

СИБИРСКИЕ ЭЛЕКТРОННЫЕ  
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 7, стр. 275–283 (2010)

УДК 519.172

MSC 05C15

ACYCLIC 3-CHOOSABILITY OF PLANAR GRAPHS  
WITH NO CYCLES OF LENGTH FROM 4 TO 11

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**ABSTRACT.** Every planar graph is known to be acyclically 7-choosable and is conjectured to be acyclically 5-choosable (Borodin et al., 2002). This conjecture if proved would imply both Borodin's acyclic 5-color theorem (1979) and Thomassen's 5-choosability theorem (1994). However, as yet it has been verified only for several restricted classes of graphs. Some sufficient conditions are also obtained for a planar graph to be acyclically 4- and 3-choosable.

In particular, a planar graph of girth at least 7 is acyclically 3-colorable (Borodin, Kostochka and Woodall, 1999) and acyclically 3-choosable (Borodin et al., 2010). A natural measure of sparseness, introduced by Erdős and Steinberg, is the absence of  $k$ -cycles, where  $4 \leq k \leq C$ . Here, we prove that every planar graph with no cycles of length from 4 to 11 is acyclically 3-choosable.

**Keywords:** acyclic coloring, planar graph, forbidden cycles

## 1. INTRODUCTION

By  $V(G)$  and  $E(G)$  denote the sets of vertices and edges of a graph  $G$ , respectively. The girth of  $G$ , i.e. the length of a shortest cycle in  $G$ , is denoted by  $g(G)$ .

A (proper)  $k$ -coloring of  $G$  is a mapping  $f : V(G) \rightarrow \{1, 2, \dots, k\}$  such that  $f(x) \neq f(y)$  whenever  $x$  and  $y$  are adjacent in  $G$ .

By the Grötzsch theorem, every planar triangle-free graph is 3-colorable. In 1976, Steinberg conjectured that every planar graph without 4-cycles and 5-cycles

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Partly supported by the grants 09-01-00244 and 08-01-00673 of the Russian Foundation for Fundamental Research.

Received August, 9, 2010, published September, 17, 2010.

is 3-colorable. This conjecture remains unsettled. Erdős (see [25]) suggested the following relaxation of this problem: does there exist a constant  $C$  such that the absence of cycles with size from 4 to  $C$  in a planar graph guarantees its 3-colorability? The best result in this direction is  $C \leq 7$  due to Borodin et al. [12].

Now suppose each vertex  $v$  of a graph  $G$  is given a list  $L(v)$  of admissible colors, represented by positive integers. The  $G$ -list  $L$  is *choosable* if there is a proper vertex coloring of  $G$  such that a color of each vertex  $v$  belongs to  $L(v)$ . A graph  $G$  is said to be *k-choosable* if every  $G$ -list  $L$  is choosable provided that  $|L(v)| \geq k$  for each  $v \in V(G)$ .

Thomassen [26] proved a famous theorem that each planar graph is 5-choosable, and Voigt [28] showed that this bound is best possible. Margit Voigt constructed the following non-3-choosable planar graphs: of girth at least 4 (in [29]), and without 4-cycles and 5-cycles (in [30]). On the other hand, it is known that a planar graph is 3-choosable if its girth is at least 5 (Thomassen [27]) or it has no cycles of length from 4 to 9 (Borodin [2]).

A proper vertex coloring of a graph is *acyclic* if every cycle uses at least three colors (Grünbaum [19]). Borodin [1] proved Grünbaum's conjecture that every planar graph is acyclically 5-colorable. This bound is best possible; moreover, there are bipartite 2-degenerate planar graphs  $G$  which are not acyclically 4-colorable (Kostochka and Mel'nikov [21]). Acyclic colorings turned out to be useful in obtaining results about other types of colorings; for a survey see monographs [20, 17].

Borodin et al. [9] proved that every planar graph is acyclically 7-choosable and conjectured a common extension of Borodin's [1] and Thomassen's results [26]:

**Conjecture 1.** *Every planar graph is acyclically 5-choosable.*

However, this challenging conjecture seems to be difficult. As yet, it has been verified only for several restricted classes of planar graphs. Some sufficient conditions are also obtained for a planar graph to be acyclically 4- and 3-colorable or choosable. The minimum  $k$  with the property that a graph  $G$  is acyclically  $k$ -colorable (acyclically  $k$ -choosable) is denoted by  $a(G)$  (by  $a^l(G)$ ).

In particular, Borodin, Kostochka and Woodall [8] showed that if  $G$  is a planar graph of girth  $g$  then  $a(G) \leq 4$  if  $g \geq 5$  and  $a(G) \leq 3$  if  $g \geq 7$ . Note that the first of these results is best possible in terms of girth due to the construction in [21]. Borodin proved  $a(G) \leq 4$  for  $G$  having neither 4- nor 6-cycles (in [3]) and neither 4- nor 5-cycles (in [4]).

Recently,  $a^l(G) \leq 4$  was proved in the following cases: if  $g \geq 5$  (Montassier [22]), or if  $G$  has no 4-, 5- and 6-cycles, or no 4-, 5- and 7-cycles, or no 4-, 5- and intersecting 3-cycles (Montassier, Raspaud and Wang [23]), or no 4-, 5- and 8-cycles (Chen and Raspaud [15]), or no 4-, 6- and 7-cycles, or else no 4-, 6- and 8-cycles (Chen, Raspaud and Wang [16]), or neither 4-cycles nor 6-cycles adjacent to a triangle (Borodin, Ivanova, and Raspaud [10]). Note that [10] obviously covers most of the above mentioned results in [8, 22, 23, 16, 3].

Moreover,  $a^l(G) \leq 3$  was proved if  $g \geq 7$  (Borodin et al. [6]) or if  $G$  has no cycles of length from 4 to 12 (Borodin [5] and, independently, Hocquard and Montassier [18]).

The purpose of this paper is to improve the result in [5, 18] as follows:

**Theorem 2.** *Every planar graph with no cycles of length from 4 to 11 is acyclically 3-choosable.*

The choosability version of Steinberg-Erdős problem is to find the smallest  $C^l$  such that each planar graph without cycles of length from 4 to  $C^l$  is 3-choosable. Borodin [2] proved that  $C^l \leq 9$  and Voigt [30] proved that  $C^l \geq 6$ . It seems natural to ask a similar question about the acyclic 3-choosability:

**Problem 1.** *Find the smallest  $C_a^l$  such that each planar graph without cycles of length from 4 to  $C_a^l$  is acyclically 3-choosable.*

It follows from [30] and Theorem 2 that  $6 \leq C_a^l \leq 11$ . Since the acyclic 3-choosability is a stronger property of a graph than the proper 3-choosability, one can expect that  $C_a^l > 6$ . Note that Borodin’s bound  $C^l \leq 9$  proved in [2] remains the best known for over fifteen years, and the proof of Theorem 2 is non-trivial, so we would like to ask the following cautious question:

**Problem 2.** *Is it true or not that each planar graph without cycles of length from 4 to 10 is acyclically 3-choosable?*

A distinctive feature of our proof of Theorem 2 is that a charge of vertices can be transferred along "feeding paths" to an unlimited distance. This kind of "global" discharging was introduced by Borodin, Ivanova, and Kostochka in [13] and used in [13, 14] to improve results in [7, 11] about homomorphisms of sparse graphs to the circulant  $C(5; 1, 2)$  and cycle  $C_5$ .

2. PROOF OF THEOREM 2

Suppose a graph  $G^*$  with a list  $L$  is a counterexample to Theorem 2 on the fewest vertices. Clearly,  $G^*$  is connected and has no 1-vertices.

If  $G^*$  is 2-connected, then we put  $G^+ = G = G^*$ . Otherwise, let  $G^+$  be a pendant block,  $w^+$  be the cut-vertex of  $G^*$  that belongs to  $G^+$ , and  $f^+$  be the face of  $G^+$  that is not a face of  $G^*$ . Clearly,  $w^+$  belongs to the boundary of  $f^+$ . Now we put  $G = G^+ - w^+$ .

By  $F(G^+)$ ,  $d(v)$ , and  $r(f)$  denote the set of faces of  $G^+$ , the degree of a vertex  $v$  in  $G^+$ , and the size of face  $f$  in  $G^+$ , respectively.

From Euler’s formula  $|V(G^+)| - |E(G^+)| + |F(G^+)| = 2$ , using well-known relations

$$\sum_{v \in V(G^+)} d(v) = 2|E(G^+)| = \sum_{f \in F(G^+)} r(f),$$

we have

$$\sum_{v \in V(G^+)} (5d(v) - 12) + \sum_{f \in F(G^+)} (r(f) - 12) = -24.$$

We set the *initial charge* of every vertex  $v \in V(G^+) - w^+$  and face  $f \in F(G^+)$  to be  $ch(v) = 5d(v) - 12$  and  $ch(f) = r(f) - 12$ , respectively. If  $w^+$  exists, then we put  $ch(w^+) = 5d(w^+) + 11$ .

Note that only 2-vertices and 3-faces of  $G^+$  except for vertex  $w^+$  have a negative initial charge. Then we use a discharging procedure leading to a *final charge*  $ch^*$  such that

$$\sum_{x \in V(G^+) \cup F(G^+)} ch^*(x) = \sum_{x \in V(G^+) \cup F(G^+)} ch(x) < 0. \tag{1}$$

Based on the structural properties of  $G$ , we shall get a contradiction with (1) by proving that  $ch^*(x) \geq 0$  for every  $x \in V(G^+) \cup F(G^+)$ .

**2.1. Structural properties of the minimum counterexample.** Recall that all degrees are considered in  $G^+$  and observe that  $d(w^+) \geq 2$  since  $G^+$  is a pendant block.

**Lemma 1.** *No 2-vertex  $v$  in  $G$  belongs to a 3-cycle.*

PROOF. It suffices to acyclically color the graph  $G^* - v$  according to list  $L$ , and then color  $v$  differently from its neighbors.  $\square$

**Lemma 2.**  *$G$  has no two adjacent 2-vertices.*  $\square$

A *triplet* is a 3-face incident with three 3-vertices of  $G$ . (So, a triplet is not incident with  $w^+$  if  $w^+$  exists.) A *garland* (see Fig. 1) is a non-empty sequence of triplets  $\mathcal{G}_k = T_1, \dots, T_k$ , where  $T_i = x_i y_i z_i$ ,  $1 \leq i \leq k \geq 1$ , such that  $z_i$  is adjacent to  $x_{i+1}$  whenever  $1 \leq i \leq k - 1$ , while  $x_1$  is adjacent to a 2-vertex  $x'_1 \neq w^+$ . The neighbor of  $x'_1$  not belonging to  $T_1$  is denoted by  $x''_1$ . By  $y'_i$  denote the neighbor of  $y_i$  such that  $y'_i \notin T_i$ , where  $1 \leq i \leq k$ , and let  $z'_k$  be the neighbor of  $z_k$  lying outside  $T_k$ .

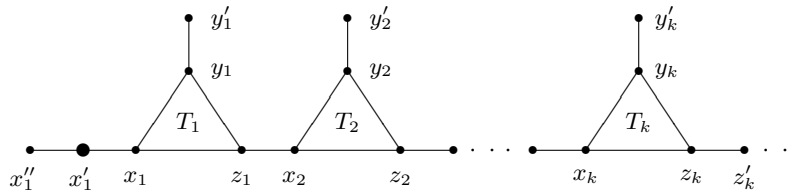


Fig. 1. Garland  $\mathcal{G}_k$

The following lemma shows the key idea in the proof of Theorem 2.

**Lemma 3.** *Suppose  $\mathcal{G}_k$  is a garland,  $c$  is an acyclic  $L$ -coloring of  $G^* - x'_1$ , and  $L(x'_1) = \{1, 2, 3\}$ ; then*

- (i) *w.l.o.g.,  $c(x''_1) = 1$ ,*
- (ii) *each of the  $3k$  vertices of  $\mathcal{G}_k$  has the same set  $\{1, 2, 3\}$  of admissible colors, and*
- (iii)  *$c(z'_k) = c(y'_i) = 1$  whenever  $1 \leq i \leq k$ .*

PROOF. Induction on  $k$ .

*STEP 1.* We are easily done if  $c(x_1) \neq c(x''_1)$ , so suppose  $c(x_1) = c(x''_1)$ . If  $c(x''_1) > 3$  then it suffices to put  $c(x'_1) \in \{1, 2, 3\} \setminus \{c(y_1), c(z_1)\}$ , so we can assume that  $c(x''_1) = 1$  and there are bicolored (1, 2)- or (1, 3)-cycles in  $G$  if we put  $c(x'_1) = 2$  or  $c(x'_1) = 3$ , respectively. By symmetry, we can assume that  $c(y_1) = 2$ ,  $c(z_1) = 3$  and  $c(y'_1) = c(x_2) = 1$ .

If, say,  $4 \in L(x_1)$  then it suffices to color  $x_1$  with 4. If  $4 \in L(y_1)$  then we can recolor  $y_1$  with 4 and put  $c(x'_1) = 2$ . Similarly, if  $4 \in L(z_1)$  then we can recolor  $z_1$  with 4 and put  $c(x'_1) = 3$ . So,  $L(x_1) = L(y_1) = L(z_1) = \{1, 2, 3\}$ .

*STEP  $i+1$ ,  $1 \leq i \leq k-1$ .* Suppose  $L(v) = \{1, 2, 3\}$  whenever  $v \in T_1 \cup T_2 \cup \dots \cup T_i$ , and  $c(v) = 1$  whenever  $v \in \{y'_1, \dots, y'_i, z'_i\}$ .

Let a coloring  $c_2$  be obtained from  $c$  by putting  $c_2(x_1) = \dots = c_2(x_i) = 1$ ,  $c_2(x'_1) = c_2(z_1) = \dots = c_2(z_i) = 2$ , and  $c_2(y_1) = \dots = c_2(y_i) = 3$ ; this can create only a  $(2, 1)$ -path through  $x'_1$ . A coloring  $c_3$  is defined by swapping numbers 2 and 3 on  $\{x'_1\} \cup T_1 \cup T_2 \cup \dots \cup T_i$  in coloring  $c_2$ .

Arguing as in Step 1, we see from  $c_2$  and  $c_3$  that: (a)  $c(x_{i+1}) = 1$ ,  $\{c(y_{i+1}), c(z_{i+1})\} = \{2, 3\}$ , (b)  $c(y'_{i+1}) = c(x_{i+2}) = 1$ , where  $x_{k+1} = z'_k$ , and (c)  $L(x_{i+1}) = L(y_{i+1}) = L(z_{i+1}) = \{1, 2, 3\}$ .  $\square$

**Corollary 1.** *No garland can close on itself; more specifically, none of vertices  $y'_k, z'_k$  in  $\mathcal{G}_k$  can coincide with one of  $y_1, \dots, y_{k-1}$ .*

PROOF. It suffices to note (see Fig. 2) that  $c(y'_k) = c(z'_k) = 1$  while none of  $y_1, \dots, y_{k-1}$  is colored 1 due to Lemma 3.  $\square$

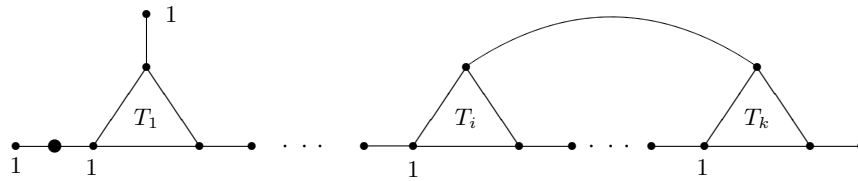


Fig. 2. Closing of  $FP$ :  $y'_k = y_i$ , where  $i < k$

**Lemma 4.** *At most one edge can join a garland to a 2-vertex of  $G$ .*

PROOF. Let a garland  $\mathcal{G}_k$  be the smallest counterexample. This means that none of  $y'_1, \dots, y'_{k-1}$  is a 2-vertex. By symmetry between  $y'_k$  and  $z'_k$ , suppose that  $d(z'_k) = 2$ , where vertices  $x'_1$  and  $z'_k$  may be distinct or coincide (see Fig. 3). Note that  $z'_k$  is distinct from  $y'_k$  due to the absence of adjacent 3-cycles in  $G$  and from  $y'_1, \dots, y'_{k-1}$  by the assumption just made.

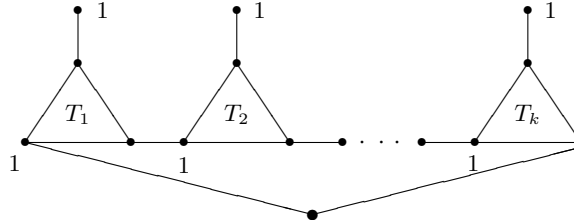


Fig. 3. Closing of  $FP$ :  $z'_k = x'_1$

Take the coloring  $c_3$  described in the proof of Lemma 3 (in which all  $x_i$ 's,  $y_i$ 's, and  $z_i$ 's, where  $1 \leq i \leq k$ , are colored with 1, 2 and 3, respectively). We see that  $c_3(x'_1) = 3$  while  $c_3(z'_k) = 1$ , which already implies that  $x'_1 \neq z'_k$ . By  $z''_k$  denote the neighbor of  $z'_k$  other than  $z_k$ . We also remember from Lemma 3 that  $c_3(z''_k) = 3$ .

Let  $c_4$  be obtained from  $c_3$  by putting  $c_4(x'_1) = c_4(z'_k) = 2$  and swapping colors on the path  $x_1 z_1 x_2 z_2 \dots x_k z_k$ . (The latter can be done since  $z'_k$  is adjacent to none of  $y'_1, \dots, y'_k$ .) It is easy to see that  $c_4$  is an acyclic  $L$ -coloring of  $G^*$ ; a contradiction.  $\square$

**Corollary 2.** *No triplet is incident with more than one edge going to a 2-vertex of  $G$ .*  $\square$

We now introduce a notion crucial for our proof. Let  $v$  be a 2-vertex of  $G$  adjacent to  $v_1$  and  $v_2$ . A *feeding path*  $FP_i(v)$  for  $v$ , where  $i \in \{1, 2\}$ , is a garland  $\mathcal{G}_k$  with the smallest  $k$  such that  $x'_1 = v$ ,  $x_1 = v_i$ , and at least one of  $y'_k, z'_k$ , say  $z'_k$ , does not belong to a triplet. Then  $z'_k$  is called a *sponsor of  $v$  along  $FP_i(v)$*  and denoted by  $S_i(v)$ .

For each of the two edges  $vv_1$  and  $vv_2$ , we start constructing a feeding path  $FP_i(v)$  for  $v$ . If, say,  $v_1$  does not belong to a triplet, then  $FP_1(v)$  is trivial and consists of just one edge  $vv_1$ . Suppose  $v_1 \in T_1$ , then we put  $x_1 = v_1$ ,  $T_1 = \{x_1, y_1, z_1\}$ . Now if at least one of  $y'_1$  and  $z'_1$  is not in a certain triplet, then this vertex is declared the (first) sponsor of  $v$ . Otherwise, using the standard notation for a garland, we consider a triplet  $T_2$  that contains  $z'_1 = x_2$ , and so on.

By Lemma 4, the feeding path we are constructing cannot come to  $v$  along edge  $vv_2$ . By Corollary 1, our garland cannot arrive at a triplet already included in it. Since  $G$  is finite, we shall eventually finish our feeding path  $FP_1(v)$  that starts with edge  $vv_1$  by finding  $S_1(v)$ . Arguing the same, we get the second feeding path  $FP_2(v)$  and sponsor  $S_2(v)$  for  $v$ .

**Corollary 3.** *No two feeding paths can intersect except possibly in their sponsors or their initial 2-vertex.*

PROOF. Suppose we have feeding paths  $FP_1 = v \dots S(v)$  and  $FP_2 = w \dots S(w)$  with a vertex  $z$  in common such that  $z$  belongs to a triplet. This contradicts Lemma 4. □

In turn, Corollary 3 implies

**Corollary 4.** *No edge can belong to more than one feeding path.* □

The following lemma has common features with Lemma 4 but is a bit more technical.

**Lemma 5.** *No 3-cycle has two 3-vertices of  $G$  that are sponsors.*

PROOF. Take a garland  $\mathcal{G}_k = T_1, \dots, T_k$  such that  $d(x'_1) = d(z'_k) = 2$  (it is not excluded that  $x'_1 = z'_k$ ) and suppose that precisely one of its vertices  $y_q$ ,  $1 \leq q \leq k$ , is "spoilt" by having  $d(y_q) \geq 4$ . Denote this object by  $\mathcal{G}_k^q$  (see Fig. 4).

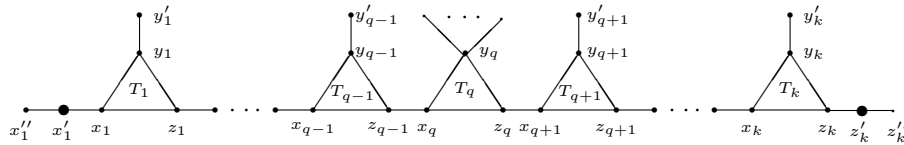


Fig. 4. Quasigarland  $\mathcal{G}_k^q$

We have to prove that  $\mathcal{G}_k^q$  does not exist. Indeed, we see a feeding path for  $x'_1$  formed by a subgarland  $\mathcal{G}_{q-1}$  of  $\mathcal{G}_k^q$ , with a sponsor  $x_q$ , and also a feeding path  $FP(z'_k)$  for  $z'_k$ , with a sponsor  $z_q$ , formed by the sequence  $T_k, \dots, T_{q+1}$  of triplets belonging to  $\mathcal{G}_k^q$ .

Let  $c$  be an  $L$ -coloring of  $G^* - x'_1$ . By Lemma 3, we can assume that  $c(x''_1) = 1$  and  $L(v) = \{1, 2, 3\}$  for each  $v \in \{x'_1\} \cup \mathcal{G}_{q-1}$ . Similarly, since  $FP(z'_k)$  is a garland, we have  $L(v) = \{\alpha, \beta, \gamma\}$  for each  $v \in \{z'_k\} \cup T_k \cup \dots \cup T_{q+1}$ .

Let a coloring  $c_t$ , where  $t \in \{2, 3\}$ , be obtained from  $c$  by putting  $c_t(x'_1) = t$  and  $c_t(x_i) = 1$ ,  $c_t(y_i) = 5 - t$ ,  $c_t(z_i) = t$  for each  $1 \leq i \leq q - 1$ . Note that coloring  $c_t$

should create a  $(1, t)$ -cycle through  $x'_1$ , for otherwise we have nothing to prove. In particular, this implies that  $c(x_q) = 1$  and  $\{c(y_q), c(z_q)\} = \{2, 3\}$ . By symmetry, we can assume that  $c(y_q) = 2, c(z_q) = 3$ .

Now the idea is to modify  $c_3$  so as to direct the dangerous  $(1, 3)$ -path along the rest of  $\mathcal{G}_k^q$  towards  $z'_k$  and there to do a final recoloring that could prevent us from any bicolored cycles in  $G^*$ .

It follows from  $c_3$  that  $c(x_{q+1}) = 1$  if  $q < k$  or  $c(z'_k) = 1$  if  $q = k$ . Hence,  $\{1, 3\} \subseteq \{\alpha, \beta, \gamma\}$  in both cases. So, for each  $q + 1 \leq i \leq k$  we have  $\{c(x_i), c(y_i), c(z_i)\} = \{1, \beta, 3\}$ .

CASE 1.  $c(y_{q+1}) = \dots = c(y_k) = \beta$ . Clearly, we are done unless  $c(z'_k) = 1$  and  $c(z''_k) = 3$ . Now if  $L(x_q) \neq \{1, 2, 3\}$ , then it suffices to modify  $c_3$  to an acyclic L-coloring  $c'_3$  by recoloring  $x_q$  with a color from  $L(x_q) \setminus \{1, 2, 3\}$ . By symmetry, we can assume  $L(x_q) = L(z_q) = \{1, 2, 3\}$ . In this case we modify  $c_3$  as follows:  $c'_3(x_i) = 3, c'_3(z_i) = 1$ , for  $1 \leq i \leq k$ , and, of course,  $c'_3(x'_1) = 2, c'_3(z'_k) = \beta$ .

CASE 2.  $c(y_{q+1}) = \dots = c(y_r) = \beta$ , where  $r < k$ , while  $c(y_{r+1}) = 3$  (see Fig. 5). Clearly, we are done unless  $c(y'_{r+1}) = 1$ . If so, our proof splits. Note that  $c(z_{r+1}) = \beta$ . By  $z^*_{r+1}$  denote vertex  $x_{r+2}$  if  $r + 2 \leq k$  or  $z'_k$  otherwise.

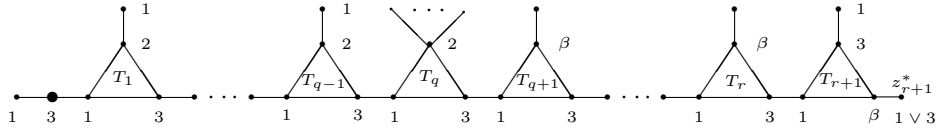


Fig. 5. Coloring  $c_3$  of  $\mathcal{G}_k^q$  in Case 2

Subcase 2.1.  $c(z^*_{r+1}) = 1$ . Here, we recolor  $y_{r+1}$  with  $\beta$  and  $z_{r+1}$  with 3. If  $r + 1 < k$  then we repeat Case 2 with bigger  $r$ . If  $r + 1 = k$ , we go to Case 1.

Subcase 2.2.  $c(z^*_{r+1}) = 3$  (see Fig. 6). Here, we obtain an acyclic L-coloring of  $G$  by recoloring  $y_{r+1}$  with  $\beta$ , all  $z_i$ 's and  $x_i$ 's for  $1 \leq i \leq r + 1$  with 1 and 3, respectively, followed by recoloring  $x'_1$  with 2.  $\square$

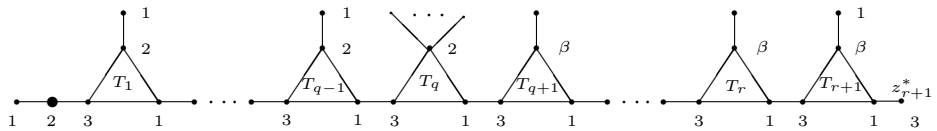


Fig. 6. Coloring  $c'_3$  in Subcase 2.2

2.2. **Completing the proof of Theorem 2.** We discharge the vertices of  $G$  as follows:

- R1:** Each 2-vertex  $v$  gets charge 1 along each feeding path started at  $v$ .
- R2:** Every 3-face  $f$  gets from each incident vertex  $v$  the following charge:
  - (i) 2 if  $d(v) = 3$  and is a sponsor,
  - (ii) 3 if  $d(v) = 3$  and  $v$  is not a sponsor,
  - (iii) 4 if  $d(v) \geq 4$ .

Finally, we discharge the cut-vertex  $w^+$  of  $G^+$  as follows:

- R3:**  $w^+$  gives charge 1 along each feeding path started at  $w^+$  and gives charge 4 to every incident 3-face of  $G^+$ .

Now check that  $ch^*(x) \geq 0$  for every  $x \in V(G^+) \cup F(G^+)$ .

*CASE 1.*  $f \in F(G^+)$ . If  $r(f) \geq 12$  then  $ch^*(f) = ch(f) = r(f) - 12 \geq 0$ . Suppose that  $f = xyz$ , so  $ch(f) = r(f) - 12 = -9$ . We have to check that  $f$  gets at least 9 from  $x$ ,  $y$ , and  $z$  by R2 and R3. Note that at most one of  $x$ ,  $y$ , and  $z$  is a sponsor of degree 3 different from  $w^+$  due to Lemma 5, thus giving as little as 2 to  $f$ . If none of these is such a sponsor, then  $ch^*(f) \geq -9 + 3 \times 3 = 0$ . Otherwise,  $f$  is not a triplet since no sponsor belongs to a triplet by definition, so  $f$  gets 4 from at least one of its incident vertices, and we have  $ch^*(f) \geq -9 + 2 + 3 + 4 = 0$ .

*CASE 2.*  $v \in V(G)$ . If  $d(v) = 2$  then  $ch^*(v) = -2 + 2 \times 1 = 0$  by R1. Recall that if  $d(v) \geq 3$  and  $v$  is adjacent to a vertex  $w$  such that edge  $vw$  does not belong to a 3-face, then  $v$  sends at most 1 along edge  $vw$  by R1, since feeding paths do not split due to Corollary 4. Also note that  $v$  is not incident with two consecutive 3-faces since  $G$  has no 4-cycles.

Suppose that  $d(v) = 3$ ; if  $v$  does not belong to a 3-face then  $ch^*(v) \geq ch(v) - 3 \times 1 = 0$ . Otherwise,  $v$  either gives 2 to its incident 3-face by R2(i) and sends at most 1 along its non-triangular edge by R1 (if  $v$  is a sponsor), or gives 3 by R2(ii), so  $ch^*(v) \geq 0$  in both cases.

Finally, suppose that  $d(v) = d \geq 4$ ; then  $ch(v) = 5d - 12 \geq 8$ . Let  $t$  be the number of 3-faces incident with  $v$ ; we know that  $t \leq \lfloor \frac{d}{2} \rfloor$ . This implies by R1 and R2(iii) that  $ch^*(v) \geq 5d - 12 - 4t - (d - 2t) \times 1 = 4d - 12 - 2t \geq 3(d - 4) \geq 0$ .

*CASE 3.*  $ch^*(w^+) \geq 5d + 11 - 4t - (d - 2t) \times 1 = 4d + 11 - 2t \geq 3d + 11 > 0$ .

So, after discharging according to rules R1–R3 the charge of each vertex and face of  $G^+$  becomes non-negative, contrary to (1).

This completes the proof of Theorem 2.

### Acknowledgement

The first author is thankful to the University of Bordeaux for inviting him as a visiting professor in the first half of 2009 and especially to André Raspaud for his cordial hospitality in Bordeaux. Also, the authors are thankful to Aleksey Glebov for carefully checking the proof and useful remarks.

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