INDECOMPOSABLE INVARIANTS OF QUIVERS FOR DIMENSION (2, . . . , 2) AND MAXIMAL PATHS, II

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Abstract. An upper bound on degrees of elements of a minimal generating system for invariants of quivers of dimension (2, . . . , 2) is established over a field of arbitrary characteristic and its precision is estimated. The proof is based on the reduction to the problem of description of maximal paths satisfying certain condition.

Keywords: representations of quivers, invariants, oriented graphs, maximal paths.

1. Introduction

We work over an infinite field $K$ of arbitrary characteristic $\text{char}(K)$. All vector spaces, algebras, and modules are over $K$ unless otherwise stated and all algebras are associative.

This paper is a completion of [11] and we use the same notations as in [11]. Let us recall some of them. A quiver $Q = (\text{ver}(Q), \text{arr}(Q))$ is a finite oriented graph, where $\text{ver}(Q)$ is the set of vertices and $\text{arr}(Q)$ is the set of arrows. The notion of quiver was introduced by Gabriel in [5] as an effective mean for description of different problems of the linear algebra.

The head (the tail, respectively) of an arrow $a$ is denoted by $a'$ ($a''$, respectively). We say that $a = a_1 \cdots a_s$ is a path in $Q$ (where $a_1, \ldots, a_s \in \text{arr}(Q)$), if $a'_1 = a''_1, \ldots, a'_{s-1} = a''_{s-1}$, and $a$ is a closed path in a vertex $v$, if $a$ is a path and $a'_s = a''_s = v$. The head of the path $a$ is $a' = a'_1$ and the tail of $a$ is $a'' = a''_1$. Denote $\text{ver}(a) = \{a'_1, a'_2, \ldots, a'_s\}$, $\text{arr}(a) = \{a_1, \ldots, a_s\}$, and $\text{deg}(a) = s$. Given a closed path $a$ and $w \in \text{ver}(Q)$, we set $\deg_w(a) = \#\{i | a'_i = w, 1 \leq i \leq s\}$. A closed
path \( a \) is called primitive if \( \deg_w(a) = 1 \) for all \( w \in \text{ver}(a) \). Denote by \( m(\mathcal{Q}) \) the maximal degree of primitive closed paths in \( \mathcal{Q} \). Closed paths \( a_1, \ldots, a_s \) in \( \mathcal{Q} \) are called incident if \( a'_1 = \cdots = a'_s \).

For a quiver \( \mathcal{Q} \) and a dimension vector \( n = (n_v \mid v \in \text{ver}(\mathcal{Q})) \) denote by \( I(\mathcal{Q}, n) \) the algebra of invariants of representations of \( \mathcal{Q} \). The algebra \( I(\mathcal{Q}, n) \) is embedded into the algebra of (commutative) polynomials \( K[x_{ij}(a) \mid a \in \text{arr}(\mathcal{Q})] \), \( 1 \leq i \leq n_{a'}, 1 \leq j \leq n_{a''} \). Denote by \( X_a = (x_{ij}(a)) \) the \( n_{a'} \times n_{a''} \) generic matrix and by \( \sigma_k(X) \) the \( k \)-th coefficient in the characteristic polynomial of an \( n \times n \) matrix \( X \), i.e.,

\[
\det(\lambda E - X) = \lambda^n - \sigma_1(X)\lambda^{n-1} + \cdots + (-1)^n\sigma_n(X).
\]

**Theorem 1.1.** (Donkin [4]) The \( K \)-algebra \( I(\mathcal{Q}, n) \) is generated by \( \sigma_k(X_{a_1} \cdots X_{a_s}) \) for all closed paths \( a = a_1 \cdots a_s \) in \( \mathcal{Q} \) (where \( a_1, \ldots, a_s \in \text{arr}(\mathcal{Q}) \)) and \( 1 \leq k \leq n_{a'} \).

Notice that \( I(\mathcal{Q}, n) \) has a grading by degrees that is given by the formula:

\[
\deg(\sigma_k(X_{a_1} \cdots X_{a_s})) = ks.
\]

Investigation of \( I(\mathcal{Q}, n) \) was originated from the partial case of a quiver with one vertex. Sibirskii [16], Razmyslov [15] and Procesi [13] described generators and relations in the case of characteristic zero field. As about the case of arbitrary characteristic, the first step was performed by Donkin in [3], where he established relations. Relations between generators of \( I(\mathcal{Q}, n) \) were established by Domokos [1] in characteristic zero case and by Zubkov [17] in arbitrary characteristic case. Theorem 1.1 was generalized to the case of action of arbitrary classical linear groups in [10] using approach from [9].

By the Hilbert–Nagata Theorem on invariants, \( I(\mathcal{Q}, n) \) is a finitely generated graded algebra. But the mentioned generating system is not finite. So it gives rise to the problem to find out a minimal (by inclusion) homogeneous system of generators (m.h.s.g.). Let \( D(\mathcal{Q}, n) \) be the least upper bound for the degrees of elements of a m.h.s.g. of \( I(\mathcal{Q}, n) \). Note that taking elements from Theorem 1.1 of the degree less or equal to \( D(\mathcal{Q}, n) \) we obtain the finite system of generators. A decomposable invariant is equal to a polynomial in elements of strictly lower degree. Obviously, \( D(\mathcal{Q}, n) \) is equal to the highest degree of indecomposable invariants.

In [11] we established an upper bound on \( D(\mathcal{Q}, n) \) for an arbitrary quiver \( \mathcal{Q} \) and \( n = (2, 2, \ldots, 2) \). In this paper we improve essentially the mentioned upper bound and estimate its precision (see Theorem 1.2 and Remark 1.3). Note that for a quiver with one vertex and \( n = (2) \) a m.h.s.g. was found in [16], [14], [2]; in case \( n = (3) \) a m.h.s.g. was described in [7], [8] and a system of parameters for a quiver with three loops was found in [6]. A m.h.s.g. for the algebra of semi-invariants of a quiver of dimension \( (2, \ldots, 2) \) was established in [12]. References to other results on generating systems for invariants are given, for example, in [11].

Without loss of generality we can assume that \( \mathcal{Q} \) is a strongly connected quiver, i.e., there exists a closed path in \( \mathcal{Q} \) that contains all vertices of \( \mathcal{Q} \) (for the details, see Section 1 of [11]).

For positive integers \( n, d, m \) define \( M(n, d, m) \) as follows:

1) if \( \text{char}(K) = 2 \), then

\[
M(n, d, m) = \begin{cases} 
2m, & \text{if } d = n = m \\
2m(d - n + \frac{1}{2}), & \text{if } d < n + 2 \left\lfloor \frac{n-1}{m} \right\rfloor \text{ and } n > m \geq 2 \\
m(d - n - 1) + 2n, & \text{otherwise}
\end{cases}
\]
2) if \( \text{char}(K) \neq 2 \), then

\[
M(n, d, m) = \begin{cases} 
2n, & \text{if } n = m \text{ and } d \in \{n, n+1\} \\
3n, & \text{otherwise}
\end{cases}
\]

Here \( [\alpha] \) stands for the greatest integer that does not exceed \( \alpha \).

Denote by \( \mathcal{Q}(n, d, m) \) the set of all strongly connected quivers \( \mathcal{Q} \) with \( \# \text{ver}(\mathcal{Q}) = n \), \( \# \text{arr}(\mathcal{Q}) = d \), and \( m(\mathcal{Q}) = m \). A criterion when \( \mathcal{Q}(n, d, m) \) is not empty is given by Lemma 2.2. For short, we write \( D(n, d, m) \) for \( \max\{ D(\mathcal{Q}, (2, \ldots, 2)) \mid \mathcal{Q} \in \mathcal{Q}(n, d, m) \} \). Our main result is the following theorem.

**Theorem 1.2.** For \( \mathcal{Q}(n, d, m) \neq \emptyset \) we have \( D(n, d, m) \leq M(n, d, m) \). Moreover,

1) if \( \text{char}(K) = 2 \), then

\[
D(n, d, m) \geq M(n, d, m) - m.
\]

2) if \( \text{char}(K) \neq 2 \), \( d \geq n + 2 \left[ \frac{n-1}{m} \right] + m \) or \( n = m \), then

\[
D(n, d, m) = M(n, d, m).
\]

As immediate corollary of this theorem we obtain that if \( \mathcal{Q} \in \mathcal{Q}(n, d, m) \), then the algebra of invariants \( I(\mathcal{Q}, (\delta_1, \ldots, \delta_n)) \) with \( \delta_1, \ldots, \delta_n \leq 2 \) is generated by elements of degree at most \( M(n, d, m) \).

**Remark 1.3.** Let \( \text{char}(K) = 2 \). In [11] we gave the following upper bound:

\[
D(n, d, m) \leq md
\]

for \( \mathcal{Q}(n, d, m) \neq \emptyset \). By Theorem 1.2, for \( m > 2 \) the deviation of this upper bound is

\[
md - D(n, d, m) \to \infty \text{ as } n, d \to \infty,
\]

where we assume that \( m \) is fixed and \( n, d \to \infty \) in such a way that at each step \( \mathcal{Q}(n, d, m) \neq \emptyset \). But the deviation of the upper bound from Theorem 1.2 is less or equal to the constant \( m \), i.e.,

\[
0 \leq M(n, d, m) - D(n, d, m) \leq m.
\]

As in [11], for a quiver \( \mathcal{Q} \) introduce an equivalence \( \equiv \) on the set of all closed paths extended with an additional symbol 0. For any paths \( a, b \) such that \( ab \) is a closed path and any incident closed paths \( a_1, a_2, \ldots \), we define

1. \( ab \equiv ba \);
2. \( a_{(1)} \cdots a_{(t)} \equiv \text{sgn}(\sigma) a_1 \cdots a_t \), where \( t \geq 2 \) and \( \sigma \in S_t \);
3. \( a_1^2 a_2 \equiv 0 \);
4. if \( \text{char}(K) = 2 \), then \( a_1^3 \equiv 0 \); if \( \text{char}(K) \neq 2 \), then \( a_1 a_2 a_3 a_4 \equiv 0 \).

We write \( M(\mathcal{Q}) \) for the maximal degree of a closed path \( a \) in \( \mathcal{Q} \) satisfying \( a \neq 0 \).

The following lemma is Lemma 1.2 of [11], which was proved using [17].

**Lemma 1.4.** Let \( a = a_1 \cdots a_s \) be a closed path in \( \mathcal{Q} \), where \( a_1, \ldots, a_s \in \text{arr}(\mathcal{Q}) \). Then \( \text{tr}(X_{a_1} \cdots X_{a_s}) \in I(\mathcal{Q}, (2, 2, \ldots, 2)) \) is decomposable if and only if \( a \equiv 0 \).

**Remark 1.5.** Let \( a = a_1 \cdots a_s \) be a closed path in \( \mathcal{Q} \), where \( a_1, \ldots, a_s \in \text{arr}(\mathcal{Q}) \). If \( q = \det(X_{a_1} \cdots X_{a_s}) \in I(\mathcal{Q}, (2, 2, \ldots, 2)) \) is indecomposable, then \( a \) is a primitive closed path and \( \deg(a) \leq m \). Thus, \( \deg(q) \leq M(n, d, m) \).
Section 2 contains necessary definitions and results from [11]. If \( \text{char}(K) \neq 2 \), then the upper bound on \( M(\mathcal{Q}) \) is calculated in Lemma 2.4; otherwise, we establish the upper bound on \( M(\mathcal{Q}) \) in Theorems 5.1 and 6.2. In Lemma 7.1 we estimate a precision of the given upper bound. Taking into account Lemma 1.4 and Remark 1.5 together with the fact that \( I(\mathcal{Q}, (2, 2, \ldots, 2)) \) is generated by indecomposable invariants, we complete the proof of Theorem 1.2.

In Sections 3–6 we assume that \( \text{char}(K) = 2 \). Sections 3, 4, and 5 are dedicated to the proof of Theorem 5.1, which consists of two steps.

At first, we introduce the set of multidegrees \( \Omega_2(\mathcal{Q}) \) with the property that if \( h \) is a closed path and \( \text{mdeg}(h) \in \Omega_2(\mathcal{Q}) \), then \( h \neq 0 \) (see Section 3 and Remark 3.2). Moreover, Lemma 2.1 implies that \( \Omega_2(\mathcal{Q}) \) is the maximal (by inclusion) set with the given property. In Theorem 3.9 of Section 3 we give some upper bound on \( |\hat{\delta}| \) for \( \hat{\delta} \in \Omega_2(\mathcal{Q}) \). Note that there can be a closed path \( h \neq 0 \) such that \( \text{mdeg}(h) \notin \Omega_2(\mathcal{Q}) \) (see Example 3.3).

During the second step we extract some information from the fact that \( h \neq 0 \) (see Lemma 4.5). Then we find out a closed subpath \( c \) in \( h \) such that for two arrows \( b_1, b_2 \) of \( c \) we have \( \text{deg}_{b_1}(h) = \text{deg}_{b_2}(h) = 1 \) and some additional properties are valid (see Lemma 5.2). The main idea of the proof of Theorem 5.1 is to substitute \( c \) with a loop in order to obtain a quiver \( \mathcal{G} \) with \( \# \text{arr}(\mathcal{G}) < \# \text{arr}(\mathcal{Q}) \) and to use induction hypothesis. The main difficulty is that we can not claim that \( c \) is a primitive closed path, thus we can not say that \( \text{deg}(c) \leq m \). To estimate \( \text{deg}(c) \) we apply Lemma 5.5.

Section 6 contains the proof of Theorem 6.2. In Section 7 we consider some examples in order to prove Lemma 7.1.

2. Auxiliary results

2.1. Notations. For a path \( a = a_1 \cdots a_s \) in a quiver \( \mathcal{Q} \), where \( a_1, \ldots, a_s \in \text{arr}(\mathcal{Q}) \), and \( b \in \text{arr}(\mathcal{Q}), \ v \in \text{ver}(\mathcal{Q}) \), we set

\[
\begin{align*}
\text{deg}_b(a) &= \#\{i \mid a_i = b, 1 \leq i \leq s\}; \\
\text{deg}_v(a) &= \max\{m_1, m_2\}, \text{ where } m_1 = \#\{i \mid a'_i = v, 1 \leq i \leq s\} \text{ and } m_2 = \#\{i \mid a''_i = v, 1 \leq i \leq s\}; \\
\text{deg}_v^a(a) &= \#\{i \mid a'_i = v, 1 \leq i \leq s - 1\}.
\end{align*}
\]

Let \( \hat{\delta} \in \mathbb{N}^{\# \text{arr}(\mathcal{Q})} \), where \( \mathbb{N} \) stands for non-negative integers. Then the path \( a \) is called \( \hat{\delta} \)-double if \( a \) is a primitive closed path and \( \delta_{a_i} \geq 2 \) for all \( i \). The definition of strongly connected components of an arbitrary quiver \( \mathcal{G} \) is well known (for example, see Section 1 of [11]). The following notions were defined in Section 5 of [11]:

- the multidegree \( \text{mdeg}(a) \) of a path \( a \);
- the empty path \( 1_v \) in a vertex \( v \);
- a subpath of a path \( a \);
- an \( h \)-restriction of \( \mathcal{Q} \) to \( V \), where \( V \subset \text{ver}(\mathcal{Q}) \) and \( h \) is a path in \( \mathcal{Q} \) (see also Example 5.1 of [11]).

Denote by \( \text{path}(\mathcal{Q}) \) the set of all paths and empty paths in \( \mathcal{Q} \). If we consider a path, then we assume that it is non-empty unless otherwise stated; if we write \( a \in \text{path}(\mathcal{Q}) \), then we assume that a path \( a \) can be empty.

Dealing with equivalences we use the following conventions. If we write \( a \equiv b \), then we assume that \( a \) and \( b \) are closed paths in \( \mathcal{Q} \). If we write \( ab \) for paths \( a \) and \( b \), then we assume that \( a' = b'' \). To explain how we apply formulas to prove some equivalence \( a \equiv b \) we split the word \( a \) into parts using dots.
For closed paths $a, b$ we write $a \sim b$ if $a = c_1c_2$ and $b = c_2c_1$ for some $c_1, c_2 \in \text{path}(Q)$. For $\theta, \theta' \in \mathbb{N}^d$ we set $\theta \equiv \theta'$ if and only if $\delta_i \equiv \delta'_i$ for all $i$ and define $|\theta| = \delta_1 + \cdots + \delta_i$.

Let $x_1, \ldots, x_s$ be all arrows in $Q$ from $u$ to $v$, where $u, v \in \text{ver}(Q)$. Then denote by $\bar{x}$ any arrow from $x_1, \ldots, x_s$, by $\{\bar{x}\}$ the set $\{x_1, \ldots, x_s\}$, and say that $\bar{x}$ is an arrow from $u$ to $v$. Schematically, we depict arrows $x_1, \ldots, x_s$ as

For a path $a$ in $Q$ denote $\deg_{\bar{x}}(a) = \sum_{i=1}^s \deg_{x_i}(a)$. As an example, an expression $\bar{x}a_1 \cdots \bar{x}a_k$ stands for a path $x_{i_1}a_1 \cdots x_{i_k}a_k$ for some $1 \leq i_j \leq s$ ($1 \leq j \leq k$). Similarly, if $x_1, \ldots, x_s$ are loops in $v \in \text{ver}(Q)$, then $\bar{x}^k$ stands for a closed path $x_{i_1} \cdots x_{i_k}$ for some $i_1, \ldots, i_k$.

The next two lemmas are well known.

**Lemma 2.1.** Suppose $Q$ is a strongly connected quiver and $\delta \in \mathbb{N}^\# \text{arr}(Q)$. Then the following conditions are equivalent:

a) There is a closed path $h$ in $Q$ such that $\text{nd}(h) = \delta$ and $\text{arr}(h) = \text{arr}(Q)$; in particular, $\text{ver}(h) = \text{ver}(Q)$.

b) We have $\delta_a \geq 1$ for all $a \in \text{arr}(Q)$ and $\sum_{a'=v} \delta_a = \sum_{a''=v} \delta_a$ for all $v \in \text{ver}(Q)$, where the sums range over all $a \in \text{arr}(Q)$ satisfying the given conditions.

We write $\delta(i, j)$ for the Kronecker symbol.

**Lemma 2.2.** For positive integers $n, d, m$ the set $Q(n, d, m)$ is not empty if and only if one of the following possibilities holds:

a) $n = m = 1$;

b) $n \geq m \geq 2$ and $d \geq n + l - \delta(0, r)$, where $n - 1 = l(m - 1) + r$, $l \geq 1$, and $0 \leq r \leq m - 2$.

**Lemma 2.3.** Suppose $Q_1, Q_2$ are strongly connected quivers and $Q_1 \subset Q_2$. Then

$$\# \text{arr}(Q_2) - \# \text{arr}(Q_1) \geq \# \text{ver}(Q_2) - \# \text{ver}(Q_1) + 1.$$  

**Proof.** For every $v \in \text{ver}(Q_2) \setminus \text{ver}(Q_1)$ there is an $a \in \text{arr}(Q_2) \setminus \text{arr}(Q_1)$ with $a' = v$. There also exists a $b \in \text{arr}(Q_2) \setminus \text{arr}(Q_1)$ satisfying $b' \in \text{ver}(Q_1)$. These remarks imply the required formula.

2.2. **Basic equivalences.** We start with the following lemma.

**Lemma 2.4.** Suppose $\text{char}(K) \neq 2$. If $Q \in Q(n, d, m)$, $h$ is a closed path in $Q$, and $h \neq 0$, then $\deg(h) \leq M(n, d, m)$.

**Proof.** We claim that $\deg(h) \leq 3n$. If $\deg(h) > 3n$, then there is a vertex $v \in \text{ver}(Q)$ such that $\deg_{\bar{x}}(h) \geq 4$. Therefore, $h \equiv h_1 \cdots h_4$ for some closed paths $h_1, \ldots, h_4$ in $v$. Thus $h \equiv 0$ by the definition of the equivalence $\equiv$; a contradiction.

To complete the proof, it is enough to consider the case of $n = m$ and $d \in \{n, n + 1\}$.

1. If $d = n$, then $\text{arr}(Q) = \{a_1, \ldots, a_n\}$, where $a = a_1 \cdots a_n$ is a primitive closed path. Then $h \equiv a^s$ for some $s > 0$. If $s \geq 3$, then $h \equiv 0$; a contradiction. Thus $\deg(h) \leq 2n$. The case of $n = 1$ and $d = n + 1$ can be treated similarly.
2. Let $n = m \geq 2$ and $d = n + 1$. In this case $Q$ is

![Diagram of a quiver](image)

where $1 \leq k \leq n$. Denote $a = a_1 \ldots a_n$ and

$$c = \begin{cases} b, & k = 1 \\ ba_k \ldots a_n, & \text{otherwise} \end{cases}$$

We have $h \equiv a^r c^s$ for some $r, s \geq 0$. If $r = 0$ or $s = 0$, then $\deg(h) \leq 2n$ (see Part 1 of the lemma). Assume that $r, s > 0$. If $r \geq 2$ or $s \geq 2$, then $h \equiv 0$; a contradiction. Hence $\deg(h) = n + \deg c \leq 2n$.

In what follows we assume that $\text{char}(K) = 2$ unless otherwise stated. We will use the following remark without references to it.

**Remark 2.5.** Suppose $f, h$ are closed paths in $Q$ and $b$ is a subpath of $f$. Let the equivalence $f \equiv h$ follows from the formulas of the form $a_{\sigma(1)} \cdots a_{\sigma(t)} \equiv a_1 \cdots a_t$, where $a_1, \ldots, a_t$ are closed paths in $v \in \text{ver}(Q)$ satisfying $\deg_v(b) = 0$, $t \geq 2$, and $\sigma \in S_t$. Then $b$ is also a subpath of $h$.

There following three lemmas are Lemmas 6.3, 6.8, and 6.9 of [11], respectively.

**Lemma 2.6.** Let $h$ be a closed path in $Q$ and $\{p\}$ be loops of $Q$ in some $v \in \text{ver}(Q)$. Then $h \equiv p^kb$, where $k \geq 0$, $b \in \text{path}(Q)$, and $\deg_p(b) = 0$.

Moreover, suppose $a \in \text{arr}(h)$ and $a' \neq a''$. If $a' = v$, then $h \equiv ap^kb_0$; if $a'' = v$, then $h \equiv p^kab_0$, where, as above, $\deg_p(b_0) = 0$.

Suppose a quiver $Q$ contains a path $a = a_1 \cdots a_s$, where $a_1, \ldots, a_s \in \text{arr}(Q)$ are pairwise different. Let $h$ be a closed path in $Q$ such that $\deg_{a_i}(h) \geq 2$ for all $i$ and there is a $b \in \text{arr}(h)$ satisfying $b \neq a_i$ for all $i$.

**Lemma 2.7.** Using the preceding notation we have $h \equiv a_1 \cdots a_s f$ for some $f \in \text{path}(Q)$. Moreover,

a) if $b' = a''_i$, then $h \equiv ba_1 \cdots a_s f$ for some $f \in \text{path}(Q)$;

b) if $b'' = a'_i$, then $h \equiv a_1 \cdots a_s bf$ for some $f \in \text{path}(Q)$.

Let $a$ and $h$ be paths as above. For $1 \leq i \leq s$ denote $v_i = a''_i$. We assume that the path $a$ is closed and primitive, $s \geq 2$, $b' \neq b''$, and $b', b'' \in \{v_2, v_k\}$ for some $k \in \{1, 3, 4, \ldots, s\}$. Schematically, this is depicted as

![Diagram of a quiver](image)
Lemma 2.8. Using the preceding notation we have $h = a_1a_2f_1a_3a_2f_2$ for some $f_1, f_2 \in \text{path}(Q)$.

Lemma 2.9. Suppose $Q$ is a quiver with $n$ vertices and $d$ arrows. Let $h$ be a closed path in $Q$ and $h \neq 0$. Then there exist pairwise different primitive closed paths $b_1, \ldots, b_r, c_1, \ldots, c_t$ in $Q$, where $r, t \geq 0$ and $r + t \leq d - n + 1$, such that

$$\text{mdeg}(h) = \sum_{i=1}^{r} \text{mdeg}(b_i) + 2 \sum_{k=1}^{t} \text{mdeg}(c_k);$$

and there are pairwise different $x_1, \ldots, x_r, y_1, \ldots, y_t, z_1, \ldots, z_t \in \text{arr}(Q)$ satisfying

(2) $y_j, z_j \in \text{arr}(e_j)$ and $\text{deg}_{y_j}(h) = \text{deg}_{z_j}(h) = 2$,

(3) $x_i \in \text{arr}(b_i)$ and $\text{deg}_{x_i}(h) - 2 \sum_{k=1}^{t} \text{deg}_{x_i}(c_k) = 1$

for any $1 \leq i \leq r, 1 \leq j \leq t$.

Proof. The statement of the lemma but the inequality $r + t \leq d - n + 1$ follows from Lemma 6.10 [11]. Applying Lemma 2.3, we can assume that $Q = Q_{\text{mdeg} h}$.

Denote by $G$ the quiver that is the union of closed paths $b_1, \ldots, b_r$, i.e., $\text{ver}(G) = \text{ver}(b_1) \cup \cdots \cup \text{ver}(b_r)$ and $\text{arr}(G) = \text{arr}(b_1) \cup \cdots \cup \text{arr}(b_r)$. Let $G_1, \ldots, G_l$ be the strongly connected components of $G$. We have $\text{arr}(G_k) = \bigcup_{i \in I_k} \text{arr}(b_i)$ for some $I_k \subset [1, r]$ and denote $\#I_k = r_k (1 \leq k \leq l)$.

We assume that $k = 1$. Consider an $i_1 \in I_1$ and let $Q_1$ be the quiver such that $\text{ver}(Q_1) = \text{ver}(b_{i_1})$ and $\text{arr}(Q_1) = \text{arr}(b_{i_1})$. If $\#I_1 > 1$, then there is an $i_2 \in I_1 \setminus \{i_1\}$ satisfying $\text{ver}(b_{i_2}) \cap \text{ver}(Q_1) \neq \emptyset$. By part a), we have $x \not\in \text{arr}(Q_1)$ for some $x \in \text{arr}(b_{i_2})$. Hence there is an $e_2 \in \text{arr}(b_{i_2})$ such that $e_2 \not\in \text{arr}(Q_1)$ and $e_2 \in \text{ver}(Q_1)$. We add the closed path $b_{i_2}$ to $Q_1$ and obtain a new quiver $Q_2$, i.e., $\text{ver}(Q_2) = \text{ver}(Q_1) \cup \text{ver}(b_{i_2})$ and $\text{arr}(Q_2) = \text{arr}(Q_1) \cup \text{arr}(b_{i_2})$. Then we repeat this procedure for $Q_2$ and so on. Finally, we obtain $Q_1, Q_2, \ldots, Q_{r_1} = G_1$ and pairwise different arrows $e_2, \ldots, e_{r_1}$ such that $e_j \in \text{arr}(Q_j) \setminus \text{arr}(Q_{j-1})$ and $e_j \in \text{ver}(Q_{j-1})$ for any $2 \leq j \leq r_1$. Then for the set $V_1 = \{e'_2, \ldots, e'_{r_1}\}$ we have $\#\{a \in \text{arr}(G_1) | a' \in V_1\} \geq \#V_1 + (r_1 - 1)$. Since for every $v \in \text{ver}(G_1) \setminus V_1$ there is at least one arrow $a \in \text{arr}(G_1)$ with $a' = v$, we have

$$\# \text{arr}(G_1) \geq \# \text{ver}(G_1) + (r_1 - 1).$$

The similar formula holds for all $k$. It follows that

(4) $\# \text{arr}(G) \geq \# \text{ver}(G) + (r - l).$

For the quiver $Q_r = G$ there is a $j_1 \in [1, t]$ satisfying $\text{ver}(c_{j_1}) \cap \text{ver}(Q_r) \neq \emptyset$. We add $c_{j_1}$ to $Q_r$ and denote the resulting quiver by $Q_{r+1}$. By (2), there exists a $g_1 \in \text{arr}(c_{j_1})$ such that $g_1 \not\in \text{arr}(Q_r)$ and $g'_1 \in \text{ver}(Q_r)$. Moreover, if the number of strongly connected components of $Q_{r+1}$ is less than the number of strongly connected components of $Q_r$, then there also exists a $g_2 \in \text{arr}(c_{j_1}) \setminus \{g_1\}$ such that $g_2 \not\in \text{arr}(Q_r)$ and $g'_2 \in \text{ver}(Q_r)$. We repeat this procedure for $Q_{r+1}$ and so on. Finally, we obtain quivers $Q_r, Q_{r+1}, \ldots, Q_{r+t} = G$ and pairwise different arrows $g_1, \ldots, g_{t+1}$ of $Q$ such that for the set $V = \{g'_1, \ldots, g'_{t+1-1}\}$ we have

$$\#\{a \in \text{arr}(Q) \setminus \text{arr}(G) | a' \in V\} \geq \#V \setminus \text{ver}(G) + (t + l - 1).$$
Therefore
\[ \# \text{arr}(Q) \setminus \text{arr}(G) \geq \# \text{ver}(Q) \setminus \text{ver}(G) + (t + l - 1) \]
and (4) completes the proof. \qed

3. Sets of multidegrees

Suppose \( Q \) is a strongly connected quiver and \( \text{char}(K) = 2 \).

The support of a non-zero vector \( \delta \in \mathbb{N}^{|\text{arr}(Q)|} \) with respect to \( Q \) is the subquiver \( Q_\delta \) of \( Q \) such that \( \text{arr}(Q_\delta) = \{a \in \text{arr}(Q) \mid \delta_a \geq 1\} \) and \( \text{ver}(Q_\delta) = \{a', a'' \mid a \in \text{arr}(Q_\delta)\} \). The following remark is extensively applied to established indecomposability of invariants.

Remark 3.1. Let \( h \) be a closed path in \( Q \). If for any \( \text{mdeg}(h) \)-double path \( a \) we have that the support of \( \text{mdeg}(h) - 2 \text{mdeg}(a) \) is not strongly connected (and is not empty), then \( h \not\equiv 0 \).

Proof. If \( h \) satisfies the condition of the lemma and \( h \equiv 0 \), then \( h \equiv a^2 f \) for some paths \( a, f \). Thus the support of \( \text{mdeg}(h) - 2 \text{mdeg}(a) = \text{mdeg}(f) \) is strongly connected; a contradiction. \qed

For a non-zero vector \( \delta \in \mathbb{N}^{|\text{arr}(Q)|} \) we say that
- \( \delta \) is indecomposable (with respect to \( Q \)) if its support is strongly connected;
- \( \delta \) is decomposable (with respect to \( Q \)) if its support is not strongly connected but is the disjoint union of strongly connected quivers.

Observe that \( \delta \) can be neither decomposable nor indecomposable. We say that \( \delta = \sum_{i=1}^{r} \delta_i \) is the decomposition of \( \delta \) with respect to \( Q \) if \( \delta_i \in \mathbb{N}^{|\text{arr}(Q)|} \) are non-zero vectors and \( Q_{\delta_1}, \ldots, Q_{\delta_r} \) are pairwise different strongly connected components of \( Q_\delta \). Obviously, if \( \delta \) is indecomposable, then \( r = 1 \); and if \( \delta \) is decomposable, then \( r \geq 2 \). Introduce the following sets:

a) the set \( \Omega_1(Q) \) consists of all \( \text{mdeg}(h) \), where \( h \) ranges over closed paths in \( Q \) with \( \text{arr}(h) = \text{arr}(Q) \);

b) the set \( \Omega_2(Q) \) consists of such \( \delta \in \Omega_1(Q) \) that for every \( \delta \)-double path \( a \) in \( Q \) we have \( \delta - 2 \text{mdeg}(a) \) is decomposable with respect to \( Q \);

c) the set \( \Omega_3(Q) \) consists of such \( \delta \in \Omega_1(Q) \) that there is no \( \delta \)-double path in \( Q \);

d) the set \( \Omega(Q) \) consists of such \( \text{mdeg}(h) \in \Omega_1(Q) \) that \( h \) is a closed path in \( Q \) and \( h \not\equiv 0 \).

For every vector \( \delta \in \Omega_1(Q) \) there exists its decomposition with respect to \( Q \) that consists of one summand. Moreover, by Lemma 2.1, for every \( \theta \in \Omega_1(Q) \) with \( \delta - \theta \geq 0 \) there also exists a decomposition of \( \delta - \theta \) with respect to \( Q \).

Remark 3.2. We have the following inclusions: \( \Omega_3(Q) \subset \Omega_2(Q) \subset \Omega(Q) \subset \Omega_1(Q) \).

Proof. The inclusion \( \Omega_2(Q) \subset \Omega(Q) \) follows from Remark 3.1. The remaining inclusions are trivial. \qed
Example 3.3. Let $h_1 = czcxyba$, $h_2 = czcyzza$ be closed paths in the quiver $Q$

Then $h_1 \equiv 0$, $h_2 \not\equiv 0$, and $\text{mdeg}(h_1) = \text{mdeg}(h_2) \in \Omega(Q) \setminus \Omega_2(Q)$.

Lemma 3.4. If $Q \in Q(n, d, m)$ and $\tilde{\delta} \in \Omega_3(Q)$, then $|\tilde{\delta}| \leq m(d - n + 1)$.

Proof. By definition, $\delta = \text{mdeg}(h)$ for some closed path $h$ in $Q$. The definition of $\Omega_3(Q)$ shows that $\tilde{h} \not\equiv 0$. Then Lemma 2.9 implies $\text{deg}(h) \leq m(r + 2t)$ and $r + t \leq d - n + 1$. Since $t = 0$, the proof is completed.

Definition (of a $\tilde{\delta}$-complete chain). A chain of paths $A = (a_1, \ldots, a_t)$ is an ordered sequence of primitive closed paths satisfying $\text{ver}(a_i) \cap \text{ver}(a_j) = \emptyset$ if $|i - j| > 1$; and $\text{ver}(a_i) \cap \text{ver}(a_j) \neq \emptyset$ otherwise. Given $\tilde{\delta} \in \Omega_2(Q)$, the chain of paths $A$ is called $\tilde{\delta}$-complete if the following holds.

1. The paths $a_1, \ldots, a_t$ are $\tilde{\delta}$-double paths.
2. For $\tilde{\theta} = \tilde{\delta} - 2 \sum_{i=1}^t \text{mdeg}(a_i)$ we have $\tilde{\theta} \geq 0$ and $|\tilde{\theta}| > 0$.
3. There is a (unique) decomposition $\tilde{\theta} = \tilde{\theta}^{(1)} + \cdots + \tilde{\theta}^{(r)}$ with respect to $Q$

and this decomposition satisfies

a) $r \geq 2$ and $\tilde{\theta}^{(i)} \in \Omega_2(Q_{\tilde{\theta}^{(i)}})$ for all $i$;

b) if $t \geq 2$, then $r = 2$ and we have $\text{ver}(Q_{\tilde{\theta}^{(i)}}) \cap \text{ver}(a_j) \neq \emptyset$ if $i = j = 1$

or $i = 2$, $j = t$.

If there is no $\tilde{\delta}$-double path in $Q$, then $A = \emptyset$ is called a $\tilde{\delta}$-complete chain. Schematically, a $\tilde{\delta}$-complete chain $A$ is depicted on Figure 1 for $t = 1$ and on Figure 2 for

$t \geq 2$, where circles stand for closed paths and rectangles stand for subquivers of $Q$:

![Figure 1](image1.png)

![Figure 2](image2.png)

Lemma 3.5. For every $\tilde{\delta} \in \Omega_2(Q)$ there exists a $\tilde{\delta}$-complete chain $A = (a_1, \ldots, a_t)$.

Proof. If there is no $\tilde{\delta}$-double path in $Q$, then $A = \emptyset$ is a $\tilde{\delta}$-complete chain; otherwise, let $a_1$ be a $\tilde{\delta}$-double path in $Q$. Consider the decomposition $\tilde{\delta} - 2 \text{mdeg}(a_1) = \tilde{\delta}^{(1)} + \cdots + \tilde{\delta}^{(r)}$ with respect to $Q$. Since $\tilde{\delta} \in \Omega_2(Q)$, we have $r \geq 2$. If $\tilde{\delta}^{(i)} \in \Omega_2(Q_{\tilde{\theta}^{(i)}})$ for all $i$, then $A = \{a_1\}$ is a $\tilde{\delta}$-complete chain. Thus without loss of generality we can assume that $\tilde{\theta}^{(2)} \not\in \Omega_2(Q_{\tilde{\theta}^{(2)}})$, i.e., there exists a $\tilde{\delta}^{(2)}$-double path $a_2$ such that $\tilde{\theta} = \tilde{\delta}^{(2)} - 2 \text{mdeg}(a_2)$ is indecomposable. But $\tilde{\delta} - 2 \text{mdeg}(a_2)$ is decomposable, since
\(\delta \in \Omega_2(Q)\). Hence we obtain \(\text{ver}(a_1) \cap \text{ver}(a_2) \neq \emptyset\) and \(\text{ver}(a_1) \cap \text{ver}(Q_2) = \emptyset\) (see the picture).

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$Q^{(a_0)}$};
\node (a1) at (0,0.5) {$a_1$};
\node (a2) at (1,0) {$Q^{(a_2)}$};
\node (a3) at (1,0.5) {$a_2$};
\node (a4) at (2,0) {$Q^{(a_3)}$};
\node (a5) at (2,0.5) {$a_3$};
\node (a6) at (3,0) {$Q^{(a_4)}$};
\node (a7) at (3,0.5) {$a_4$};
\end{tikzpicture}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$Q^{(a_0)}$};
\node (a1) at (0,0.5) {$a_1$};
\node (a2) at (1,0) {$Q^{(a_2)}$};
\node (a3) at (1,0.5) {$a_2$};
\node (a4) at (2,0) {$Q^{(a_3)}$};
\node (a5) at (2,0.5) {$a_3$};
\node (a6) at (3,0) {$Q^{(a_4)}$};
\node (a7) at (3,0.5) {$a_4$};
\end{tikzpicture}
\end{array}
\]

If \(r \geq 3\), then we consider \(a_2\) instead of \(a_1\) and obtain that \(\delta - 2 \text{mdeg}(a_2) = \theta' + \theta\) is the decomposition of \(\delta - 2 \text{mdeg}(a_2)\), where \(\theta' = \delta^{(1)} + \delta^{(3)} + \cdots + \delta^{(r)} + 2 \text{mdeg}(a_1)\) is indecomposable. Thus without loss of generality we can assume that \(r = 2\).

We have the decomposition \(\delta - 2 \text{mdeg}(a_1) - 2 \text{mdeg}(a_2) = \theta^{(1)} + \theta^{(2)}\), where \(\theta^{(1)} = \delta^{(1)}\) and \(\theta^{(2)} = \theta\). Then we consider \(\theta^{(1)}\) and \(\theta^{(2)}\) in the same way as we have considered \(\delta^{(2)}\); and so on. Finally, we obtain a \(\delta\)-complete chain. \(\square\)

**Definition** (of a \(\delta\)-tree). For \(\delta \in \mathbb{N}^\#\text{arr}(Q)\) a triple \((T, \delta^{(v)}, A_v \mid v \in \text{ver}(T))\) is called a \(\delta\)-tree if the following holds:

1. \(T\) is an oriented rooted tree, i.e., there is no closed path in \(T\), there is a unique \(v_0 \in \text{ver}(T)\) with \(a' \neq v_0\) for all \(a \in \text{arr}(T)\), and for each other vertex \(v\) of \(T\) there is a unique \(a \in \text{arr}(T)\) with \(a' = v\). The vertex \(v_0\) is called the root and a vertex \(v \in \text{ver}(T)\) with \(v \neq a''\) for all \(a \in \text{arr}(T)\) is called a leaf.
2. Suppose \(v \in \text{ver}(T)\), then
   a) \(\delta^{(v)} \in \mathbb{N}^\#\text{arr}(Q)\) and \(\delta^{(v_0)} = \delta\);
   b) \(A_v = (a_1, \ldots, a_t)\) is a \(\delta^{(v)}\)-complete chain;
   c) if \(A_v \neq \emptyset\), then \(\delta - 2 \sum_{i=1}^t \text{mdeg}(a_i) = \delta^{(b_1)} + \cdots + \delta^{(b_t)}\) is the decomposition with respect to \(Q\), where \(b_1, \ldots, b_t\) are all arrows of \(T\) whose tails are equal to \(v\); otherwise \(v\) is a leaf.

In particular, the conditions that \(v \in \text{ver}(T)\) is a leaf, \(A_v = \emptyset\), and \(\delta^{(v)} \in \Omega_3(Q_{\delta^{(v)}})\) are equivalent. Note that \(\#\text{ver}(T) = 1\) if \(\delta \in \Omega_3(Q_2)\). By Lemma 3.5, there exists a \(\delta\)-tree for every \(\delta \in \Omega_2(Q)\). Observe that for different \(u, v \in \text{ver}(T)\) and closed paths \(a \in A_u, b \in A_v\) we have \(a \neq b\).

**Lemma 3.6.** Suppose \(\delta \in \Omega_2(Q)\setminus\Omega_3(Q)\) and \((T, \delta^{(v)}, A_v \mid v \in \text{ver}(T))\) is a \(\delta\)-tree. Denote \(l = \#\{v \in \text{ver}(T) \mid v\) is not a leaf\} and define a set \(A = \{a \mid a \in A_v\) for some \(v \in \text{ver}(T)\}\). Then there are pairwise different \(c_1, \ldots, c_t \in A\) such that \(A \setminus \{c_1, \ldots, c_t\} = B_1 \cup \cdots \cup B_{t_2}\) is a disjoint union, where \(B_1, \ldots, B_{t_2}\) are some chains of paths, \(0 \leq t_1 < t\), and \(1 \leq t_2 \leq t\).

**Proof.** We assume that \(i = 1\). Suppose \(v \in \text{ver}(T)\) is not a leaf, \(A_v = (a_1, \ldots, a_t)\), and \(b_1, \ldots, b_r\) are arrows of \(T\) whose tails are equal to \(v\). If \(t = 1\) and there is a \(1 \leq j \leq r\) such that \(A_{b_j} \neq \emptyset\), then we define \(c_i = a_1\), assign \(b_j\) to \(c_i\), and increase \(i\) by one.
If $t \geq 2$, then $r = 2$ by the definition of a complete chain. If we also have $A_{b_i} \neq \emptyset$, then we define $c_i = a_1$, assign $b'_1$ to $c_i$, and increase $i$ by one. If $A_{b_i} \neq \emptyset$, then we define $c_i = a_1$, assign $b'_2$ to $c_i$, and increase $i$ by one.

Repeat this procedure for all vertices of $T$ that are not leaves and obtain a set of pairwise different closed paths $C = \{c_1, \ldots, c_t\}$. Since we have defined an injection $C \to \{v \in \text{ver}(T) \mid v \text{ is neither a leaf nor the root}\}$, the inequality $l_1 < l$ holds. The claim of the lemma follows from the construction. \hfill \Box

**Lemma 3.7.** Suppose $Q \in Q(n, d, m)$ and $A = (a_1, \ldots, a_t)$ is a chain of paths such that for $\delta = 2 \sum_{i=1}^t \text{mdeg}(a_i)$ we have $\delta \in \Omega_1(Q)$ and $t \geq 1$. Then $|\delta| - mt - n_0 \leq n$, where

$$n_0 = \begin{cases} 0, & \text{if } t = 1 \\ \#(\text{ver}(a_1) \cap \text{ver}(a_2)), & \text{if } t = 2 \\ \#(\text{ver}(a_2) \cup \cdots \cup \text{ver}(a_{t-1})), & \text{if } t \geq 3 \end{cases}.$$

**Proof.** If $t = 1$, then $\text{deg}(a_1) = n$ and $m = n$. Thus $|\delta| - mt - n_0 = n$.

If $t \geq 2$, then $\frac{1}{2} |\delta| \leq n + n_0$. Therefore $|\delta| - mt - n_0 = \sum_{i=1}^t (\text{deg}(a_i) - m) + (\frac{1}{2} |\delta| - n_0) \leq n$, since $\text{deg}(a_i) \leq m$. \hfill \Box

**Lemma 3.8.** Suppose $Q \in Q(n, d, m)$, $\delta \in \Omega_2(Q)$, $A = (a_1, \ldots, a_t) \neq \emptyset$ is a $\delta$-complete chain, and $\bar{\theta} = \delta - 2 \sum_{i=1}^t \text{mdeg}(a_i)$. Let $\theta = \bar{\theta}^{(1)} + \cdots + \bar{\theta}^{(r)}$ be the decomposition with respect to $Q$. We define $k = n - \text{# ver}(Q_{\theta})$ and assume that

$$|\theta^{(j)}| \leq m(d_j - n_j) + n_j + \rho_j$$

for any $1 \leq j \leq r$, where $d_j = \text{# arr}(Q_{\theta^{(j)}})$, $n_j = \text{# ver}(Q_{\theta^{(j)}})$, and $\rho_j \in \mathbb{Z}$. Then

$$|\delta| \leq m(d - n) + n + \sum_{j=1}^r \rho_j + \rho,$$

where $\rho = 2 \sum_{i=1}^t \text{deg}(a_i) - m(t + 1) - k$.

**Proof.** We define a quiver $G$ by $\text{ver}(G) = \text{ver}(Q)$ and $\text{ver}(G) = \text{ver}(Q_{\theta})$. Let $G_1, \ldots, G_t$ be all strongly connected components of $G$. Then $l = k + r$ and for any $1 \leq i \leq k + r$ there is an arrow $b$ in $\text{arr}(Q) \setminus \text{arr}(Q_{\theta})$ such that $b' \in \text{ver}(G_i)$. Moreover, for any $1 \leq i \leq t - 1$ there are at least two arrows in $\text{arr}(Q) \setminus \text{arr}(Q_{\theta})$ whose heads are in $\text{ver}(a_i) \cap \text{ver}(a_{i+1})$ and every vertex from $\text{ver}(a_i) \cap \text{ver}(a_{i+1})$ is a strongly connected component of $G$. These two remarks imply that

$$d \geq \sum_{j=1}^r d_j + (k + r) + (t - 1).$$

Since $r \geq 2$, we have $\sum_{j=1}^r d_j \leq d - k - t - 1$ and $\sum_{j=1}^r n_j = n - k$. Clearly,

$$|\delta| \leq m \sum_{j=1}^r d_j + (1 - m) \sum_{j=1}^r n_j + \sum_{j=1}^r \rho_j + 2 \sum_{i=1}^t \text{deg}(a_i),$$

and the above formulas complete the proof. \hfill \Box

**Theorem 3.9.** Suppose $Q \in Q(n, d, m)$ is a quiver and $\delta \in \Omega_3(Q)$. Then $|\delta| \leq m(d - n - 1) + 2n$.

**Proof.** If $\delta \in \Omega_3(Q)$, then the required formula follows from Lemma 3.4.
Suppose $\delta \not\in \Omega_3(Q)$ and $(T, \delta^{(v)}, A_v \mid v \in \text{ver}(T))$ is a $\delta$-tree. Define the set $I = \{v \in \text{ver}(T) \mid v$ is not a leaf$\}$. For $v \in \text{ver}(T)$ denote $m_v = m(Q_{\delta^{(v)}}) \leq m$, $n_v = \# \text{ver}(Q_{\delta^{(v)}})$, and $d_v = \# \text{ver}(Q_{\delta^{(v)}})$. If $v \in \text{ver}(T) \setminus I$, then $\delta^{(v)} \in \Omega_3(Q)$ and Lemma 3.4 together with the inequalities $m_v \leq m$ and $n_v \leq d_v$ implies

$$|\delta^{(v)}| \leq m_v(d_v - n_v) + m_v \leq m(d_v - n_v) + n_v.$$  

For $v \in I$ let $A_v = (a_{v,1}, \ldots, a_{v,t_v})$. We define $\delta^{(v)} = 2 \sum_{i=1}^{t_v} \text{md}(a_{v,i})$ and $k_v = n_v - \# \text{ver}(Q_{\delta^{(v)}})$. By (5), we can apply Lemma 3.8 to all vertices of $I$ starting from elements of the set $\{v \in I \mid a' \text{ is a leaf for every } a \in \text{arr}(T)$ with $a'' = v\}$. Hence we obtain

$$|\delta| \leq m(d - n) + n + \rho,$$

where $\rho = \sum_{v \in I} \rho_v$ and $\rho_v = 2 \sum_{i=1}^{t_v} \text{deg}(a_{v,i}) - m(t_v + 1) - k_v$.

We consider closed paths $c_1, \ldots, c_{l_1}$ from Lemma 3.6, where $l_1 \leq \#I - 1$. For every $v \in I$ we define $J_v \subset [1, t_v]$ by the equality $C_v = A_v \setminus \{c_1, \ldots, c_{l_1}\} = \{a_{v,i} \mid i \in J_v\}$ and denote $I_0 = \{v \in I \mid C_v \neq \emptyset\}$. Therefore,

$$\rho = \left(2 \sum_{v \in I_0} \sum_{i \in J_v} \text{deg}(a_{v,i}) - m(t + \#I) - \sum_{v \in I} k_v\right) + 2 \sum_{v \in I} \text{deg}(c_i),$$

where $t$ stands for $\sum_{v \in I} t_v = l_1 + \sum_{v \in I_0} \#J_v$. Since $\text{deg}(c_i) \leq m$ and $l_1 - \#I \leq -1$, we have

$$\rho \leq \sum_{v \in I_0} \left(2 \sum_{i \in J_v} \text{deg}(a_{v,i}) - m \#J_v - k_v\right) - m.$$  

For all $v \in I_0$ define $n_v$ for the chain of paths $C_v$ in the same way as we have defined $n_0$ in Lemma 3.7 and let $s_v$ be the number of vertices in $C_v$. Lemma 3.7 together with the inequality $-k_v \leq -n_v$ implies $\rho \leq \sum_{v \in I_0} s_v - m$. Since there is no $u \in \text{ver}(Q)$ that belongs to $C_{v_1}$ and $C_{v_2}$ for different $v_1, v_2 \in I_0$, we have $\sum_{v \in I_0} s_v \leq n$ and $\rho \leq n - m$. 

\[\square\]

4. Properties of a closed path $h$ with $h \neq 0$

In this section $Q$ is a strongly connected quiver and $\text{char}(K) = 2$. Let $a = a_1 \cdots a_s$ be a primitive closed path in $Q$ and $v_1 = a'_1, \ldots, v_s = a'_s$, where $a_1, \ldots, a_s \in \text{arr}(Q)$ and $s \geq 2$. Suppose $h$ is a closed path in $Q$.

**Definition (of good subpaths).** A subpath $b$ in $h$ is called good, if

a) $b', b'' \in \{v_1, \ldots, v_s\}$;

b) $\text{deg}_{a_i}(b) = 0$ for all $i$;

c) $b \neq a_i$ for all $i$.

Suppose $h \sim b_1 g_1 b_2 g_2$, where $b_1, b_2$ are good subpaths in $h$ and $g_1, g_2$ are paths in $Q$. Then we say that $b_1$ and $b_2$ are different subpaths in $h$.

If we change part c) of the definition of a good path into

\[c') b \neq a_i\]  

for every $i$ satisfying $\text{deg}_{a_i}(h) \leq 2$,

then we obtain the definition of a semi-good subpath $b$ in $h$.

**Definition (of good components).** A subset $I \subset \{v_1, \ldots, v_s\}$ is called a good component with respect to $h$, if the following conditions are valid:

a) For every good subpath $b$ in $h$ we have $b' \in I$ if and only if $b'' \in I$. 

b) There is a good subpath $b$ in $h$ such that $b' \in I$.

c) The set $I$ is a minimal (by inclusion) subset of $\{v_1, \ldots, v_s\}$ that satisfies (a) and (b).

Taking semi-good subpaths instead of good subpaths in the above definition, we obtain the definition of a *semi-good component*.

Let $I_1, \ldots, I_r$ be all good components with respect to $h$. Obviously,

$$\{v_1, \ldots, v_s\} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_r$$

for some $I_0 \subset \{v_1, \ldots, v_s\}$. Formula (6) is called the *decomposition into good components* with respect to $h$ and $I_0$ is called the *null component* with respect to $h$.

In what follows we assume that $\deg_{a_i}(h) = 2$ for all $i$ unless otherwise stated.

**Lemma 4.1.** 1. For every good subpath $b$ in $h$ we have $b' \notin I_0$ and $b'' \notin I_0$.

2. For all $u, w \in I_j$, where $j > 0$, there are pairwise different good subpaths $b_1, \ldots, b_l$ in $h$ such that $b_1 \cdots b_l$ is a closed path in $Q$, $b_1 = u$, and $b_k = w$ for some $k$ with $1 \leq k \leq l$.

**Proof.** Part 1 follows from the definition. Let $G$ be the $h$-restriction of $Q$ to the vertices $v_1, \ldots, v_s$ (see Section 2.1 for the definition). We consider $h$ as a path in $G$ and define $\theta \in \mathbb{N}^{arr(G)}$ as $\theta = \deg(h) - 2 \deg(a)$. Since $\deg_{a_i}(h) = 2$ for all $i$, it is not difficult to see that for the decomposition $\bar{\theta} = \theta^{(1)} + \cdots + \theta^{(r)}$ with respect to $Q_2$, we have $\ver(Q_{\bar{\theta}^{(j)}}) = I_j$ for any $1 \leq j \leq r$. To conclude the proof, we apply Lemma 2.1 to $\bar{\theta}^{(j)}$ and $Q_{\bar{\theta}^{(j)}}$. \hfill $\Box$

**Lemma 4.2.** If $h \neq a^2f$ for all $f \in \path(Q)$, then the number of good components with respect to $h$ is equal or greater than two.

**Proof.** Let $r$ be the number of good components and $\delta = \deg(h)$. If $r = 0$, then

$$\delta_b = \begin{cases} 
0, & \text{if } b \neq a_i \text{ for all } i \\
2, & \text{otherwise}
\end{cases}$$

for $b \in \arr(Q)$. Hence $h \sim a^2$ and we have a contradiction.

Suppose $r = 1$. If $v_i \in I_0$, then $h \sim a_{i-1}a_i f_1 a_{i-1} a_i f_2$ for some paths $f_1, f_2$ that do not contain $a_{i-1}$ and $a_i$. Substitute a new arrow $a_{i+1}$ for the path $a_{i-1}a_i$. Repeat this procedure for all elements of $I_0$. Thus we can assume that $I_0 = \emptyset$ and $I = \{v_1, \ldots, v_s\}$ is the only good component.

If $s = 1$, then Lemma 2.6 implies a contradiction. Otherwise, we consider the $h$-restriction of $Q$ to $v_1, \ldots, v_s$, remove arrows $a_1, \ldots, a_s$ from this restriction, and denote the resulting quiver by $\mathcal{G}$. Let $T$ be a spanning tree for $\mathcal{G}$, i.e.,

a) $\ver(T) = \{v_1, \ldots, v_s\}$ and $\arr(T) \subset \arr(\mathcal{G})$;

b) If we consider $T$ as a graph without orientation, then it is a tree.

Consider a leaf $v_i$ of $T$ together with the unique arrow $b \in \arr(T)$ satisfying $v_i \in \{b', b''\}$. Then the condition of Lemma 2.8 is true and we have $h \equiv a_{i-1}a_i f_1 a_{i-1}a_i f_2$ for some $f_1, f_2 \in \path(Q)$. We remove the vertex $v_i$ and the arrow $b$ from $T$ and denote the resulting quiver by $T_i$. As above, we consider some leaf of $T_i$, apply Lemma 2.8, and so on. Finally, we obtain $h \equiv a f_1 a f_2 \equiv a^2 f_1 f_2$ for some paths $f_1, f_2 \in \path(Q)$; a contradiction. \hfill $\Box$
Lemma 4.3. Suppose \( \{v_1, \ldots, v_s\} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_r \) is the decomposition into good components with respect to \( h \), \( r \geq 2 \), and \( V \subset \{v_1, \ldots, v_s\}\). Let \( b, c, e \) be pairwise different good subpaths in \( h \) such that

a) \( b' \in I_1 \) and \( c', e' \in V \);

b) \( v \in \text{ver}(b) \cap \text{ver}(c) \cap \text{ver}(e) \) for some \( v \).

Then there exists a closed path \( h_0 \) in \( Q \) such that \( h_0 \equiv h \) and

\[
\{v_1, \ldots, v_s\} = I_0 \sqcup T_1 \sqcup \bigcup_{k \in D} I_k \sqcup J_1 \sqcup \cdots \sqcup J_l
\]
is the decomposition into good components with respect to \( h_0 \), where \( l \geq 0 \) and \( D = \{2, \ldots, r\}\) for \( c' \in I_1, c' \in I_j \). Moreover, \( \#T_1 > \#I_1 \).

Proof. We have \( b = b_1b_2, \ c = c_1c_2, \) and \( e = e_1e_2 \) for some paths \( b_i, c_i, e_i \) in \( Q \) \((i = 1, 2)\) with \( b'_i = c'_i = e'_i = v \). There are two possibilities:

1. If \( h \sim b_2f_1c_1 \cdot c_2f_2c_1 \cdot c_2f_3b_1 \) for some \( f_1, f_2, f_3 \in \text{path}(Q) \), then we define \( h_0 = b_2f_1c_1 \cdot e_2f_2b_1 \cdot e_2f_3c_1 \) and we have \( h_0 \equiv h \). Let \( S_0 \) and \( S \) be the sets of good subpaths in \( h_0 \) and \( h \), respectively. Then \( S_0 = (S \cup \{b_1c_2, c_1e_2, e_1b_2\}) \) \( \setminus \{b, c, e\} \).

Clearly, \( I_k \) is a good component with respect to \( h_0 \), where \( 2 \leq k \leq r \) and \( k \neq i, j \), and \( I_0 \) is the null component with respect to \( h_0 \). By part 2 of Lemma 4.1, the set \( I_1 \) and the vertices \( c' \) and \( e'' \) belong to one and the same good component with respect to \( h_0 \), which we denote by \( T_1 \). Thus \( \#T_1 > \#I_1 \) and the claim is proven.

2. If \( h \sim b_2f_1e_1 \cdot e_2f_2c_1 \cdot c_2f_3b_1 \) for some \( f_1, f_2, f_3 \in \text{path}(Q) \), then the proof is analogous.

\[
\square
\]

Lemma 4.4. Suppose \( h \neq a'^2f \) for all \( f \in \text{path}(Q) \). Then there exists a closed path \( h_0 \) in \( Q \) and a good component \( I \) with respect to \( h_0 \) such that \( h_0 \equiv h \) and if good subpaths \( b, c \) in \( h_0 \) and \( v \in \text{ver}(Q) \) satisfy the following condition:

\[
(\text{7}) \quad b' \in I, \ c' \notin I, \ \text{and} \ v \in \text{ver}(b) \cap \text{ver}(c),
\]

then

a) \( b \) is the unique good subpath in \( h_0 \) satisfying (7), i.e., \( h_0 \sim bf_0 \), where \( f_0 \)

b) do not contain a good subpath \( b_1 \) with \( b'_1 \in I \) and \( v \in \text{ver}(b_1) \);

\[
\text{deg}_u(b) = 1.
\]

Proof. The proof consists of two parts. At first we find \( h_0 \) and \( I \) that satisfy condition a), then we change \( h_0 \) to make condition b) valid.

a) For a good component \( I \) with respect to \( h \) and \( V \subset \{v_1, \ldots, v_s\}\), we write \( I > V \) if the condition of Lemma 4.3 does not hold for \( I_1 = I \) and \( V \).

Suppose \( \{v_1, \ldots, v_s\} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_r \) is the decomposition into good components with respect to \( h \). If \( I_1 \nexists I_2 \sqcup \cdots \sqcup I_r \), then Lemma 4.3 implies that there is an \( h^{(0)} \equiv h \) such that \( \#T_1 > \#I_1 \) for a good component \( T_1 \) with respect to \( h^{(0)} \). Repeat this procedure for \( T_1 \) and so on. Finally, we obtain \( h_1 \equiv h \) such that \( I_1, \ldots, I_{r_1} \)

are all good components with respect to \( h_1 \) and \( I_{r_1} > I_{r_1} \sqcup \cdots \sqcup I_{r_1} \). Note that \( r_1 \geq 2 \) by Lemma 4.2.

If \( I_{r_2} \nexists I_{r_1} \sqcup \cdots \sqcup I_{r_3} \), then we act as above; and so on. Finally, we obtain \( h_0 \equiv h \) such that \( I_{r_1}, \ldots, I_{r_1} \)

are all good components with respect to \( h_0 \) and \( I_{r_1} \nexists I_{i+1} \sqcup \cdots \sqcup I_{r_1} \) for any \( 1 \leq i < r_1 \). Then condition a) holds for \( h_0 = h \) and \( I = I_{r_1} \).

b) Consider \( h_0 \) and \( I \) that have been constructed in part a) of the proof. Suppose \( b, c \) are good subpaths with respect to \( h_0 \), \( b' \in I, \ c' \notin I, \) and \( v \in \text{ver}(Q) \). If
\( \deg_v(b) \geq 2 \), then \( b = b_1q_2b_2 \) for some paths \( b_1, q, b_2 \) satisfying \( q' = q'' = v \) and \( \deg_v(b_1b_2) = 0 \). Assume that \( c = c_1c_2 \) for paths \( c_1, c_2 \) with \( c'_1 = c'_2 = v \) and \( h \sim b_1f_2 \). Then

\[
  h_0 \sim b_2f_1c_1 \cdot c_2f_2b_1 \cdot q \equiv c_2f_2b_1 \cdot b_2f_1c_1 \cdot q,
\]

and we define \( h_1 = c_2f_2b_1 \cdot b_2f_1c_1 \cdot q \). Let \( S_1 \) and \( S_0 \) be the sets of good subpaths in \( h_1 \) and \( h_0 \), respectively. Then \( S_1 = (S_0 \cup \{b_1b_2, c_1c_2\}) \setminus \{b, c\} \). It is not difficult to see that every good component with respect to \( h_1 \) is diminished by a positive number. Hence we finally obtain Lemma 5.2.

If conditions a) and b) of the lemma does not hold for \( h_1 \) and some paths \( b \) and \( c \), then we repeat the above procedure for \( h_1 \); and so on. Denote by \( k \) the sum \( \sum \deg b \) that ranges over all \( b \in \arr(Q) \) with \( b' \in I \). After each step of the procedure \( k \) is diminished by a positive number. Hence we finally obtain \( h_0 \) that satisfies conditions a) and b).

Now we assume that \( h \) is a closed path in \( Q \) with \( \deg_{a_i}(h) \geq 2 \) for all \( i \).

**Lemma 4.5.** Suppose \( h \neq a^2f \) for all \( f \in \path(Q) \). Then there exists a closed path \( h_0 \) in \( Q \) and a semi-good component \( I \) with respect to \( h_0 \) such that \( h_0 \equiv h \) and if good subpaths \( b, c \) in \( h_0 \) and \( v \in \ver(Q) \) satisfy (7) then conditions a) and b) of Lemma 4.4 are valid.

**Proof.** Suppose \( h \sim a_1c_1 \cdots a_sc_s \) for some \( 1 \leq i \leq s \), where \( l = \deg_{a_i}(h) \geq 3 \). Then we add a new arrow \( b_1 \) to \( Q \) and define \( b'_1 = a'_1, b''_1 = a''_1 \). Moreover, we substitute \( a_1a_2b_1c_3 \cdots b_sc_s \) for \( h \). After performing this procedure for all \( i \) we obtain a strongly connected quiver \( \mathcal{G} \) and a closed path \( h_1 \) in \( \mathcal{G} \) satisfying \( \deg_{a_i}(h_1) = 2 \) for all \( i \). Lemma 4.4 completes the proof.

## 5. The main upper bound

Suppose \( Q \in \mathcal{Q}(n, d, m) \) is a quiver and \( \text{char}(K) = 2 \). The set \( \Omega_2(Q) \) has been defined in Section 3. This section is dedicated to the proof of the following theorem.

**Theorem 5.1.** If \( h \neq 0 \) is a closed path in \( Q \), then \( \deg(h) \leq m(d - n - 1) + 2n \).

**Lemma 5.2.** Suppose \( h \) is a closed path in \( Q \) such that \( h \neq 0 \) and \( \text{mdeg}(h) \notin \Omega_2(Q) \). Then \( h \equiv cf \), where \( f \) and \( c = c_1b_1c_2b_2c_3 \) are closed paths in \( Q \), paths \( c_1, c_2, c_3 \) can be empty, \( b_1, b_2 \in \arr(Q) \), and the following conditions hold:

a) \( \deg_{c_1}(h) = \deg_{c_2}(h) = 1 \);

b) \( \ver(c_1) \cap \ver(c_2) = \emptyset, \ver(c_2) \cap \ver(c_3) = \emptyset, \) and \( \ver(c_1) \cap \ver(c_3) = c'_3 = v_0 \). Schematically, this condition is depicted as

\[
  v_0 \quad \cdots
\]

c) for all \( v \in \ver(c) \) with \( \deg_v(c) \geq 2 \) we have \( \deg_v(c) = \deg_v(h) \). In particular, \( \deg_{v_0}(c) = 1 \);

d) for all \( v \in \ver(c_2) \) we have \( \deg_v(h) > \deg_v(c) = 1 \).
Proof. Since mdeg(h) \not\in \Omega_2(Q), there is a mdeg(h)-double path h in Q such that mdeg(Q) - 2mdeg(a) is indecomposable. We apply Lemma 4.5 to h and a to obtain a closed path h_0 in Q and a semi-good component I satisfying the conditions from Lemma 4.5. Without loss of generality we can assume that h = h_0. In what follows, all good subpaths in h will be considered with respect to a. Define the subset

V \subset \text{ver}(Q) \setminus \text{ver}(a)

that contains v if and only if there is a good subpath e in h such that e' \not\in I and v \in \text{ver}(e). Since mdeg(Q) - 2mdeg(a) is indecomposable, there is a good subpath b = b_1 \cdots b_l in h satisfying b' \in I and \text{ver}(b) \cap V \not= \emptyset, where b_1, \ldots, b_l \in \text{arr}(Q). Since b'_l \not\in V and b'_l \not\in V, we can define

i = \min \{1 \leq k \leq l \mid b'_k \in V\} and j = \min \{i < k \leq l \mid b'_k \not\in V\}.

Let \text{deg}_b(h) > 1, then there is a good subpath e in h with b_i \in \text{arr}(e) and h \sim b_1 e f_2 for some f_1, f_2 \in \text{path}(Q) because \text{deg}_{b'}(b) = 1 by Lemma 4.5. If e' \in I, then we obtain a contradiction to Lemma 4.5. If e' \not\in I, then b''_l \in V; a contradiction. Therefore, \text{deg}_b(h) = 1 and, similarly, \text{deg}_{b''}(h) = 1.

If \text{ver}(b_1 \cdots b_{i-1}) \cap \text{ver}(b_{i+1} \cdots b_l) \not= \emptyset, then h \sim c f for c = c_1 b_1 e_2 b_3, satisfying part b) of the lemma, where f is a path in Q, e_2 = b_{i+1} \cdots b_{j-1} \in \text{path}(Q), and c_1, c_3 \in \text{path}(Q) are subpaths in b_1 \cdots b_{i-1}, b_{i+1} \cdots b_l, respectively. Moreover, we can assume that \text{deg}_a(c_1) = \text{deg}_v(c_3) = 1.

If \text{ver}(b_1 \cdots b_{i-1}) \cap \text{ver}(b_{i+1} \cdots b_l) = \emptyset, then, taking into account Lemma 2.7, we have h \equiv c f, where c satisfies the same conditions as above and f is a path in Q.

If there is a v \in \text{ver}(c_1) such that \text{deg}_v(h) > \text{deg}_v(c_1) \geq 2, then c_1 = c_1 e_2 e_3, where e_1, e_2, e_3 \in \text{path}(Q), e'_2 = e'_3 = v, and \text{deg}_v(c_1) = \text{deg}_v(c_3) = 1. We have h \sim e_1 e_2 e_3 b_1 c_2 b_3 c_3 \cdot f_1 \cdot \text{deg}_v(c_1) = \text{deg}_v(c_3) = 1. We have h \sim e_1 e_2 e_3 b_1 c_2 b_3 c_3 \cdot f_1 \cdot \text{deg}_v(c_1) = \text{deg}_v(c_3) = 1. We have h \sim e_1 e_2 e_3 b_1 c_2 b_3 c_3 \cdot f_1 \cdot \text{deg}_v(c_1) = \text{deg}_v(c_3) = 1. We have h \sim e_1 e_2 e_3 b_1 c_2 b_3 c_3 \cdot f_1 \cdot \text{deg}_v(c_1) = \text{deg}_v(c_3) = 1. We have}

We assume that h \equiv c f \not= 0 is a closed path from Lemma 5.2, where c = c_1 b_1 c_2 b_2 c_3. Define sets

S_{arr} = \{a \in \text{arr}(c_3 c_1) \mid \text{deg}_a(h) = \text{deg}_a(c)\},

S_{ver} = \{v \in \text{ver}(c_3 c_1) \mid \text{deg}_v(h) = \text{deg}_v(c)\}.

In this section we will use the next remark.

Remark 5.3.

1. Since f is not an empty path, we have v_0 \not\in S_{ver}.
2. For all v \in \text{ver}(c_1), a \in \text{arr}(c_1) the equalities \text{deg}_v(c) = \text{deg}_v(c_1), \text{deg}_a(c) = \text{deg}_a(c_1) hold (i = 1, 2, 3).

Lemma 5.4. For all v \in \text{ver}(c_3 c_1), a \in \text{arr}(c_3 c_1) we have

a) if a' \in S_{ver} or a'' \in S_{ver}, then a \in S_{arr};

b) if a' \not\in S_{ver} or a'' \not\in S_{ver}, then \text{deg}_a(c) = 1;

c) if v \not\in S_{ver}, then \text{deg}_v(c) = 1.
Proof. Part a) is trivial. Part c) follows from part c) of Lemma 5.2. If \( a' \notin S_{\text{ver}} \), then \( \deg_{a'}(c) = 1 \) by part c); hence \( \deg_{a}(c) = 1 \) and part b) is proven. \( \square \)

The following lemma will help us to perform the induction step in the proof of Theorem 5.1.

**Lemma 5.5.** We assume that for every strongly connected quiver \( G \) with \( \# \text{arr}(G) < d \) the assertion of Theorem 5.1 is valid. Then \( \deg(c) \leq m(\# S_{\text{arr}} - \# S_{\text{ver}} + 1) + 2\# S_{\text{ver}} \).

**Proof.** Let \( \# S_{\text{ver}} = \emptyset \). Then \( c \) is a primitive closed path by parts c) and d) of Lemma 5.2. Hence \( \deg(c) \leq m \).

Let \( \# S_{\text{ver}} \geq 1 \). Using the fact that \( v_0 \notin S_{\text{ver}} \) and part b) of Lemma 5.2 we obtain

\[
S_{\text{ver}} = S_{\text{ver}}^1 \cup S_{\text{ver}}^3 \quad \text{and} \quad S_{\text{arr}} = S_{\text{arr}}^1 \cup S_{\text{arr}}^3
\]

for \( S_{\text{ver}}^1 = S_{\text{arr}} \cap \text{arr}(c_1), S_{\text{arr}}^3 = S_{\text{arr}} \cap \text{arr}(c_3), S_{\text{ver}}^1 = S_{\text{ver}} \cap \text{ver}(c_1) \), and \( S_{\text{ver}}^3 = S_{\text{ver}} \cap \text{ver}(c_3) \).

Suppose \( \# S_{\text{ver}}^1 \geq 1 \). We assume that \( c_1 = x_1 \cdots x_s \in \text{arr}(Q) \), where for \( i \neq j \) we have \( x_i = x_j \). We define \( p = \min\{1 \leq k \leq s | x_k \in S_{\text{ver}}^1 \} \) and \( q = \max\{1 \leq k \leq s | x_k \in S_{\text{ver}}^1 \} \). Then \( c_1 = e_1 e_2 e_3 \), where paths \( e_1 = x_1 \cdots x_p, e_2 = x_{p+1} \cdots x_q, \) and \( e_3 = x_{q+1} \cdots x_s \) can be empty. We claim that

\[
\deg(e_2) \leq m(\# S_{\text{arr}}^1 - \# S_{\text{arr}}^3) + 2\# S_{\text{ver}}^1.
\]

To prove the claim we consider the \( e_2 \)-restriction of \( Q \) to \( S_{\text{ver}}^1 \), add a new arrow \( z \) from \( e_2' \) to \( e_2'' \), and denote the resulting quiver by \( G \). In other words, \( \text{ver}(G) = S_{\text{ver}}^1 \) and \( a \in \text{arr}(G) \) has one of the following types:

1. \( a = \tilde x_i \), where \( 1 \leq i \leq s \) and \( x_i, x_i'' \in S_{\text{ver}}^1 \);
2. \( a = x_i \cdots x_j \) for \( 1 \leq i < j \leq s \), where \( x_i'', x_j' \in S_{\text{ver}}^1 \) and \( x_i', \ldots, x_{j-1}'' \notin S_{\text{ver}} ;
3. \( a = z \).

Note that for an arrow \( a = \tilde x_i \) of type 1 we have \( x_i \in S_{\text{arr}} \) by part a) of Lemma 5.4 and we say that \( x_i \) is assigned to \( a \). Similarly, for an arrow \( a = x_i \cdots x_j \) of type 2 we have \( x_i, x_j \in S_{\text{arr}} \) and we say that \( x_i, x_j \) are assigned to \( a \); moreover,

a) \( \deg_{x_k}(e_2) = 1 \) for any \( i \leq k \leq j \) (see part b) of Lemma 5.4).

b) \( \deg_{x_k'}(e_2) = \deg_{x_k'}(e_2) = 1 \) for any \( i \leq k \leq j - 1 \) (see part c) of Lemma 5.4).

In particular, \( x_i \cdots x_j \) is either a primitive closed path or it is a subpath of \( c \) without self-intersections; thus, \( \deg(x_i \cdots x_j) \leq m \).

Let \( y \) be the unique path in \( G \) that corresponds to the path \( e_2 \) in \( Q \). The quiver \( G \) is strongly connected, since \( yz \) is a closed path in \( G \) that contains all arrows and all vertices of \( G \). Moreover, we have \( yz \neq 0 \), since \( \deg_y(y) = 1 \) for every arrow \( a \) of type 2, \( \deg_y(y) = 0 \), and \( b \neq 0 \).

For every arrow \( a \) of type 1 there is an arrow from \( S_{\text{arr}}^1 \) that is assigned to \( a \); and for every arrow \( b \) of type 2 there are two arrows from \( S_{\text{arr}}^1 \) that are assigned to \( b \). But the arrow \( x_p \in S_{\text{arr}}^1 \) is not assigned to any arrow of \( G \). Therefore,

\[
\# \text{arr}(G) - 1 \leq \# S_{\text{arr}} - l - 1,
\]

where \( l \) stands for the number of arrows of type 2. Since \( b_1, b_2 \notin S_{\text{arr}}^1 \), it follows that \( \# \text{arr}(G) \leq \# S_{\text{arr}} ^1 < \text{arr}(Q) = d \). Applying Theorem 5.1 to \( G \), we obtain

\[
\deg(yz) \leq m(G)(\# \text{arr}(G) - \# \text{ver}(G)) + 2\# \text{ver}(G) .
\]
It is not difficult to see that $m(G) \leq m$. Thus
\[
\text{deg}(y) \leq m(#S_{arr}^1 - #S_{ver}^1 - l) + 2#S_{ver}^1 - 1.
\]
By property b) of paths of type 2, we have
\[
\text{deg}(e_2) \leq \text{deg}(y) + l(m - 1).
\]
The last two formulas conclude the proof of (9).

If $#S_{ver}^3 \geq 1$, then we rewrite $c_3$ in a form $c_3 = g_1g_2g_3$ in the same way as we have done for $c_1 = e_1e_2e_3$. Then the proof of the formula
\[
(10) \quad \text{deg}(g_2) \leq m(#S_{arr}^3 - #S_{ver}^3) + 2#S_{ver}^3
\]
is similar to the proof of (9).

Suppose $S_{ver}^1 \neq \emptyset$ and $S_{ver}^3 \neq \emptyset$. Then
\[
\text{deg}(c) = \text{deg}(c_1c_2b_2c_3) = \text{deg}(e_2) + \text{deg}(g_2) + \text{deg}(f_1) + \text{deg}(f_2),
\]
where $f_1 = g_3e_1$ and $f_2 = e_3b_1c_2b_2g_1$. Parts c) and d) of Lemma 5.2 imply that
\[
\begin{align*}
&\text{a) for every } v \in (\text{ver}(f_1) \cup \text{ver}(f_2)) \setminus \{f'_1, f''_1, f'_2, f''_2\} \text{ we have } \text{deg}_v(c) = 1; \\
&\text{b) } (\text{ver}(f_1) \cup \text{ver}(f_2)) \cap (\text{ver}(e_2) \cup \text{ver}(g_2)) = \{f'_1, f''_1, f'_2, f''_2\}.
\end{align*}
\]
It follows that there are paths $d_1, d_2$ in $Q$ such that $f_1d_1f_2d_2$ is a primitive closed path in $Q$. In particular, $\text{deg}(f_1) + \text{deg}(f_2) \leq m$. Formulas (9) and (10) conclude the proof of the lemma.

The cases $S_{ver}^1 \neq \emptyset, S_{ver}^3 = \emptyset$ and $S_{ver}^1 = \emptyset, S_{ver}^3 \neq \emptyset$ can be treated in the similar fashion. If $S_{ver}^3 = \emptyset$ and $S_{ver}^3 = \emptyset$, then $S_{ver} = \emptyset$; a contradiction.

\begin{proof}
Suppose $Q_{m\text{deg}(h)} \in Q(n_0, d_0, m_0)$ for some $n_0, d_0, m_0$. We assume that the theorem is proven for the case $Q = Q_{m\text{deg}(h)}$. Then we have $\text{deg}(h) \leq m_0(d_0 - n_0 - 1) + 2n_0$. Lemma 2.3 implies
\[
\text{deg}(h) \leq m(d - n - 1) + 2n - m_0
\]
and we obtain the required upper bound. Therefore, without loss of generality we can assume that $Q = Q_{m\text{deg}(h)}$.

We prove the theorem by induction on $\# \text{arr}(Q)$.

\textbf{Induction base.} If $\# \text{arr}(Q) = 1$, then $\text{ver}(Q) = \{v\}$ and the only arrow of $Q$ is a loop in $v$. Then $\text{deg}(h) = 1$ and the required upper bound on $\text{deg}(h)$ holds.

\textbf{Induction step.} If $m\text{deg} \in \Omega_2(Q)$, then see Theorem 3.9; otherwise we apply Lemma 5.2 to $h$ and obtain $h \equiv cf$, where $f$ and $c = c_1c_2b_2c_3$ are closed paths in some vertex $v_0$. By Lemma 5.5, we have
\[
(11) \quad \text{deg}(c) \leq m(#S_{arr} - #S_{ver} + 1) + 2#S_{ver}.
\]
We define the quiver $G$ by $\text{ver}(G) = \{v \in \text{ver}(Q) \mid \text{deg}_v(h) > \text{deg}_v(c)\}$ and $\text{arr}(G) = \{a \in \text{arr}(Q) \mid \text{deg}_a(h) > \text{deg}_a(c)\} \cup \{x\}$, where $x$ is a new loop in the vertex $v_0$. Then $xf$ is a closed path in $G$ that contains all vertices and arrows of $G$. In particular, $G$ is a strongly connected quiver and $G = G_{m\text{deg}(xf)}$. Since $cf \not\equiv 0$, we have $xf \not\equiv 0$. By parts a) and d) of Lemma 5.2, we have $\# \text{arr}(G) \leq d - #S_{arr} - 1$ and $\text{ver}(G) = n - #S_{ver}$. Applying induction hypothesis to the closed path $xf$ in $G$ and using the inequalities $\text{arr}(G) > \text{ver}(G)$ and $m(G) \leq m$, we obtain
\[
\text{deg}(xf) \leq m(d - n - 1) + 2n - m(#S_{arr} - #S_{ver} + 1) - 2#S_{ver}.
\]
Formula (11) implies the required upper bound on $\text{deg}(h)$.
\end{proof}
6. The upper bound for the case of small $d$

Let $\text{char}(K) = 2$. The following lemma is a stronger version of Lemma 2.9.

**Lemma 6.1.** Suppose $Q \in Q(n, d, m)$. Then using the notation of Lemma 2.9 we have $r \geq 1$.

**Proof.** Suppose $r = 0$. Then $m \deg(h) = 2 \sum_{j=1}^{t} m \deg(c_j)$, where $t \geq 1$, and we have two possibilities.

1. Let $m \deg(h) \not\in \Omega_2(Q)$. Then there exists a primitive closed path $a = a_1 \cdots a_s$ in $Q (a_1, \ldots, a_s \in \text{arr}(Q))$ such that $m \deg(h) - 2m \deg(a)$ is indecomposable and $\deg_{a_i}(h) \geq 2$ for all $i$. It is not difficult to see that $\text{ver}(a) = I \sqcup J$, where

\[\deg_v(h) = 2\text{ for all } v \in I;\]
\[\text{for every } u, v \in J \text{ with } u \neq v \text{ there is a path } g \text{ in } Q \text{ from } u \text{ to } v;\]
\[\text{for every } e \in \text{arr}(g) \text{ we have } \deg_{e}(h) \geq 2, \text{ if } e \not\in \text{arr}(a); \text{ and } \deg_{e}(h) \geq 4, \text{ if } e \in \text{arr}(a).\]

Lemma 2.7 implies that $h \equiv gf$ for some path $f$.

If $s > 1$, then, using Lemma 2.8, we have $h \equiv a_1a_2f_1a_1a_2f_2 \equiv a_1a_2a_3f_3a_1a_2a_3f_4 \equiv \cdots \equiv a_{2s-3}a_{2s-2}$ for some paths $f_1, \ldots, f_{2s-2}$. Lemma 2.6 gives $h \equiv 0$ for $s \geq 1$; a contradiction.

2. If $m \deg(h) \in \Omega_2(Q)$, then we consider a $m \deg(h)$-tree $(T, \delta^{(v)}, A_v | v \in \text{ver}(T))$ constructed in Section 3. For a leaf $v \in \text{ver}(T)$ we have $\delta^{(v)} \in \Omega_3(Q_{\delta^{(v)}})$; a contradiction. \hfill \square

**Theorem 6.2.** Suppose $Q \in Q(n, d, m)$, $h$ is a closed path in $Q$, and $h \not\equiv 0$. Then $\deg(h) \leq 2m(d - n) + m$.

**Proof.** Using the notation of Lemma 2.9 we have $\deg h \leq m(r + 2t)$ and $r + t \leq d - n + 1$. Lemma 6.1 implies

\[r + 2t \leq 2r - 1 + 2t \leq 2(d - n) + 1\]

and we obtain the required upper bound. \hfill \square

7. Examples

**Lemma 7.1.** Suppose $Q(n, d, m) \neq \emptyset$. Then there is a $Q \in Q(n, d, m)$ and a closed path $h$ in $Q$ such that $h \not\equiv 0$ and

1. $\deg(h) \geq M(n, d, m) - m$, if $\text{char}(K) = 2$;
2. $\deg(h) = M(n, d, m)$, if $\text{char}(K) \not= 2$, $d \geq \lfloor n - 2 \rfloor + m$ or $n = m$;

where the definition of $M(n, d, m)$ was given in Section 1.

**Proof.** Suppose $\text{char}(K) = 2$.

a) If $m = 1$, then $n = 1$. For the quiver $Q$ with one vertex $v$ and loops $a_1, \ldots, a_d$ in $v$ we have $h = a_1 \cdots a_d \not\equiv 0$ and $\deg(h) = d$. 


b) If $m \geq 2$ and $n = m$, then we consider the quiver $Q \in Q(n, d, m)$:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics{quiver1.png}
\end{array}
\end{array}
\]

where $t = d - n + 1 \geq 1$. For $h = a_1 b \cdots a_t b$, where $b = b_1 \cdots b_{n-1}$, we have $\deg(h) = tn$ and $h \neq 0$.

c) We assume that $d \geq n + 2 \left\lceil \frac{n-1}{m} \right\rceil$ and $n > m \geq 2$. Then $n - 1 = lm + r$ for $l = \left\lfloor \frac{n-1}{m} \right\rfloor \geq 1$ and $0 \leq r \leq m - 1$. Consider the quiver $Q \in Q(n, d, m)$:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics{quiver2.png}
\end{array}
\end{array}
\]

where there are $t = d - n - 2l + 1 \geq 1$ arrows from $u$ to $v$, the right primitive closed path contains $r + 1$ arrows, any other primitive closed path contains $m$ arrows, and $s = t + 2$. Define $\delta \in \Omega_1(Q)$ in such a way that if a number $k$ is assigned to an arrow $a \in \text{arr}(Q)$, then $\delta_a = k$. Since $\delta \in \Omega_2(Q)$, there is a closed path $h$ in $Q$ with $\text{mdeg}(h) = \delta$ and $h \neq 0$ by Remark 3.2. It is not difficult to see that $\deg(h) = |\delta| = m(d - n - 1) + 2n - (r + 1)$.

d) We assume that $d < n + 2 \left\lceil \frac{n-1}{m} \right\rceil$ and $n > m \geq 2$. As above, we have $n - 1 = lm + r$ for $l \geq 1$ and $0 \leq r \leq m - 1$. Consider the quiver $Q \in Q(n, d, m)$:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics{quiver3.png}
\end{array}
\end{array}
\]

where every primitive closed path contains $m$ arrows, $i, j \geq 0$, $1 \leq t < m$, and

\[
\begin{aligned}
n &= m(i + j + 2) - j - t, \\
d &= m(i + j + 2) + 2i - t + 1.
\end{aligned}
\]

It is not difficult to see that there exist $i, j, t$ satisfying the given conditions. We define $\delta \in \Omega_2(Q)$ in a similar way as in part c). Hence $|\delta| = 2m(2i + j + 1)$ and $M(n, d, m) - |\delta| = m$.

e) Suppose $\text{char}(K) \neq 2$ and the condition from part 2) of the lemma holds.

If $m = 1$, then we construct the required $h$ similarly to part a).

If $n = m \geq 2$, then we consider the quiver from part b). We set $h = a_1 b a_1 b$ if $d \in \{n, n+1\}$ and $h = a_1 b a_2 b a_3 b$ if $d > n + 1$. Obviously, $\deg(h) = M(n, d, m)$ and $h \neq 0$. 

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics{quiver4.png}
\end{array}
\end{array}
\]
Let \( n > m \geq 2 \). We define \( l \) and \( r \) in the same way as in part c) and consider the quiver \( Q \in Q(n, d, m) \):

Here we assume that we have not depicted some loops in \( Q \). Namely, for \( s = d - n - 2l - r - 1 \geq 0 \) there are loops \( a, b_1, \ldots, b_s \) in the vertex \( u \) and loops \( c_1, \ldots, c_r \) in vertices \( v_1, \ldots, v_r \), respectively. We assign number 1 to loops \( a, c_1, \ldots, c_r \) and number 0 to \( b_1, \ldots, b_s \). Define \( \delta \in \Omega_1(Q) \) in such a way that if a number \( k \) is assigned to an arrow \( x \in \text{arr}(Q) \), then \( \delta_x = k \). Let \( h \) be a closed path in \( Q \) with \( \text{mdeg}(h) = \delta \). Since \( \text{deg}_w(h) = 3 \) for all \( w \in \text{ver}(Q) \), we have \( \text{deg}(h) = 3n \). Lemma 7.2 (see below) completes the proof.

Given a closed path \( a = a_1 \cdots a_s \) in \( Q \), where \( a_i \in \text{arr}(Q) \), we write \( \text{tr}(X_a) \) for \( \text{tr}(X_{a_1} \cdots X_{a_s}) \).

**Lemma 7.2.** Using notation from part e) of the proof of Lemma 7.1, we have \( h \neq 0 \).

**Proof.** Since the construction of \( Q \) and \( h \) depend on \( l \), we write \( Q_l \) for \( Q \) and \( h_l \) for \( h \) (\( l \geq 1 \)).

Assume that \( h_l \equiv 0 \). By Lemma 1.4, \( \text{tr}(X_{h_l}) \equiv 0 \). Denote \( I = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \) and \( J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \). We set \( X_a = I \), \( X_e = J \), and \( X_g = E \) for every arrow \( g \notin \{a, e, f\} \) from the left rhombus of \( Q_l \). Since \( \text{tr}(I) = \text{tr}(J) = \text{tr}(IJ) = 0 \), it is not difficult to see that \( \text{tr}(X_{h_{l-1}}) \equiv 0 \) in \( I(Q_{l-1}, (2, \ldots, 2)) \), where \( h_0 \) is defined below. Repeating this procedure, we obtain that \( \text{tr}(X_{h_0}) \equiv 0 \) in \( I(Q_0, (2, \ldots, 2)) \) for

\[
\mathcal{h}_0 = x_1 y_1 \cdots x_{r+1} y_{r+1} \cdot x_1 \cdots x_{r+1},
\]

where \( x_1, \ldots, x_{r+1} \in \text{arr}(Q_0) \), \( x_1 \cdots x_{r+1} \) is a closed primitive path in \( Q_0 \), \( y_i \) is a loop in \( x'_i \) (\( 1 \leq i \leq r + 1 \)). For \( j = 1, 2 \) we denote

\[
z_{ij} = \begin{cases} y_i, & \text{if } j = 1 \\ x'_i, & \text{otherwise} \end{cases}
\]

Since for all \( \pi_1, \ldots, \pi_{r+1} \in S_2 \)

\[
x_1 z_{1, \pi_1(1)} \cdots x_{r+1} z_{r+1, \pi_{r+1}(1)} \cdot x_1 z_{1, \pi_1(2)} \cdots x_{r+1} z_{r+1, \pi_{r+1}(2)} \equiv
\]

\[
\equiv \text{sgn}(\pi_1) \cdots \text{sgn}(\pi_{r+1}) h_0,
\]

we obtain that \( h_0 \neq 0 \). Lemma 1.4 implies a contradiction. \( \square \)
References


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