\langle W, \leq, *\rangle$ to define

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semantics for Nelson's logic with constructive negation, but the set of possible worlds should be partitioned in this case into two parts, $W = U \cup U^*$, of "un-starred" (U) and "starred" (U^*) worlds. The $*$-operator should be a bijection between starred and un-starred worlds, moreover, the validity of formulas was defined in different ways for starred and un-starred worlds. In case of the logic N^* we have more direct combination of Kripke semantics for intutionistic connectives with Routley's $*$-operator. It looks rather unexpected that logic obtained via a combination of two very popular kinds of semantics was not considered earlier in the literature, however the author cannot point out a reference to the work, where the system equivalent to N^* was introduced.

The aim of this paper is to develop the algebraic semantics for the logic N^* and for the class of its normal extensions. This question is interesting in its own right, the adequate algebraic semantics for N^* and its normal extensions is provided by the variety of algebras, which we call Heyting-Ockham algebras. On one hand, these algebras are Heyting algebras if we forget on negation, on the other hand, they are Ockham lattices (if we consider implication-free reducts). Ockham lattices are bounded distributive lattices satisfying De Morgan laws, but unlike De Morgan algebras they need not satisfy the double negation elimination law. The study of these lattices has been initiated by J. Berman [3] and continued by A. Urquhart [15, 16], who suggested the name "Ockham lattices" (see [15, p.202] for the motivation of this choice).

Studying the algebraic semantics of N^* we have in mind also applications to the investigation of WFS. The most popular kinds of semantics for logic programs with negation such as stable models, partial stable models, and WFS, are based on distinguishing a special class of models of a program. As a result they generate non-monotonic consequence relations. To work effectively with a non-monotonic consequence it is useful to have a monotonic logic, whose properties are closely connected with the considered non-monotonic inference. This interplay of monotonic and non-monotonic consequences is catched by the notion of deductive base [8](see also [6]). Let $\mid\!\sim$ be a non-monotonic consequence relation and L be a logic with monotonic inference relation $\vdash_L$. Let $\Pi_1 \equiv_L \Pi_2$ mean that $\Pi_1 \vdash_L \Pi_2$ and $\Pi_2 \vdash_L \Pi_1$, and let $\Pi_1 \approx \Pi_2$ denote the nonmonotonic equivalence, i.e., $\Pi_1 \mid\!\sim \varphi \Leftrightarrow \Pi_2 \mid\!\sim \varphi$ for any φ. We say that a logic L is a *deductive base* for $\mid\!\sim$ iff

(i) $\vdash_L \subseteq \mid\!\sim$;
(ii) if $\Pi \mid\!\sim \varphi$ and $\varphi \vdash_L \psi$, then $\Pi \mid\!\sim \psi$;
(iii) if $\Pi_1 \equiv_L \Pi_2$, then $\Pi_1 \approx \Pi_2$.

We will say that the deductive base L is *strong* if it satisfies the additional condition:

$$\Pi_1 \not\equiv_L \Pi_2 \quad \Rightarrow \quad \text{there exists } \Gamma \text{ such that } \Pi_1 \cup \Gamma \not\approx \Pi_2 \cup \Gamma.$$

It was proved in [5] that the logic HT^2, a finite valued extension of N^*, is a strong deductive base for WFS. Having information on the lattice of HT^2-extensions we can pose a question on whether HT^2 provides a maximal deductive base for WFS.

The paper is structured as follows. In Section 2 we survey the results of [5, 7] on the logic N^*, to be self contained we include also the proofs. We discuss here whether the set of N^* axioms is independent and connections of the negation operator in N^* with negative modal operators of impossibility and unnecessity. Section 3 is devoted to the algebraic semantics of N^*. Finally, in Section 4 we give the first application of the developed technique and describe the lattice of normal HT^2-extensions. This result will be used in the subsequent work to study maximality of HT^2 as a deductive base for WFS.

2. Logic N^*

The main object of investigations in this paper is the logic N^*, introduced in [5] as an extension of logic N introduced by K. Dosen in [10] (see also [11]). Dosen's aim was to study logics weaker than Johansson's minimal logic and he suggested to interpret negation as a modal operator of impossibility. Recall basic definitions and facts concerning logic N. Formulas of N are built-up in the usual way using propositional variables from a given set *Prop* and the standard logical constants: $\wedge$, $\vee$, $\rightarrow$, $\neg$, respectively standing for conjunction, disjunction, implication and negation. The rules of inference for N are *modus ponens* and the contraposition rule for negation

$$\frac{\alpha \rightarrow \beta}{\neg\beta \rightarrow \neg\alpha}.$$

The set of axioms contains the axiom schemata of positive logic:

P1) $\alpha \rightarrow (\beta \rightarrow \alpha)$; **P2**) $(\alpha \wedge \beta) \rightarrow \alpha$;
P3) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$; **P4**) $(\alpha \wedge \beta) \rightarrow \beta$;
P5) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \wedge \gamma)))$; **P6**) $\alpha \rightarrow (\alpha \vee \beta)$;
P7) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$; **P8**) $\beta \rightarrow (\alpha \vee \beta)$;

and the only axiom scheme for negation:

$$\neg\alpha \wedge \neg\beta \rightarrow \neg(\alpha \vee \beta). \tag{1}$$

We use $\varphi \leftrightarrow \psi$ as abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. By a *logic* we mean a set of formulas closed under substitution and *modus ponens*. A logic is said to be a *normal logic* if it is closed additionally under the rule of contraposition. Identifying N with the set of its theorems we can consider it as a normal logic. For a logic Δ, we denote by $\mathbf{NExt}\Delta$ the class of its normal extensions, i.e. the class of normal logics Δ' such that $\Delta \subseteq \Delta'$.

Proposition 2.1. *Every logic $\Delta \in \mathbf{NExt}N$ is closed under the replacement rule*

$$\frac{\alpha \leftrightarrow \beta}{\gamma(\alpha) \leftrightarrow \gamma(\beta)}.$$

Proof. This statement can be proved in a standard way using axioms of positive logic and the contraposition rule for negation.

□

Definition 2.2. *A* frame *for N (N-*frame*) is a triple $\mathcal{W} = \langle W, \leq, R\rangle$ such that:*

(i) *W is a non empty set (of worlds),*
(ii) *$\leq$ is a partial ordering on W,*
(iii) *$R \subseteq W^2$ is an accessibility relation among worlds verifying $(\leq R) \subseteq R$*[1].

*An N-*model *$\mathcal{M} = \langle W, \leq, R, v\rangle$ is an N-frame $\mathcal{W} = \langle W, \leq, R\rangle$ augmented with a valuation function $v : Prop \longrightarrow 2^W$ satisfying the persistency condition:*

$$u \in v(p)\ \ \&\ \ u \leq w \ \ \Rightarrow\ \ w \in v(p). \tag{2}$$

We say in this case that $\mathcal{M}$ is a model over *$\mathcal{W}$.*

[1]In [10], K. Dosen defined more general class of frames. What we have defined was called in [10] *cohesive* frames. However, this narrower class of frames defines the same logic.

The validity of formulas at worlds of $\mathcal{M}$ is defined by induction as follows:

- $\mathcal{M}, w \models p \iff w \in v(p)$;
- $\mathcal{M}, w \models \varphi \wedge \psi \iff \mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$;
- $\mathcal{M}, w \models \varphi \vee \psi \iff \mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$;
- $\mathcal{M}, w \models \varphi \to \psi \iff \forall w'(w \leq w' \Rightarrow (\mathcal{M}, w' \models \varphi \Rightarrow \mathcal{M}, w' \models \psi))$;
- $\mathcal{M}, w \models \neg\varphi \iff \forall w'(wRw' \Rightarrow \mathcal{M}, w' \not\models \varphi)$.

In what follows we will write $w \models \varphi$ instead of $\mathcal{M}, w \models \varphi$ if it does not lead to a confusion.

As the reader may have already observed, for positive connective the validity is defined in exactly the same way as for Kripke models of intuitionistic logic, whereas the negation is treated as the modal operator of impossibility and the validity of negative formulas is defined with the help of the accessibility relation R.

A formula φ is said to be *true* in an N-model $\mathcal{M} = \langle W, \leq, R, v\rangle$, and we write $\mathcal{M} \models \varphi$, if $\mathcal{M}, w \models \varphi$ for all $v \in W$. We say that φ is *true* in an N-frame $\mathcal{W}$ if φ is true in every N-model $\mathcal{M} = \langle \mathcal{W}, v\rangle$ over $\mathcal{W}$. We write in this case $\mathcal{W} \models \varphi$. And finally, a formula φ is N-*valid*, in symbols $\models_N \varphi$, if it is true in every N-model (or, equivalently, in every N-frame). It is easy to prove by induction that the analog of condition (2) above holds for any formula φ, i.e.,

$$\mathcal{M}, u \models \varphi \ \& \ u \leq w \Rightarrow \mathcal{M}, w \models \varphi. \tag{3}$$

Moreover N is complete for this semantics.

Theorem 2.3. [10] *A formula φ is N-valid iff φ is a theorem of N.*

Let us consider now the normal logic N^* obtained by adding to N the following axiom schemes:

$$\neg(\alpha \to \alpha) \to \beta; \tag{4}$$

$$\neg((\alpha \to \alpha) \to \neg(\beta \to \beta)); \tag{5}$$

$$\neg(\alpha \wedge \beta) \to \neg\alpha \vee \neg\beta. \tag{6}$$

Axioms (1) and (6) together with the contraposition rule allow to prove that both De Morgan laws hold in N^*:

$$N^* \vdash \neg(\alpha \wedge \beta) \leftrightarrow \neg\alpha \vee \neg\beta, \ N^* \vdash \neg(\alpha \vee \beta) \leftrightarrow \neg\alpha \wedge \neg\beta.$$

Lemma 2.4. *Let $\mathcal{W} = \langle W, \leq, R\rangle$ be an N-frame. The following statements hold.*

i) [7] *We have $\mathcal{W} \models \neg(\alpha \to \alpha) \to \beta$ iff R is serial.*

ii) [7] *We have $\mathcal{W} \models \neg(\alpha \wedge \beta) \to \neg\alpha \vee \neg\beta$ iff for every $x \in W$ the set of all R-accessible from x elements is directed wrt $\leq$:*

$$\forall x, y, z \in W((xRy \wedge xRz) \Rightarrow \exists t \in W(xRt \wedge y \leq t \wedge z \leq t)). \tag{7}$$

iii) *If the relation R is serial, then $\mathcal{W} \models \neg((\alpha \to \alpha) \to \neg(\beta \to \beta))$.*

Proof. It is clear that the validity of the above schemes is equivalent to the validity of formulas $\neg(p \to p) \to q$, $\neg((p \to p) \to \neg(q \to q))$, and $\neg(p \wedge q) \to \neg p \vee \neg q$.

i) Consider a model $\mathcal{M} = \langle \mathcal{W}, v\rangle$ such that $v(q) = \varnothing$. Taking into account the definition of implication the relation $\mathcal{M} \models \neg(p \to p) \to q$ is equivalent to $w \not\models \neg(p \to p)$ for all w , i.e., for every w, there is u such that wRu and $u \models p \to p$. Since $p \to p$ is a valid formula, the latter is exactly equivalent to the fact that for every w, there is u such that wRu, i.e., that R is serial.

ii) Let R meet condition (7). Consider a model $\mathcal{M}$ over $\mathcal{W}$. Let $w \models \neg(p \wedge q)$ for some $w \in W$. Then

$$\forall u \in W(wRu \Rightarrow \mathcal{M}, u \not\models p \wedge q). \tag{8}$$

Assume that $w \not\models \neg p \vee \neg q$, then there are $u, v \in W$ such that wRu and $u \models p$; wRv and $v \models q$. According to (7) there exists $t \in W$ such that wRt, $u \leq t$, and $v \leq t$. By (3) we have $t \models p \wedge q$, which contradicts (8). We have thus proved $\mathcal{M} \models \neg(p \wedge q) \rightarrow \neg p \vee \neg q$.

Assume now that the condition (7) fails. Consequently, there are x, y, and z in W such that xRy and xRz, but there is no t such that $y \leq t$, $z \leq t$, and xRt. Consider a model $\mathcal{M} = \langle \mathcal{W}, v \rangle$, where $v(p) = \{w \in W \mid y \leq w\}$ and $v(q) = \{w \in W \mid z \leq w\}$. Note that $v(p) \cap v(q) \cap \{w \in W \mid xRw\} = \varnothing$, which means that for any w, if xRw, then $w \not\models p \wedge q$. This means that $x \models \neg(p \wedge q)$. At the same time, $x \not\models \neg p$, since xRy and $y \in v(p)$; $x \not\models \neg q$, since xRz and $z \in V(q)$. Thus, $x \not\models \neg p \vee \neg q$ and we have proved $\mathcal{W} \not\models \neg(p \wedge q) \rightarrow \neg p \vee \neg q$.

iii) Since R is serial, the formula $\neg(q \rightarrow q)$ fails to be valid for every $w \in W$. Consequently, $\mathcal{M}, w \not\models (p \rightarrow p) \rightarrow \neg(q \rightarrow q)$, and so $\mathcal{M} \models \neg((p \rightarrow p) \rightarrow \neg(q \rightarrow q))$. $\square$

Corollary 2.5. *For every N-frame $\mathcal{W}\langle W, \leq, R\rangle$ the following equivalence holds:*
$$\mathcal{W} \models N^* \quad \Leftrightarrow \quad R \text{ is serial and satisfies (7).}$$

We have thus distinguished the class of N-frames, which validates the logic N^*. It turns out that N^* is complete wrt to a narrower class of N^*-frames, which we define now.

An N-frame $\mathcal{W} = \langle W, \leq, R\rangle$ is called an N^*-*frame* if the relation R satisfies the following condition:

$$\forall x \in W \exists x^* \in W(xRx^* \wedge \forall y \in W(xRy \Rightarrow y \leq x^*)), \tag{9}$$

i.e., if for every world x in $\mathcal{M}$ there is the $\leq$-greatest world x^* in the set of worlds accessible from x via R.

It is obvious that in N^*-frames the relation R is serial and satisfies (7), therefore the last lemma implies

Corollary 2.6. *For every N-frame $\mathcal{W}\langle W, \leq, R\rangle$, we have $\mathcal{W} \models N^*$.*

The completeness of N^* wrt the class of N^*-frames will follow from Theorem 2.13 (see below). Item iii) of Lemma 2.4 saying that axiom (5) holds in all serial frames gives rise to the question whether axiom (5) is independent of other axioms of the logic N^*. The affirmative answer to this question is given by Corollary 2.19.

If $\mathcal{M} = \langle \mathcal{W}, v\rangle$ is a model over N^*-frame $\mathcal{W} = \langle W, \leq, R\rangle$, $x \in W$ and x^* is the greatest among R-accessible from x elements, then in view of condition (3) the validity $\mathcal{M}, x \models \neg\varphi$ is equivalent to $\mathcal{M}, x^* \not\models \varphi$. This simple observation allows us to define a Routley style semantics [13] for extensions of N^*.

Definition 2.7. *A* Routley frame *is a triple $\langle W, \leq, *\rangle$, where W is a set, $\leq$ a partial order on W and $* : W \rightarrow W$ is such that $x \leq y$ implies $y^* \leq x^*$. A Routley model $\mathcal{M} = \langle \mathcal{W}, v\rangle$ is a Routley frame $\mathcal{W}$ together with a valuation $v : Prop \rightarrow 2^W$ satisfying the persistency condition (2). As above, we say that $\mathcal{M}$ is a model over $\mathcal{W}$.*

The validity relation is defined in just the same way as for N except for the case of negative formulas:

$$\mathcal{M}, x \models \neg\varphi \;\;\Leftrightarrow\;\; \mathcal{M}, x^* \not\models \varphi. \tag{10}$$

For Routley frames, the validity of formulas also is persistent wrt $\leq$.

Lemma 2.8 (Persistence)**.** [7] *If $\mathcal{M} = \langle W, \leq, *, v\rangle$ is a Routley model and $u, w \in W$, then for any formula φ,*

$$\mathcal{M}, u \models \varphi \;\;\&\;\; u \leq w \;\;\Rightarrow\;\; \mathcal{M}, w \models \varphi. \tag{11}$$

Proof. It follows by induction on the structure of formulas and we have to check only the case of negation. If $u \models \neg\varphi$ and $u \leq w$, then $u^* \not\models \varphi$ and $w^* \leq u^*$. By induction hypothesis $w^* \not\models \varphi$, which means $w \models \neg\varphi$.

□

As usual, a formula φ is valid in a Routley model if it is valid at every world of this model, and it is valid in a Routley frame if it is valid in any model over this frame.

Completeness proofs for N^* and for its normal extensions can be obtained via the method of canonical models. Let S be some normal logic extending N^*, i.e., some set of formulas containing N^* and closed under substitution, contraposition rule, and *modus ponens*. First we say that a set of formulas Γ is a *theory* wrt S (*S-theory*) if it contains S and is closed under *modus ponens* and a *prime S-theory* if it additionally is non-trivial (does not equal to the set of all formulas) and satisfies the disjunction property:

$$\alpha \vee \beta \in \Gamma \Rightarrow \alpha \in \Gamma \;\text{ or }\; \beta \in \Gamma.$$

Let Σ and Δ be sets of formulas. A relation $\Sigma \vdash_S \Delta$ means that for some $\varphi_0, \ldots, \varphi_n \in \Delta$ the disjunction $\varphi_0 \vee \ldots \vee \varphi_n$ can be obtained from elements of S and Σ using the rule of *modus ponens*.

Next we prove in a standard way the following statement.

Lemma 2.9. *For any normal extension S of N^*, any sets of formulas Σ and Δ, if $\Sigma \not\vdash_S \Delta$, then there is a prime S-theory $\Gamma \supseteq \Sigma$ such that $\Gamma \not\vdash_S \Delta$.*

On this basis one defines canonical models as follows.

Definition 2.10 (Canonical model)**.** *Let S be a normal extension of N^*. The* canonical S-frame *is the triple $\mathcal{W}^c = \langle W^c, \leq^c, *^c\rangle$ where*

1. *W^c is the set of all prime theories wrt S;*
2. *$\Gamma \leq^c \Delta$ iff $\Gamma \subseteq \Delta$;*
3. *$\Gamma^{*^c} := \{\alpha \mid \neg\alpha \notin \Gamma\}$.*

The canonical S-model $\mathcal{M}^c$ *is the canonical S-frame $\mathcal{W}^c$ together with the valuation function v^c such that*

$$\Gamma \in v^c(p) \;\text{ iff }\; p \in \Gamma.$$

Proposition 2.11. [7] *For every normal N^*-extension S, the canonical model $\mathcal{M}^c$ is a Routley model.*

Proof. The only non-trivial item is to prove that the $*$-function is well defined, i.e., we have to verify that for any prime S-theory Γ, the set Γ^{*^c} is also a prime S-theory.

Since $\neg((\alpha \to \alpha) \to \neg(\beta \to \beta)) \in \Gamma$, we have $(\alpha \to \alpha) \to \neg(\beta \to \beta) \notin \Gamma^{*c}$, and so Γ^{*c} is non-trivial.

For $\varphi \in S$, we have $\neg\varphi \notin \Gamma$, otherwise Γ is trivial. Indeed, $\varphi \leftrightarrow (\alpha \to \alpha) \in S$ and we can replace $\alpha \to \alpha$ by φ in axiom (4). In this way we obtain that $\beta \in \Gamma$ for any β. We have thus proved that $S \subseteq \Gamma^{*c}$.

Let φ and $\varphi \to \psi$ be in Γ^{*^c}, i.e., $\neg\varphi, \neg(\varphi \to \psi) \notin \Gamma$. By the disjunction property of Γ we have $\neg\varphi \vee \neg(\varphi \to \psi) \notin \Gamma$, and by De Morgan law $\neg(\varphi \wedge (\varphi \to \psi)) \notin \Gamma$. The equivalence $\varphi \wedge (\varphi \to \psi) \leftrightarrow \varphi \wedge \psi$ holds in the positive intuitionistic logic and so in S. Applying the replacement rule we obtain $\neg(\varphi \wedge \psi) \notin \Gamma$. Since $\neg\psi \to \neg(\varphi \wedge \psi) \in N^*$, $\neg\psi \notin \Gamma$, i.e., $\psi \in \Gamma^{*^c}$. Thus, Γ^{*^c} is closed under *modus ponens*. The disjunction property of Γ^{*^c} follows by De Morgan laws. If $\varphi \vee \psi \in \Gamma^{*^c}$, then $\neg(\varphi \vee \psi) \notin \Gamma$ and we have the chain of equivalences:

$$\neg(\varphi \vee \psi) \notin \Gamma \ \text{ iff } \ \neg\varphi \wedge \neg\psi \notin \Gamma \ \text{ iff } \ \neg\varphi \notin \Gamma \text{ or } \neg\psi \notin \Gamma.$$

The latter means $\varphi \in \Gamma^{*^c}$ or $\psi \in \Gamma^{*^c}$.

□

The central point of completeness proof is the following.

Lemma 2.12 (Canonical model lemma)**.** [7] *Let S be an N^*-extension. In the canonical S-model $\mathcal{M}^c$, for every $\Gamma \in W^c$ and every φ,*

$$\mathcal{M}^c, \Gamma \models \varphi \ \Leftrightarrow \ \varphi \in \Gamma.$$

Proof. By induction on the complexity of φ. The base of induction is due to the definition of v^c. The case of positive connectives is treated in exactly the same way as for positive intuitionistic logic.

For negation, we have $\mathcal{M}^c, \Gamma \models \neg\varphi$ iff $\mathcal{M}^c, \Gamma^{*^c} \not\models \varphi$. This is equivalent by the induction hypothesis to $\varphi \notin \Gamma^{*^c}$, i.e., $\neg\varphi \in \Gamma$ by definition of Γ^{*^c}.

□

The completeness property of N^* follows by noting that if $\not\vdash_{N^*} \varphi$ then by Lemma 2.9 there is a prime N^*-theory Γ such that $\varphi \notin \Gamma$. It follows from Lemma 2.12 that φ does not hold at the world Γ of the canonical N^*-model and therefore is not N^*-valid.

Theorem 2.13. [7] *For any formula φ, $\vdash_{N^*} \varphi$ iff φ is valid in every Routley model.*

Corollary 2.14. *For any formula φ, we have:*

$$\varphi \in N^* \quad \Leftrightarrow \varphi \text{ it true in every } N^*\text{-model.}$$

Proof. Due to Corollary 2.6 it remains to prove the inverse implication. Let $\varphi \notin N^*$ and Routley model $\mathcal{M} = \langle W, \leq, *, v\rangle$ is such that $\mathcal{M} \not\models \varphi$. Consider N-model $\mathcal{M}^R = \langle W, \leq, R, v\rangle$, where $R := \{(x, y) \mid x, y \in W,\ y \leq x^*\}$. It is easy to check that $\mathcal{M}^R$ is an N^*-model. Moreover, since x^* is the greatest among the elements R-accessible from x, for every formula ψ and every $w \in W$, the equivalence holds:

$$\mathcal{M}, w \models \psi \ \ \Leftrightarrow \ \ \mathcal{M}^R, w \models \psi. \tag{12}$$

In particular, $\mathcal{M}^R \not\models \varphi$.

□

Now we turn to the axiom (4). It can be used to define an intuitionistic negation in N^*. Let us consider $-\alpha$ as abbreviation for $\alpha \to \neg(p_0 \to p_0)$. From axiom (4) it follows that for any Routley model $\mathcal{M} = \langle W, \leq, *, v\rangle$, the formula $\neg(p_0 \to p_0)$ is not valid at any world $w \in W$. Therefore, the validity of the derived expression $-\alpha$ coincides exactly with the interpretation of negation in intuitionistic logic:

$$\mathcal{M}, w \models -\alpha \ \Leftrightarrow \ \forall w'(w \leq w' \ \Rightarrow \ \mathcal{M}, w' \not\models \alpha).$$

Let N^{*-} denote the definitional extension of N^* obtained by adding the connective $-$ to its language and the scheme $-\alpha \leftrightarrow (\alpha \to \neg(p_0 \to p_0))$ to the set of its axioms.

Proposition 2.15. [7] *The* $\langle \vee, \wedge, \to, - \rangle$*-fragment of* N^{*-} *coincides with intuitionistic logic.*

Proof. The fact that the validity of positive connectives and of the new connective $-$ is defined in exactly the same way as in intuitionistic logic implies that every intuitionistic tautology written in the language $\langle \vee, \wedge, \to, - \rangle$ is provable in N^{*-}. If φ is not an intuitionsitic tautology, then $\mathcal{W} \not\models \varphi$ for some Kripke frame $\mathcal{W} = \langle W, \leq \rangle$. Take some $a \in W$ and define $* : W \to W$ by $x^* = a$ for all $x \in W$. Obviously, $\mathcal{W}' := \langle W, \leq, * \rangle$ is a Routley frame, and since $\neg$ does not occur in φ, we still have $\mathcal{W}' \not\models \varphi$.

□

Let us compare logic N^{*-} with systems $HK\Diamond'$ and $HK\Box'$, which was introduced by K. Dosen in [9] and formalize properties of intuitionistic negative modal operators. Here $\Diamond'$ stand for impossibility operator and $\Box'$ for unnecessity operator. Both systems $HK\Diamond'$ and $HK\Box'$ have the rule of *modus ponens* and the antimonotonicity rule for modal operator

$$\frac{\varphi \to \psi}{\Diamond'\psi \to \Diamond'\varphi} \text{ (resp., } \frac{\varphi \to \psi}{\Box'\psi \to \Box'\varphi}\text{)}.$$

Axiomatics include axioms schemes of intuitionistic logic. Additionally $HK\Diamond'$ has the axiom schemes

$\Diamond'1$ $\quad \Diamond'\alpha \wedge \Diamond'\beta \to \Diamond'(\alpha \vee \beta)$;
$\Diamond'2$ $\quad \Diamond' - (\alpha \to \alpha)$;

and $HK\Box'$ include the axiom schemes

$\Box'1$ $\quad \Box'(\alpha \wedge \beta) \to \Box'\alpha \vee \Box'\beta$;
$\Box'2$ $\quad -\Box'(\alpha \to \beta)$.

Let us replace in both systems connectives $\Diamond'$ and $\Box'$ by $\neg$. After such replacement the antimonotonicity rule turns to the contraposition rule for $\neg$, axiom $\Diamond'1$ turns to axiom (1), and $\Box'1$ to axiom (6). It is easy to see that axiom $\Diamond'2$ becomes equivalent to axiom (5), and $\Box'2$ to axiom (4). Thus, we can see that the negation of N^{*-} combines the properties of both negative modal operators of impossibility and unnecessity.

Point out some further properties of N^*.

Lemma 2.16. *The following statements hold.*

1. *For any* α *and* β, $N^* \vdash \alpha$ *implies* $N^* \vdash \neg\alpha \to \beta$.
2. *There is no* α *such that* $N^* \vdash \alpha$ *and* $N^* \vdash \neg\alpha$.
3. *For every* α, *if* $N^* \vdash \alpha$, *then* $N^* \vdash \neg\neg\alpha$.

Proof. (1) It follows from axiom (4) by replacement rule.

(2) If $N^* \vdash \alpha$ and $N^* \vdash \neg\alpha$, then by the previous item we have $N^* \vdash \beta$ for any β. But N^* is non-trivial due to completeness theorem.

(3) Let $N^* \vdash \alpha$. By completeness theorem α is true in all Routley models. Check the validity of $\neg\neg\alpha$. The condition $\mathcal{M}, x \models \neg\neg\alpha$ is equivalent to $\mathcal{M}, x^{**} \models \alpha$. In this way, $\neg\neg\alpha$ is also true in all Routley models.

□

Concluding this section we prove that axiom (5) cannot be inferred from other axioms of N^*. To this end we use the sysntactic method known as "Kleene's slash" (see [1, 12]).

Let N' denote the extension of N via axioms (4) and (6). By induction on the length of formula φ we define a new predicate $|_{N'}\,\varphi$ ("Kleene's slash") on the set of formulas as follows (further on, instead of "$N' \vdash \varphi$ and $|_{N'}\,\varphi$" we write $\Vdash_{N'} \varphi$):

$$
\begin{array}{lcl}
|_{N'}\, p & \Leftrightarrow & N' \vdash p,\ \ p \in Prop; \\
|_{N'}\, \varphi \wedge \psi & \Leftrightarrow & |_{N'}\, \varphi \ \text{ and } \ |_{N'}\, \psi; \\
|_{N'}\, \varphi \vee \psi & \Leftrightarrow & \Vdash_{N'} \varphi \ \text{ or } \ \Vdash_{N'} \psi; \\
|_{N'}\, \varphi \to \psi & \Leftrightarrow & (\Vdash_{N'} \varphi \ \Rightarrow \ |_{N'}\, \psi); \\
|_{N'} \neg\varphi & \Leftrightarrow & \vdash_{N'} \ \varphi \text{ and } \vdash_{N'} \ \neg\varphi.
\end{array}
$$

By Item (2) of Lemma 2.16, which obviously remains true for N', the definition of $|_{N'}$ for negative formulas is degenerate, $|_{N'}\neg\varphi$ is false for any φ. This allows to prove that negative formulas are not inferable in N', as well as the disjunctive property and the constructive negation property for N'. Recall that a set of formulas Φ has the constructive negation property if $\neg(\varphi \wedge \psi) \in \Phi$ implies $\neg\varphi \in \Phi$ or $\neg\psi \in \Phi$.

Lemma 2.17. *For any formula φ, the provability $N' \vdash \varphi$ implies $|_{N'}\,\varphi$.*

Proof. Let $N' \vdash \varphi$. By induction on the length of proof we show that $|_{N'}\,\varphi$. In the proof we omit the lower index N'.

Prove that this statement holds for axioms of N'. For axioms of positive logic we argue as in [1].

Consider axiom (1). By definition of $|_{N^*}$ we have to prove that $\Vdash \neg\alpha \wedge \neg\beta$ implies $|\neg(\alpha \vee \beta)$. The condition $|\,\neg\alpha \wedge \neg\beta$ is equivalent to $|\alpha$ and $|\beta$. Thus, the premiss of the implication is false and the very implication is true.

Axioms (4) and (6) are implications with negative premisses, therefore, the validity of $|\,\varphi$, where φ is either of these axioms, is equivalent to the implication with false premiss.

Now we consider the rules of inference.

Let φ be obtained by *modus ponens* from $\psi \in N^*$ and $\psi \to \varphi \in N^*$. By induction hypothesis $|\,\psi$ and $|\,\psi \to \varphi$. Consequently, $\Vdash \psi$ implies $|\,\varphi$, whence $|\,\varphi$.

It remains to consider the contraposition rule, but $|\,\neg\psi \to \neg\varphi$ is true in any case, since the premiss of the implication $\neg\psi \to \neg\varphi$ is negative.

□

Proposition 2.18. *The following statements hold.*

(1) *For any φ, the formula $\neg\varphi$ is not inferable in N'.*
(2) *The logic N' possesses the disjunction property.*
(3) *The logic N' possesses the constructive negation property.*

Proof. (1) If $N' \vdash \neg\varphi$, then $|_{N'}\,\neg\varphi$ is true, which is impossible by definition of $|_{N'}$.

(2) Let $N' \vdash \varphi \vee \psi$. By Lemma 2.17 the condition $|_{N'}\,\varphi \vee \psi$ holds, consequently, $\Vdash_{N'} \varphi$ or $\Vdash_{N'} \psi$.

(3) The constructive negation property is valid for N' in a trivial way, since in view of Item 1 formulas of the form $\neg(\varphi \wedge \psi)$ are not provable in N'.

□

Since axiom (5) is a negative formula, we obtain

Corollary 2.19. *Axiom (5) is not inferable from other axioms of N^*.*

Finally, we pose the question: does the logic N^* possess the disjunctive property and the constructive negation property?

3. Algebraic semantics for N^*

We turn now to the study of an algebraic semantics for N^*.

Definition 3.1. *An algebra $\mathcal{A} = \langle A, \wedge, \vee, \to, \neg, 0, 1\rangle$ is called a* Heyting-Ockham algebra (HO-algebra) *if the following conditions are satisfied:*

(1) *The $\neg$-free reduct of $\mathcal{A}$, $\mathcal{A}^H := \langle A, \wedge, \vee, \to, 0, 1\rangle$, is a Heyting algebra, i.e. it is a restricted lattice with the least element 0, the greatest element 1, and the implication operation $\to$ satisfying the equivalence*
$$x \leq a \to b \quad \textit{iff} \quad a \wedge x \leq b.$$
(2) *The $\to$-free reduct of $\mathcal{A}$, $\mathcal{A}^O := \langle A, \wedge, \vee, \neg, 0, 1\rangle$, is an Ockham lattice, i.e. it is a bounded distributive lattice satisfying the identities:*
$$\neg(x \vee y) = \neg x \wedge \neg y, \ \neg(x \wedge y) = \neg x \vee \neg y, \ \neg 0 = 1, \ \neg 1 = 0.$$

The intuitionistic negation $-a$ on an HO-algebra $\mathcal{A}$ is an abbreviation for $a \to 0$. By $\leq_{\mathcal{A}}$ we denote the lattice ordering on $\mathcal{A}$. The subscript in $\leq_{\mathcal{A}}$ is omitted if it does not lead to a confusion. The expression $(a \to b) \wedge (b \to a)$ is abbreviated as $a \leftrightarrow b$.

Lemma 3.2. *Let $\mathcal{A}$ be an HO-algebra. For every $a, b, c \in \mathcal{A}$, the following holds:*

(1) *$a \to b = 1$ iff $a \leq b$;*
(2) *$a \leftrightarrow b = 1$ iff $a = b$;*
(3) *$(a \vee b) \to c = (a \to c) \wedge (b \to c)$;*
(4) *$a \to (b \wedge c) = (a \to b) \wedge (a \to c)$;*
(5) *if $a \leq b$, then $\neg b \leq \neg a$.*

Proof. Items (1)–(4) hold since $\mathcal{A}^H$ is a Heyting algebra. Let $a \leq b$, i.e. $b = a \vee b$. We calculate using De Morgan law:
$$\neg b = \neg(a \vee b) = \neg a \wedge \neg b,$$
whence $\neg b \leq \neg a$. □

Proposition 3.3. *The class $\mathcal{V}^*$ of all HO-algebras forms a variety.*

Proof. It follows easily from the facts that Heyting algebras as well as Ockham lattices form varieties. Joining the defining identities for Heyting algebras and for Ockham lattices yields the set of defining identities for the variety of HO-algebras. □

Recall that a variety $\mathcal{V}$ is called *congruence distributive* if for any algebra $\mathcal{A} \in \mathcal{V}$, the lattice of congruences of algebra $\mathcal{A}$ is distributive. A variety $\mathcal{V}$ is called *congruence permutable* if for any algebra $\mathcal{A} \in \mathcal{V}$, its congruences are permutable wrt composition. An *arithmetic variety* is a variety, which is congruence permutable and congruence distributive. According to Pixley's theorem (see [4]) a variety $\mathcal{V}$ is arithmetic if and only if there exists a term $m(x, y, z)$ such that the identities
$$m(x, y, x) = m(x, y, y) = m(y, y, x) = x$$
hold in $\mathcal{V}$.

Proposition 3.4. *The variety of $\mathcal{V}^*$ of HO-algebras is arithmetic.*

Proof. In case of HO-algebras, as well as in case of Heyting algebras (see [4]), we can use the term

$$m(x,y,z) := ((x \to y) \to z) \wedge ((z \to y) \to x) \wedge (x \vee z)$$

to establish that the variety of HO-algebras is arithmetic. The verification is straightforward.

□

For a variety of algebras V, denote by $Eq(V)$ the equational theory of this variety, i.e., the set of all identities that hold on all algebras of V.

For an HO-algebra $\mathcal{A}$, we define an $\mathcal{A}$-valuation v in a standard way as a homomorphism from the algebra of formulas to $\mathcal{A}$. A formula φ is said to be *true* on $\mathcal{A}$, symbolically $\mathcal{A} \models \varphi$, if $v(\varphi) = 1$ for any $\mathcal{A}$-valuation v, or, equivalently, if the identity $\varphi = 1$ holds on $\mathcal{A}$. Put $L\mathcal{A} := \{\varphi \mid \mathcal{A} \models \varphi\}$ for an HO-algebra $\mathcal{A}$ and $L\mathcal{K} := \bigcap\{L\mathcal{A} \mid \mathcal{A} \in \mathcal{K}\}$ for a class $\mathcal{K}$ of HO-algebras. In a standard way one can prove the following

Proposition 3.5. *For any HO-algebra $\mathcal{A}$ and for any class of HO-algebras $\mathcal{K}$, the sets $L\mathcal{A}$ and $L\mathcal{K}$ are normal logics extending N^*.*

The fact that N^* and its normal extensions are closed under the replacement rule allows to define in a standard way Lindenbaum algebras of these logics. To an arbitrary logic $\Delta \in \mathbf{NExt}N^*$ we assign the equivalence relation $\equiv_\Delta$ on the set of formulas. For any φ and ψ, we put

$\varphi \equiv_\Delta \psi$ iff $\Delta \vdash \varphi \leftrightarrow \psi$.

The abstract class of φ wrt $\equiv_\Delta$ is denoted as $[\varphi]_\Delta$, i.e., $[\varphi]_\Delta := \{\psi \mid \psi \equiv_\Delta \varphi\}$. Put $L_\Delta := \{[\varphi]_\Delta \mid \varphi \in For\}$. Since Δ is closed under the replacement rule, $\equiv_\Delta$ is a congruence on the algebra of formulas. Due to this fact we can define on the set L_Δ the operations $\vee, \wedge, \to$ and $\neg$ as follows:

$[\varphi]_\Delta * [\psi]_\Delta := [\varphi * \psi]_\Delta$, where $* \in \{\vee, \wedge, \to\}$;
$\neg[\varphi]_\Delta := [\neg\varphi]_\Delta$.

Put $1_\Delta = [p_0 \to p_0]_\Delta$ and $0_\Delta = [\neg(p_0 \to p_0)]_\Delta$. The algebra

$$\mathcal{L}(\Delta) := \langle L_\Delta, \vee, \wedge, \to, \neg, 0_\Delta, 1_\Delta\rangle$$

is called *Lindenbaum algebra* of Δ. Positive logic axioms and axiom (4) allow to prove that $\langle L_\Delta, \vee, \wedge, \to, \neg, 0_\Delta, 1_\Delta\rangle$ is a Heyting algebra. In Heyting algebras, the equality $a = b$ is equivalent to $a \leftrightarrow b = 1$, and the inequality $a \leq b$ to $a \to b = 1$ (see Lemma 3.2), therefore, De Morgan laws guarantee that the identities $\neg(x \vee y) = \neg x \wedge \neg y$ and $\neg(x \wedge y) = \neg x \vee \neg y$ hold on $\mathcal{L}(\Delta)$. Axiom (4) implies that $\neg 1_\Delta$ is the least element of $\mathcal{L}(\Delta)$, i.e., $\neg 1_\Delta = 0_\Delta$. The equality $\neg 0_\Delta = 1_\Delta$ follows from Item (3) of Lemma 2.16. Thus, the reduct $\langle L_\Delta, \vee, \wedge, \neg, \neg, 0_\Delta, 1_\Delta\rangle$ is an Ockham lattice and we have proved

Proposition 3.6. *For every logic $\Delta \in \mathbf{NExt}N^*$, its Lindenbaum algebra $\mathcal{L}(\Delta)$ is an HO-algebra.*

The lattice of subvarieties of the variety $\mathcal{V}^*$ we denote by $Sub(\mathcal{V}^*)$. For $\Delta \in \mathbf{NExt}N^*$, put

$$V(\Delta) := \{\mathcal{A} \mid \Delta \subseteq L\mathcal{A}\}.$$

It is obvious that $V(\Delta) \in Sub(\mathcal{V}^*)$.

We show that every logic $\Delta \in \mathbf{NExt}N^*$ is characterized by the variety $V(\Delta)$.

Theorem 3.7. *For every logic $\Delta \in \mathbf{NExt}N^*$ and formula φ the following equivalence holds:*

$$\varphi \in \Delta \quad \textit{iff} \quad \varphi = 1 \in Eq(V(\Delta)).$$

In particular,

$$\varphi \in N^* \quad \textit{iff} \quad \varphi = 1 \in Eq(\mathcal{V}^*).$$

Proof. Consider a logic $\Delta \in \mathbf{NExt}N^*$. If $\varphi \in \Delta$, then $\varphi = 1 \in Eq(V(\Delta))$ by definition of $V(\Delta)$.

To prove the inverse implication check that the Lindenbaum algebra $\mathcal{L}(\Delta)$ belongs to $V(\Delta)$. Let $\varphi \in \Delta$ and let v be an $\mathcal{L}(\Delta)$-valuation. If $\varphi = \varphi(p_1, \ldots, p_n)$ and $v(p_1) = [\psi_1]_\Delta, \ldots, v(p_n) = [\psi_n]_\Delta$, then $v(\varphi) = [\varphi(\psi_1, \ldots, \psi_n)]_\Delta = 1_\Delta$ in view of $\varphi(\psi_1, \ldots, \psi_n) \in \Delta$. Thus, the identity $\varphi = 1$ holds on $\mathcal{L}(\Delta)$ and $\mathcal{L}(\Delta) \in V(\Delta)$.

Assume that $\varphi \notin \Delta$ and an $\mathcal{L}(\Delta)$-valuation v is such that $v(p) = [p]_\Delta$ for all $p \in Prop$. Then $v(\varphi) = [\varphi]_\Delta \neq 1_\Delta$. In this way, $\varphi = 1 \notin Eq(V(\Delta))$.

□

Now we consider the inverse mapping from $Sub(\mathcal{V}^*)$ to $\mathbf{NExt}N^*$. For a subvariety $V \in Sub(\mathcal{V}^*)$, put

$$L(V) := \{\varphi \mid \varphi = 1 \in Eq(V)\}.$$

Proposition 3.8. *For every $V \in Sub(\mathcal{V}^*)$, we have $L(V) \in \mathbf{NExt}N^*$. Moreover, $V = V(L(V))$.*

Proof. Theorem 3.7 implies that $N^* = L(\mathcal{V}^*)$. Since $V \subseteq \mathcal{V}^*$, we obtain $N^* = L(\mathcal{V}^*) \subseteq L(V)$. By definition the set $L(V)$ is closed under the substitution rule. Check that $L(V)$ is closed under the rules of *modus ponens* and contraposition. Let φ, $\varphi \to \psi \in L(V)$. Take an arbitrary algebra $\mathcal{A} \in V$ and an arbirary $\mathcal{A}$-valuation v. Then $v(\varphi) = 1$ and $v(\varphi) \to v(\psi) = 1$. The last equality is equivalent to $v(\varphi) \leq v(\psi)$ by Lemma 3.2. In this way, $v(\psi) = 1$. Consequently, $\psi \in L(V)$. Assume that $\varphi \to \psi \in L(V)$, i.e., $v(\varphi) \leq v(\psi)$ for any $\mathcal{A} \in V$ and $\mathcal{A}$-valuation v. By Lemma 3.2 the inequality $v(\varphi) \leq v(\psi)$ implies $v(\neg\psi) \leq v(\neg\varphi)$. Thus, $\neg\psi \to \neg\varphi \in L(V)$ and we have proved that $L(V) \in \mathbf{NExt}N^*$.

Check the equality $V = V(L(V))$. By definition $Eq(V(L(V))) \subseteq Eq(V)$. Prove the inverse inclusion. Let $\varphi = \psi \in Eq(V)$. By Lemma 3.2 $\varphi \leftrightarrow \psi = 1 \in Eq(V)$. Consequently, $\varphi \leftrightarrow \psi \in L(V)$ and $\varphi \leftrightarrow \psi = 1 \in Eq(V(L(V)))$. Applying again Lemma 3.2 we obtain $\varphi = \psi \in Eq(V(L(V)))$.

□

Theorem 3.9. *The mappings $V : \mathbf{NExt}N^* \to Sub(\mathcal{V}^*)$ and $L : Sub(\mathcal{V}^*) \to \mathbf{NExt}N^*$ are mutually inverse dual lattice isomorphisms between $\mathbf{NExt}N^*$ and $Sub(\mathcal{V}^*)$.*

Proof. The equality $V = V(L(V))$ was established in the last proposition. The equality $\Delta = L(V(\Delta))$, where $\Delta \in \mathbf{NExt}N^*$, follows from the definition of $L(V)$ and Theorem 3.7. Thus, $V : \mathbf{NExt}N^* \to Sub(\mathcal{V}^*)$ and $L : Sub(\mathcal{V}^*) \to \mathbf{NExt}N^*$ are mutually inverse bijections. It is obvious that both mappings V and L inverse the orderings. This means that $\mathbf{NExt}N^*$ and $Sub(\mathcal{V}^*)$ are dually isomorphic as orders, consequently, they are dually isomorphic as lattices too.

□

The last statement means that HO-algebras provide the adequate algebraic semantics to study the lattice of normal N^*-extensions. To go ahead we need some basis of algebraic theory of HO-algebras.

Definition 3.10. *A non-empty subset F of an HO-algebra $\mathcal{A}$ is said to be a $*$-*filter *on $\mathcal{A}$ if the following conditions hold: 1) if $a, b \in F$, then $a \wedge b \in F$; 2) if $a \in F$ and $a \leq b$, then $b \in F$; 3) if $a \to b \in F$, then $\neg b \to \neg a \in F$.*

In other words, a $*$-filter on an HO-algebra $\mathcal{A}$ is a filter on the Heyting algebra $\mathcal{A}^H$ satisfying additionally condition 3).

Denote by $\mathcal{F}^*(\mathcal{A})$ the lattice of $*$-filters on $\mathcal{A}$ and by $Con(\mathcal{A})$ the lattice of congruences on $\mathcal{A}$. For $F \in \mathcal{F}^*(\mathcal{A})$ and $\theta \in Con(\mathcal{A})$, put

$$\theta_F := \{(a, b) \mid a \leftrightarrow b \in F\} \text{ and } F_\theta := \{a \mid (a, 1) \in \theta\}.$$

Proposition 3.11. *Let $\mathcal{A}$ be an HO-algebra.*

1. *For every $F \in \mathcal{F}^*(\mathcal{A})$, the relation θ_F is a congruence on $\mathcal{A}$.*
2. *For every $\theta \in Con(\mathcal{A})$, the set F_θ is a $*$-filter on $\mathcal{A}$.*
3. *The mappings $F \mapsto \theta_F$, $F \in \mathcal{F}^*(\mathcal{A})$, and $\theta \mapsto F_\theta$, $\theta \in Con(\mathcal{A})$ determine mutually inverse isomorphisms of the lattices $\mathcal{F}^*(\mathcal{A})$ and $Con(\mathcal{A})$.*

Proof. 1. Since $\mathcal{A}^H$ is a Heyting algebra, θ_F is a congruence wrt positive connectives. Let $a\theta_F b$, i.e., $a \to b$ and $b \to a$ are in F. Definition of $*$-filters implies that $\neg b \to \neg a$ and $\neg a \to \neg b$ are in F. The latter means $\neg a \theta_F \neg b$.

2. If $\theta \in Con(\mathcal{A})$, then θ is a congruence on Heyting algebra $\mathcal{A}^H$, and so F_θ is a filter on $\mathcal{A}^H$. It remains to check condition 3) of Definition 3.10. If $a \to b \in F_\theta$, then $(a \wedge b) \leftrightarrow a \in F_\theta$, i.e., $(a \wedge b)\theta a$. Indeed, $(a \wedge b) \to a = 1 \in F_\theta$ and $a \to (a \wedge b) = (a \to a) \wedge (a \to b) = a \to b$. Since $\theta \in Con(\mathcal{A})$, we have $\neg(a \wedge b)\theta\neg a$. Consequently, $\neg(a \wedge b) \to \neg a \in F_\theta$. Using De Morgan law and Lemma 3.2 we obtain

$$\neg(a \wedge b) \to \neg a = (\neg a \vee \neg b) \to \neg a = (\neg a \to \neg a) \wedge (\neg b \to \neg a) = \neg b \to \neg a.$$

Thus, $\neg b \to \neg a \in F_\theta$.

3. That $F = F_{\theta_F}$ and $\theta = \theta_{F_\theta}$ follows from the facts that F is a filter and θ is a congruence on a Heyting algebra $\mathcal{A}^H$. Thus, the mappings $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ are mutually inverse. Obviously, these mappings preserve the inclusion relation, therefore, they are order isomorphisms, in which case they are lattice isomorphisms too.

□

The notion of $*$-filter can be simplified as follows.

Proposition 3.12. *Let $\mathcal{A}$ be an HO-algebra and $\varnothing \neq F \subseteq \mathcal{A}$. The set F is a $*$-filter on $\mathcal{A}$ iff it is a filter on $\mathcal{A}^H$ and satisfies the condition $3')$ $-\neg a \in F$ for $a \in F$.*

Proof. Let F be a $*$-filter on $\mathcal{A}$. If $a = 1 \to a \in F$, then by condition 3) we have $\neg a \to \neg 1 = \neg a \to 0 = -\neg a \in F$.

To prove the inverse implication we need the following

Lemma 3.13. *The formula $-\neg(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ belongs to N^*.*

Proof. Let $\mathcal{M} = \langle W, \leq, *, v\rangle$ be a Routley model. Check that for every $w \in W$, $w \models -\neg(\varphi \to \psi)$ implies $w \models \neg\psi \to \neg\varphi$. Note that

$$w \models \neg\psi \to \neg\varphi \Leftrightarrow \forall u \geq w(u \models \neg\psi \Rightarrow u \models \neg\varphi) \Leftrightarrow \forall u \geq w(u^* \not\models \psi \Rightarrow u^* \not\models \varphi)$$

$$\Leftrightarrow \forall u \geq w(u^* \models \varphi \Rightarrow u^* \models \psi).$$

Using the resulting equivalence we obtain

$$w \models -\neg(\varphi \to \psi) \Leftrightarrow \forall u \geq w(u \not\models \neg(\varphi \to \psi)) \Leftrightarrow \forall u \geq w(u^* \models \varphi \to \psi) \Longrightarrow$$
$$\Longrightarrow \forall u \geq w(u^* \models \varphi \Rightarrow u^* \models \psi) \Leftrightarrow w \models \neg\psi \to \neg\varphi.$$

□

Let F be a filter on $\mathcal{A}^H$ satisfying condition $3')$. If $a \to b$, then $-\neg(a \to b) \in F$ by condition $3')$. It follows from the last lemma and completeness theorem for N^* that $-\neg(a \to b) \leq \neg b \to \neg a$. Consequently, $\neg b \to \neg a \in F$.

□

Following [17] we define a $\Box$-algebras as follows.

An algebra $\mathcal{A} = \langle A, \vee, \wedge, \to, \Box, 0, 1\rangle$ is said to be a *$\Box$-algebra* if its $\Box$-free reduct $\mathcal{A}^H = \langle A, \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra and the operator $\Box$ satisfies the properties:

$$\Box 1 = 1,\ \Box(a \wedge b) = \Box a \wedge \Box b$$

for all $a, b \in A$. The operator $\Box$ is an algebraic interpretation of the intuitionistic modal operator of necessity.

To any HO-algebra $\mathcal{A} = \langle A, \vee, \wedge, \to, \neg, 0, 1\rangle$ we assign the algebra

$$\mathcal{A}^{\Box} := \langle A, \vee, \wedge, \to, -\neg, 0, 1\rangle.$$

Proposition 3.14. *For any HO-algebra $\mathcal{A}$, the algebra $\mathcal{A}^{\Box}$ is a $\Box$-algebra, i.e., the operator "$-\neg$" satisfies the properties:*

$$-\neg 1 = 1 \text{ and } -\neg(a \wedge b) = -\neg a \wedge -\neg b$$

for all $a, b \in \mathcal{A}$.

Proof. It is clear that $-\neg 1 = -0 = 1$. For $a, b \in \mathcal{A}$, we calculate

$$-\neg(a \wedge b) = -(\neg a \vee \neg b) = (\neg a \vee \neg b) \to 0 = (\neg a \to 0) \wedge (\neg b \to 0) = -\neg a \wedge -\neg b.$$

□

According to [17] a subset F of a $\Box$-algebra $\mathcal{A}$ is called a $\Box$-filter if F is a filter on the corresponding Heyting algebra $\mathcal{A}^H$ and F is closed under the rule $a/\Box a$. It was proved in [17] that $\Box$-filters are in one-to-one correspondence with congruences on the $\Box$-algebra $\mathcal{A}$, and this correspondence is established via the mappings $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ defined in the same way as for $*$-filters and congruences on HO-algebras. Using this fact and Proposition 3.12 we obtain

Proposition 3.15. *Let $\mathcal{A}$ be an HO-algebra. The congruence lattices of $\mathcal{A}$ and of the $\Box$-algebra $\mathcal{A}^{\Box}$ coincide.*

Since subdirectly irreducible algebras are defined as algebras whose congruence lattice has the least non-zero element, we infer

Corollary 3.16. *An HO-algebra $\mathcal{A}$ is subdirectly irreducible iff the $\Box$-algebra $\mathcal{A}^{\Box}$ is subdirectly irreducible.*

In [17, Proposition 1.6], the description of subdirectly irreducible $\Box$-algebras was obtained. Using this description and the last corollary we can describe subdirectly irreducible HO-algebras. Denote

$$(-\neg)^0 a := a;\ (-\neg)^{n+1} a := (-\neg)^n - \neg a \text{ for } n > 0;$$
$$(-\neg)^{(n)} a := \bigwedge\{(-\neg)^m a \mid m \leq n\}.$$

Proposition 3.17. *An HO-algebra $\mathcal{A}$ is subdirectly irreducible iff there exists an element $a \in \mathcal{A}$ with $a \neq 1$ such that for all $b \in \mathcal{A}$ with $b \neq 1$ there is $n \in \omega$ such that $(-\neg)^{(n)}b \leq a$.*

The duality between HO-algebras and Routley frames can be defined in a similar way to the duality of Heyting algebras and Kripke frames. We omit here topological aspects of this duality.

Let $\mathcal{W} = \langle W, \leq, * \rangle$ be a Routley frame. Define its *algebra of cones* $\mathcal{A}(\mathcal{W})$ as follows:

$$\mathcal{A}(\mathcal{W}) := \langle \langle W, \leq \rangle^+, \cap, \cup, \Rightarrow, \neg, \varnothing, W \rangle,$$

where

- $\langle W, \leq \rangle^+$ is the set of cones (upward closed sets) of the partial ordering $\langle W, \leq \rangle$;
- $\cap$ and $\cup$ are the intersection and the sum of sets;
- $X \Rightarrow Y := \{w \in W \mid \forall u \geq w (u \in X \Rightarrow u \in Y)\}$ for $X, Y \in \langle W, \leq \rangle^+$.
- $\neg X := \{w \in W \mid w^* \notin X\}$ for $X \in \langle W, \leq \rangle^+$.

Since for a valuation function $v : Prop \to 2^W$ we have $v(p) \in \langle W, \leq \rangle^+$ by persistence condition, we can consider v as an $\mathcal{A}(\mathcal{W})$-valuation. Conversely, every $\mathcal{A}(\mathcal{W})$-valuation can be used to define a model over $\mathcal{W}$.

Proposition 3.18. *Let $\mathcal{W}$ be a Routley frame.*

(1) *The algebra of cones $\mathcal{A}(\mathcal{W})$ is an HO-algebra.*
(2) *For any Routley model $\langle \mathcal{W}, v \rangle$ over $\mathcal{W}$, a formula φ, and $w \in W$, holds the equivalence*

$$w \models \varphi \quad \text{iff} \quad w \in v(\varphi).$$

In particular, $\langle \mathcal{W}, v \rangle \models \varphi$ iff $v(\varphi) = 1_{\mathcal{A}(\mathcal{W})}$.

Proof. 1. Since the $\neg$-free reduct of $\mathcal{A}(\mathcal{W})$, $\mathcal{A}(\mathcal{W})^H := \langle \langle W, \leq \rangle^+, \cap, \cup, \Rightarrow, \varnothing, W \rangle$ is defined in exactly the same way as the algebra of cones of the Kripke frame $\langle W, \leq \rangle$, we conclude that $\mathcal{A}(\mathcal{W})^H$ is a Heyting algebra. It remains to check the identities of Ockham lattices for $\neg$-operation. For $X, Y \in \langle W, \leq \rangle^+$, we have

$$\neg(X \cup Y) = \{w \in W \mid w^* \notin X \cup Y\} =$$
$$= \{w \in W \mid w^* \notin X\} \cap \{w \in W \mid w^* \notin Y\} = \neg X \cap \neg Y.$$

Similarly, $\neg(X \cap Y) = \neg X \cup \neg Y$. Obviously, $\neg W = \varnothing$ and $\neg \varnothing = W$.

2. This item can be proved by an easy induction on the structure of formula.

□

Since classes of $\mathcal{A}(\mathcal{W})$-valuations and valuations in a frame $\mathcal{W}$ coincide, we obtain

Corollary 3.19. *For any Routley frame $\mathcal{W}$, the associated algebra of cones $\mathcal{A}(\mathcal{W})$, and a formula φ, holds the equivalence*

$$\mathcal{W} \models \varphi \quad \text{iff} \quad \mathcal{A}(\mathcal{W}) \models \varphi.$$

For an arbitrary HO-algebra $\mathcal{A}$, we construct now a frame $\mathcal{W}^{\mathcal{A}}$ such that $\mathcal{A}$ is embedded into the algebra of cones $\mathcal{A}(\mathcal{W}^{\mathcal{A}})$.

Recall that a filter F on $\mathcal{A}$ is *prime* if F is a proper subset of $\mathcal{A}$ and $a \vee b \in F$ is equivalent to $a \in F$ or $b \in F$. Denote by $W(\mathcal{A})$ the set of prime filters on $\mathcal{A}$ (we consider here namely filters, not $*$-filters) and put

$$\mathcal{W}^{\mathcal{A}} := \langle W(\mathcal{A}), \subseteq, * \rangle,$$

where

- $\subseteq$ is a set theoretical inclusion;
- $F^* := \{a \mid \neg a \notin F\}$.

Proposition 3.20. *For any HO-algebra $\mathcal{A}$, $\mathcal{W}^{\mathcal{A}}$ is a Routley frame.*

Proof. In fact, we have to check that the $*$-operator is well defined on $W(\mathcal{A})$ and that $F_1^* \subseteq F_2^*$ whenever $F_2 \subseteq F_1$. The last statement is obvious. So let us prove that F^* is a prime filter on $\mathcal{A}$.

We have $\neg 0 = 1 \in F$, whence $0 \notin F^*$. Thus, F^* is a proper subset of $\mathcal{A}$.

Let $a, b \in F^*$. Then $\neg a \notin F$ and $\neg b \notin F$. If $\neg(a \wedge b) = \neg a \vee \neg b \in F$, then since F is prime, either $\neg a \in F$ or $\neg b \in F$. Both cases conflict with our assumption. Consequently, $\neg(a \wedge b) \notin F$ and $a \wedge b \in F^*$.

Let $a \in F^*$ and $a \leq b$. If $\neg b \in F$, then $\neg b \leq \neg a \in F$, which contradicts to $a \in F^*$. Thus, $\neg b \notin F$, i.e., $b \in F^*$.

Finally, assume that $a \vee b \in F^*$. If $a \notin F^*$ and $b \notin F^*$, then $\neg a, \neg b \in F$. Consequently, $\neg a \wedge \neg b = \neg(a \vee b) \in F$, which conflicts with the assumption. $\square$

Proposition 3.21. *For any HO-algebra $\mathcal{A}$, the mapping $a \mapsto X^a$, where $X^a = \{F \in W(\mathcal{A}) \mid a \in F\}$, $a \in \mathcal{A}$, is an embedding of $\mathcal{A}$ into $\mathcal{A}(\mathcal{W}^{\mathcal{A}})$. If $\mathcal{A}$ is finite, then it is an isomorphism.*

Proof. It is well known that the mapping $a \mapsto X^a$ is an embedding of Heyting algebra $\mathcal{A}^H$ into Heyting algebra $(\mathcal{A}(\mathcal{W}^{\mathcal{A}}))^H$. Moreover, this embedding is onto if $\mathcal{A}$ is finite. It remains to check that this mapping preserve $\neg$-operation:

$$\neg X^a = \{F \mid F^* \notin X^a\} = \{F \mid a \notin F^*\} = \{F \mid \neg a \in F\} = X^{\neg a}.$$

$\square$

From the last proposition and Corollary 3.19 we obtain

Proposition 3.22. *For any HO-algebra and formula φ, the implication holds:*

$$\mathcal{W}^{\mathcal{A}} \models \varphi \quad \Rightarrow \quad \mathcal{A} \models \varphi.$$

If $\mathcal{A}$ is finite, then

$$\mathcal{W}^{\mathcal{A}} \models \varphi \quad \Leftrightarrow \quad \mathcal{A} \models \varphi.$$

4. The lattice of HT^2-extensions

In this section we give the first application of algebraic semantics for N^* developed in the previous section. Namely, we describe completely the class of normal extensions of the logic HT^2. The logic HT^2 was introduced in [2] as a monotonic deductive base [8] for WFS, later ([5]) it was identified as a finite valued extension of N^* and axiomatized. More natural axiomatics for HT^2 was found in [7].

The logic HT^2 can be defined in terms of N^*-frames as follows.

An HT^2-*frame* is a Routley frame $\mathcal{W}^{HT^2} = \langle W^{HT^2}, \leq, * \rangle$ such that (i) W^{HT^2} comprises 4 worlds denoted by h, h', t, t', (ii) $\leq$ is a partial ordering on W satisfying $h \leq t$, $h \leq h'$, $h' \leq t'$, and $t \leq t'$, (iii) $h^* = t^* = t'$, $(h')^* = (t')^* = t$.

The ordering of the HT^2-frame and the action of $*$ at this frame are presented at the diagram of Figure 1.

The logic HT^2 is the set of formulas true in HT^2-frame,

$$HT^2 := \{\varphi \mid \mathcal{W}^{HT^2} \models \varphi\}.$$

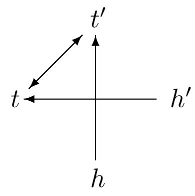
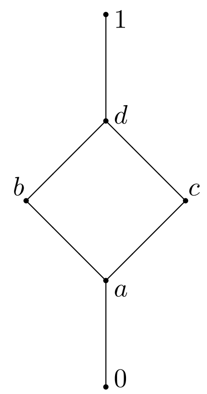


Figure 1.

It was proved in [7] that the logic HT^2 equals to the least normal N^*-extension containing the following axioms:

A1) $p \vee (p \to (q \vee (q \to (r \vee -r))))$;
A2) $p \to \neg\neg p$;
A3) $(p \wedge \neg p) \to (\neg q \vee \neg\neg q)$;
A4) $(p \wedge \neg p) \to (q \vee (q \to r) \vee -r)$;
A5) $\neg\neg(q \vee (q \to r) \vee -r)$;
A6) $(\neg\neg p \wedge \neg\neg q) \to ((p \to q) \vee (q \to p))$.

It is known from [5] that HT^2 coincides with the logic of six-element HO-algebra $\mathbf{6} := \langle\{0, a, b, c, d, 1\}, \wedge, \vee, \to, \neg, 0, 1\rangle$. The lattice structure of $\mathbf{6}$ and the truth table for negation look as follows.



x	$\neg x$
0	1
a	c
b	0
c	c
d	0
1	0

Figure 2.

In fact, this is the algebra of cones of HT^2-frame.

Proposition 4.1. *The algebra* $\mathbf{6}$ *is isomorphic to* $\mathcal{A}(\mathcal{W}^{HT^2})$.

Proof. It is routine to check that the desired isomorphism is given by the following mapping from $\mathbf{6}$ to $\langle W^{HT^2}, \leq\rangle^+$:

$$0 \mapsto \varnothing,\ a \mapsto \{t'\},\ b \mapsto \{t, t'\},\ c \mapsto \{h', t'\},\ d \mapsto \{t, h', t'\},\ 1 \mapsto W^{HT^2}.$$

□

Since every variety of HO-algebras is determined by its subdirectly irreducible elements, prior to describe the lattice of normal extensions of the logic HT^2 we have to find out all subdirectly irreducible HO-algebras modelling HT^2.

It can be easily checked that $(-\neg)^2 p \leftrightarrow -\neg p \in HT^2$. Taking into account this fact and Proposition 3.17 we can describe subdirectly irreducible HO-algebras modelling HT^2 as follows.

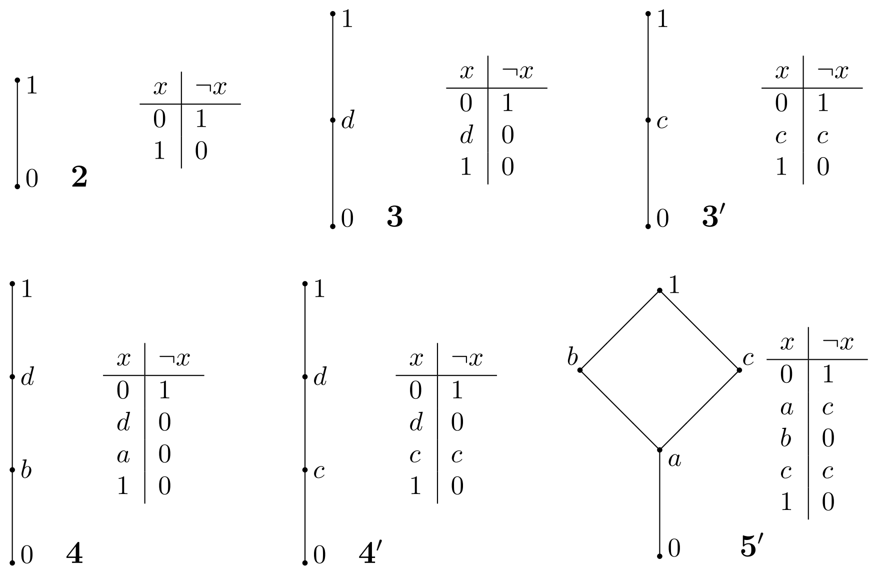
Proposition 4.2. *Let $\mathcal{A}$ be an HO-algebra and $HT^2 \subseteq L\mathcal{A}$. Then $\mathcal{A}$ is subdirectly irreducible iff there exists an element $a \in \mathcal{A}$ with $a \neq 1$ such that for all $b \in \mathcal{A}$ with $b \neq 1$, holds $b \wedge -\neg b \leq a$.*

Since HO-algebras have lattice operations, the variety $\mathcal{V}^*$ of HO-algebras is congruence distributive. The following result by B. Jonsson [4, p. 168, Corollary 6.10] is well known. In a congruence distributive variety, all subdirectly irreducible elements of the subvariety generated by the finite set $\mathcal{K}$ of finite algebras belong to $\mathsf{HS}(\mathcal{K})$. Here $\mathsf{H}(\mathcal{K})$ denotes the class of all algebras isomorphic to homomorphic images of algebras from $\mathcal{K}$; $\mathsf{S}(\mathcal{K})$ is the class of all algebras isomorphic to subalgebras of algebras from $\mathcal{K}$. For finite subdirectly irreducible models of N^*, the above result implies the equivalence:

$$L\mathcal{A} \subseteq L\mathcal{B} \ \text{ iff } \ \mathcal{B} \in \mathsf{HS}(\mathcal{A}).$$

In this way, to list all subdirectly irreducible HT^2-models we have to calculate the set $\mathsf{HS}(\mathbf{6})$ and to distinguish in this set all subdirectly irreducible algebras using Proposition 4.2.

By $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3'}$, $\mathbf{4}$, $\mathbf{4'}$, and $\mathbf{5'}$ we denote HO-algebras with lattice structures and truth tables for negations presented at diagrams of Figure 3.



2

x	$\neg x$
0	1
1	0

3

x	$\neg x$
0	1
d	0
1	0

3′

x	$\neg x$
0	1
c	c
1	0

4

x	$\neg x$
0	1
d	0
a	0
1	0

4′

x	$\neg x$
0	1
d	0
c	c
1	0

5′

x	$\neg x$
0	1
a	c
b	0
c	c
1	0

Figure 3.

Proposition 4.3. *Up to isomorphism* $\mathsf{HS}(\mathbf{6})$ *equals to*

$$\{\mathbf{2}, \mathbf{3}, \mathbf{3'}, \mathbf{4}, \mathbf{4'}, \mathbf{5'}, \mathbf{6}\}.$$

All elements of $\mathsf{HS}(\mathbf{6})$ *are subdirectly irreducible.*

Proof. Since algebra **6** is finite all its subalgebras can be found via an exhaustive search, which can be restricted by the observation that a cannot belong to a proper subalgebra of **6**. Indeed, $\neg a = c$, $c \to a = b$, $c \vee b = d$. We obtain

$$\mathsf{S}(\mathbf{6}) = \{\mathbf{2}, \mathbf{3}, \{0, b, 1\}, \mathbf{3'}, \mathbf{4}, \mathbf{4'}, \mathbf{6}\},$$

where the algebra with universe $\{0, b, 1\}$ is isomorphic to **3**.

The action of $-\neg$-operator on **6** is given by the following truth-table:

x	$-\neg x$
0	0
a	0
b	1
c	0
d	1
1	1

Since $-\neg c = 0$, the element c cannot belong to a proper $*$-filter on **6**. Thus, there are two proper non-trivial $*$-filters on **6**: $F_1 := \{d, 1\}$ and $F_2 := \{b, d, 1\}$. It is easy to see that $\mathbf{6}/F_1 \cong 5'$ and $\mathbf{6}/F_1 \cong 3'$. Thus, up to isomorphism we have

$$\mathsf{H}(\mathbf{6}) = \{\mathbf{3'}, \mathbf{5'}, \mathbf{6}\}.$$

In a similar way, we calculate

$$\mathsf{H}(\mathbf{2}) = \{\mathbf{2}\},\ \mathsf{H}(\mathbf{3}) = \{\mathbf{2}, \mathbf{3}\},\ \mathsf{H}(\mathbf{3'}) = \{\mathbf{3'}\},\ \mathsf{H}(\mathbf{4}) = \{\mathbf{2}, \mathbf{3}, \mathbf{4}\},\ \mathsf{H}(\mathbf{4'}) = \{\mathbf{3'}, \mathbf{4'}\},$$

where all equalities are up to isomorphism. Thus, the only algebra belonging to $\mathsf{HS}(\mathbf{6}) \setminus \mathsf{S}(\mathbf{6})$ is $\mathbf{5'}$.

Using Proposition 4.2 we easily check that all elements of $\mathsf{HS}(\mathbf{6})$ are subdirectly irredicible. For example, the element b of $\mathbf{5'}$ is such that $x \wedge -\neg x \leq b$ for all $x \in \mathbf{5'} \setminus \{1\}$.

$\square$

As in the previous proof we can calculate $\mathsf{HS}(\mathcal{A})$ for all $\mathcal{A} \in \mathsf{HS}(\mathbf{6})$. This allows to obtain the graph of the relation $\sqsubseteq$ on $\mathsf{HS}(\mathbf{6})$, where $\mathcal{A} \sqsubseteq \mathcal{B}$ iff $\mathcal{B} \in \mathsf{HS}(\mathcal{A})$.

The following diagram presents the ordering of $\mathsf{HS}(\mathbf{6})$ by the relation $\sqsubseteq$.

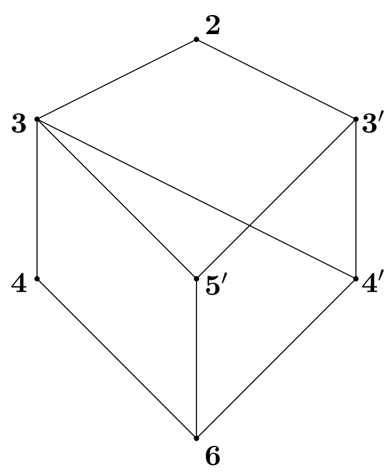


Figure 4.

Theorem 4.4. *The lattice* $\mathbf{NExt}HT^2$ *of* HT^2*-extensions is isomorphic to the lattice of cones of the ordering presented at Figure 4. It contains trivial logic* For *and 13 different non-trivial logics:*

$$L\mathbf{2},\ L\mathbf{3},\ L\mathbf{3}', L\mathbf{3}\cap L\mathbf{3}',\ L\mathbf{4},\ L\mathbf{4}',\ L\mathbf{5}',$$

$$L\mathbf{4}\cap L\mathbf{3}',\ L\mathbf{4}\cap L\mathbf{4}',\ L\mathbf{4}\cap L\mathbf{5}',\ L\mathbf{4}'\cap L\mathbf{5}',\ L\mathbf{4}\cap L\mathbf{4}'\cap L\mathbf{5}',\ L\mathbf{6}=HT^2.$$

The ordering of $\mathbf{NExt}HT^2$ *is presented at Figure 5.*

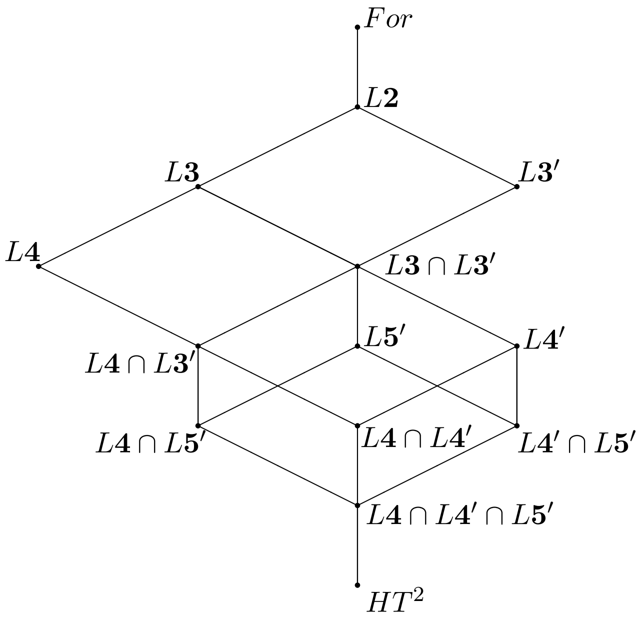


Figure 5.

Proof. If $L \in \mathbf{NExt}HT^2$, then it is obvious that the set $V(L)\cap \mathsf{HS}(\mathbf{6})$ is upward closed under the relation $\sqsubseteq$. On the other hand, every cone of algebras in $\langle \mathsf{HS}(\mathbf{6}), \sqsubseteq\rangle$ determines a variety of HO-algebras and these varieties are different for different cones, because they have different families of subdirectly irreducible algebras. According to Theorem 3.9 to every variety $V \in Sub(V(HT^2))$ corresponds a logic $L(V) \in \mathbf{NExt}HT^2$ and different varieties determine different logics.

We have thus proved that the lattice $\mathbf{NExt}HT^2$ is isomorphic to the lattice of cones of the ordering presented at Figure 4. Now it can be checked directly that the lattice $\mathbf{NExt}HT^2$ has the structure presented at Figure 5.

$\square$

In conclusion we point out some well known logics presented at Figure 5. Clearly, that $L\mathbf{2}$ is nothing else that classical logic (written in the language $\langle \vee, \wedge, \to, -\rangle$), where Routley negation is identical to classical negation, i.e.,

$$L\mathbf{2} = Cl + \{-p \leftrightarrow \neg p\}.$$

In a similar way, operations "$-$" and "$\neg$" coincides on the algebra $\mathbf{3}$ and it is well known that the three element Heyting algebra defines the logic of here-at-there $HT = Int + \{p \vee (p \to q) \vee -q\}$, so

$$L\mathbf{3} = Int + \{p \vee (p \to q) \vee -q,\ -p \leftrightarrow \neg p\}.$$

The logic of the algebra $\mathbf{3}'$ is the so called logic HT^* of total models from [6], where it was axiomatized as follows:

$$L\mathbf{3}' = HT^* = N^* + \{p \vee (p \to q) \vee -q,\ \neg\neg p \leftrightarrow p,\ (p \wedge \neg p) \to (\neg q \vee \neg\neg q)\}.$$

The axiomatization of all logics from $\mathbf{NExt}HT^2$ is not needed for the intended application of the results of this section, namely for the studying whether HT^2 is a maximal deductive base for WFS.

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