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THE INCREASING SMOOTHNESS PROPERTY OF SOLUTIONS TO SOME HYPERBOLIC PROBLEMS IN TWO INDEPENDENT VARIABLES

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Abstract. The initial-boundary problems for first-order hyperbolic systems and for the wave equation are considered in the half-strip $\Pi = \{(x,t) : 0 < x < 1,\ t > 0\}$. Boundary conditions which guarantee the increasing of smoothness of the solutions to the considered problems as t grows are formulated.

Keywords: first-order hyperbolic systems on the plane, wave equation, initial-boundary problems, increasing smoothness of the solutions.

Dedicated to the memory of professor T.I.Zelenyak

1. Introduction

Mixed problems for hyperbolic systems in two independent variables arise in the mathematical modeling of physical and chemical processes in connection with the phenomena of warm-and mass-transfer. An extensive literature (see the references in [1]–[6]) is devoted to studying the qualitative properties of solutions to these problems (existence of global (in t) solutions, existence of periodic solutions, propa gation of discontinuities, bifurcation of solutions, stability of solutions, and so on). This article is a survey of the results established by the authors, concerning the increasing smoothness properties of solutions to some mixed problems for hyperbolic systems in two variables x, t. Full proofs of the results stated below one can found in the references to this article.

Lyulko N.A. The increasing smoothness property of solutions to some hyperbolic problems in two independent variables.

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It is known that the smoothness of a solution to the Cauchy problem for the simplest hyperbolic system of two equations with constant coefficients

$$u_t + u_x = au + bv, \quad v_t - v_x = cu + dv,$$

$$u|_{t=0} = u_0(x), \quad v|_{t=0} = v_0(x)$$

is not higher than the smoothness of the initial data u_0 and v_0. If the point $x = x_0$ is the jump discontinuity of the initial data u_0 and v_0, this discontinuity will persist for function $u(x,t)$ along the characteristic $t = x - x_0$ and for the function $v(x,t)$ along the characteristic $t = -x + x_0$ of system, starting at that point. This discontinuity will exist for any period of time t. A similar situation takes place for the Cauchy problem for the wave equation

$$u_{tt} - u_{xx} = 0, \quad u|_{t=0} = u_0(x),\, u_t|_{t=0} = u_1(x).$$

The discontinuity of the initial data u_0 and u_1 at $x = x_0$ for the solution $u(x,t)$ propagates along both mentioned characteristics simultaneously.

For initial-boundary value problems in the half-strip $\Pi = \{(x,t) : 0 < x < 1, t > 0\}$ it is possible to set boundary conditions on lateral sides of Π such that the smoothness of solutions to the corresponding boundary problems increases as t grows. Here we give an example, which belongs to T.I.Zelenyak, that illustrate the increasing smoothness of solutions to the hyperbolic problem

$$u_t + u_x = 2v, \qquad v_t - v_x = 0, \quad (x,t) \in \Pi, \tag{1}$$

$$u|_{x=0} = \alpha v|_{x=0}, \qquad v|_{x=1} = \beta u|_{x=1}, \tag{2}$$

$$u|_{t=0} = u_0(x), \qquad v|_{t=0} = \frac{dv_0(x)}{dx}, \tag{3}$$

where $u_0(x)$, $\frac{dv_0(x)}{dx} \in C^1[0,1]$; α, β are constants.

The general solution to the system (1) has the form

$$u(x,t) = \varphi(t+x) + \psi(t-x), \qquad v(x,t) = \frac{d}{d\xi}\varphi(\xi)|_{\xi=t+x}, \tag{4}$$

where $\varphi(\xi) \in C^2[0,\infty)$, $\psi(\xi) \in C^1[-1,\infty)$ are arbitrary functions.

We consider the case $\alpha\beta = 0$. Let $\alpha = 0$, $\beta = 1$. Substituting (4) into (2), we obtain the relations for $\xi \geq 0$

$$\frac{d}{d\xi}\varphi(\xi+1) = \varphi(\xi+1) + \psi(\xi-1), \quad \varphi(\xi) + \psi(\xi) = 0. \tag{5}$$

From (3) we have for $0 \leq \xi \leq 1$

$$\varphi(\xi) = v_0(\xi), \quad \psi(-\xi) = u_0(\xi) - v_0(\xi). \tag{6}$$

One can see from (5) that the smoothness of functions $\varphi(\xi)$ and $\psi(\xi)$ increases by 1 over every interval with respect to ξ of the length 2. By (4), for solutions to the problem (1)-(3) the smoothness increases in t over intervals of the length 2.

The case $\alpha \neq 0$, $\beta = 0$ is treated similarly. For $\alpha = \beta = 0$, obviously, $v(x,t) \equiv 0$ for $t + x > 1$ and $u(x,t) \equiv 0$ for $t - x > 1$.

Let $\alpha = \beta = 1$, then from boundary conditions (2), by virtue of (4), we have the relations for $\xi \geq -1$

$$\psi(\xi+2) = \psi(\xi), \qquad \frac{d}{d\xi}\varphi(\xi+2) - \varphi(\xi+2) = \psi(\xi),$$

which imply that the function $\psi(\xi)$ is 2-periodic. From (4) and (6) we obtain that the smoothness of solutions to (1)-(3) is not higher than the smoothness of the initial data.

So, the above example shows that in the case of mixed problem (1)-(3) the increasing of the smoothness of solutions $u(x,t)$, $v(x,t)$ up to any order k, $k \in N$, takes place, if the coefficients in boundary conditions (2) satisfy $\alpha\beta = 0$.

In this work we consider the well-posed initial-boundary value problems for first-order hyperbolic systems and for the wave equation in the half-strip $\Pi = \{(x,t) : 0 < x < 1, t > 0\}$. We formulate the boundary conditions which guarantee the increasing smoothness property of the solutions to the considered problems as t grows.

Henceforth, the containment of $F(x,t)$ in $C^{k,m}_{x,t}(\bar{\Pi})$ is understood as follows: $F(x,t) \in C^{k,m}_{x,t}(\bar{\Pi}_T)$ for any $T > 0$, where $\bar{\Pi}_T = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$. We denote by K and A the constants depending on the coefficients of problem and independent of t and $U_0(x)$.

2. Initial-boundary value problems for hyperbolic systems

In the half-strip $\Pi = \{(x,t) : 0 < x < 1, t > 0\}$ we consider the following problem:

$$U_t - L_{\mathcal{A}}U = F(x,t), \quad (x,t) \in \Pi, \tag{7}$$

$$I_0U(0,t) + I_1U(1,t) = 0, \qquad U(x,0) = U_0(x). \tag{8}$$

Here $U(x,t) = [u_1(x,t), ..., u_n(x,t)]^T$ is an n-dimensional vector of unknown functions, $F(x,t)$ is the n-dimensional vector of right hand part,

$$L_{\mathcal{A}}U = -\mathcal{K}(x)U_x + \mathcal{A}(x)U, \quad \mathcal{A}(x) = (a_{ij}(x))_{i,j=1,\dots,n},$$

$\mathcal{K}(x)$ is the diagonal matrix with entries $k_i(x) \ne k_j(x)$ $(i \ne j)$, the first p of them are positive, and the rest $n-p$ of them are negative, moreover $1 \le p < n$, $n \ge 2$.

The boundary conditions are splitted, i.e. the matrices I_0 and I_1 have the form

$$I_0 = \begin{pmatrix} 1 & . & 0 & \alpha_{1,p+1} & . & \alpha_{1,n} \\ . & . & . & . & . & . \\ 0 & . & 1 & \alpha_{p,p+1} & . & \alpha_{p,n} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{p+1,1} & . & \beta_{p+1,p} & 1 & . & 0 \\ . & . & . & . & . & . \\ \beta_{n,1} & . & \beta_{n,p} & 0 & . & 1 \end{pmatrix}.$$

If $\mathcal{K}(x) \in C^1[0,1]$, then through each point $(x_0,t_0) \in \bar{\Pi}$ pass n various characteristics $x = \varphi_i(t; x_0, t_0)$ of system (7) defined by the equations

$$\frac{d\varphi_i}{dt} = k_i(\varphi_i), \quad \varphi_i|_{t=t_0} = x_0, \qquad i = 1, ..., n.$$

Following [1], [7], we denote $\chi_i(x_0,t_0) = \inf\{t : (\varphi_i(t; x_0,t_0), t) \in \bar{\Pi}\}$, then, obviously, $0 \le \chi_i(x_0,t_0) \le t_0$ and if $\chi_i(x_0,t_0) > 0$, then $\varphi_i(\chi_i(x_0,t_0); x_0,t_0)$ will be equal 0 or 1. We consider the following sets for any i, $1 \le i \le n$:

$$\Pi_i = \{(x,t) \in \bar{\Pi} : \chi_i(x,t) = 0\},$$

$$\Pi_i^0 = \{(x,t) \in \bar{\Pi} : \chi_i(x,t) > 0 \quad \text{and} \quad \varphi_i(\chi_i(x,t); x,t) = 0\},$$

$$\Pi_i^1 = \{(x,t) \in \bar{\Pi} : \chi_i(x,t) > 0 \quad \text{and} \quad \varphi_i(\chi_i(x,t); x,t) = 1\}.$$

Obviously, $\Pi_i^1 = \emptyset$, $i = p+1, ..., n$, and $\Pi_i^0 = \emptyset$, $i = 1, ..., p$.

We integrate the i-th equation of system (7) along the corresponding characteristic $x = \varphi_i(t; x_0, t_0)$. From (8) we obtain the following system of integral equations:

$$u_i(x_0, t_0) = \int_{\chi_i(x_0,t_0)}^{t_0} (\sum_{j=1}^{n} a_{ij}(x) u_j(x,t) + f_i(x,t))\Big|_{t=\tau,\, x=\varphi_i(\tau;x_0,t_0)} d\tau + v_i(x_0, t_0),$$

$$(x_0, t_0) \in \Pi, \qquad i = 1, ..., n, \tag{9}$$

$$v_i(x,t) = \begin{cases} u_{0i}(\varphi_i(0; x,t)), & (x,t) \in \Pi_i, \quad 1 \le i \le n; \\ -\sum\limits_{j=p+1}^{n} \alpha_{ij} u_j(0, \chi_i(x,t)), & (x,t) \in \Pi_i^0, \quad 1 \le i \le p; \\ -\sum\limits_{j=1}^{p} \beta_{ij} u_j(1, \chi_i(x,t)), & (x,t) \in \Pi_i^1, \quad p+1 \le i \le n, \end{cases}$$

$U_0 = [u_{01}, ..., u_{0n}]^T$.

In [1] the question of well-posedness of problem (7), (8) in spaces of continuous, continuously differentiable and integrable functions is investigated. Let $\mathcal{A}(x)$, $\mathcal{K}(x)$, $U_0(x) \in C^1[0,1]$, $F(x,t) \in C^{1,0}_{x,t}(\bar{\Pi})$, then one can see that the continuously differentiable function $U(x,t)$ is a classical solution of the problem if and only if it is a solution of integral system (9). A necessary condition for the continuity of a solution in the half-strip Π is the fulfillment of the zero-order compatibility conditions

$$I_0 U_0(0) + I_1 U_0(1) = 0, \tag{10}$$

and a necessary condition for the differentiability of a solution is the fulfillment of the zero-order compatibility conditions (10) and the first-order compatibility conditions

$$I_0 U_1(0) + I_1 U_1(1) = 0, \quad \text{where} \quad U_1(x) = L_{\mathcal{A}} U_0(x) + F(x,0). \tag{11}$$

The sequential approximation method applied to the system (9) is proved in [1]

Theorem 1. *Let $\mathcal{A}(x)$, $\mathcal{K}(x) \in C^1[0,1]$, $F(x,t) \in C^{1,0}_{x,t}(\bar{\Pi})$ and the initial data $U_0(x) \in C^1[0,1]$ satisfies the compatibility conditions* (10), (11). *Then problem* (7), (8) *has a unique continuously differentiable solution $U(x,t)$ in the half-strip* Π*; moreover, for $t > 0$ it satisfies the estimate*

$$||U(x,t)||_{C^1[0,1]} \le K e^{At} (||U_0||_{C^1[0,1]} + \max_{0 \le \tau \le t} ||F(x,\tau)||_{C^1[0,1]}).$$

If $\mathcal{K}(x) \in C^1[0,1]$, $\mathcal{A}(x) \in C[0,1]$, $F(x,t) \in C(\bar{\Pi})$ and the initial data $U_0 \in C[0,1]$ satisfies the zero-order compatibility conditions (10), then in [1] the existence in Π of continuous solution of integral system (9) is proved. This solution is called a continuous generalized solution of problem (7), (8).

In [7] the notion of piecewise smooth solution (PSS) of problem (7), (8) was introduced by the author. This solution is a solution of integral system (9) if $\mathcal{A}(x), U_0(x)$, $\mathcal{K}(x) \in C^1[0,1]$, $F(x,t) \in C^{1,0}_{x,t}(\bar{\Pi})$. If the compatibility conditions (10), (11) are not fulfilled, then PSS $U(x,t)$ of problem and its derivatives are discontinuous on the set of characteristics of system (7) which is at most a countable. At the points where the derivatives exist, the function $U(x,t)$ satisfies system (7). If the compatibility conditions (10), (11) are fulfilled, then the piecewise smooth solution is a classical solution of problem (7), (8).

Definition 1. *We say that the problem* (7), (8) *possesses the increasing smoothness property up to the order k if there exists a number $T(k) > 0$ such that every PSS $U(x,t)$ to the problem* (7), (8) *is k times continuously differentiable for $t > T(k)$.*

In [7] a class of the so-called B-regular boundary conditions was described. The mentioned conditions are necessary and sufficient for the linear homogeneous problem (7), (8) to possess the increasing smoothness property up to the order k, which is determined by the smoothness of the matrices $\mathcal{K}(x)$ and $\mathcal{A}(x)$ and does not depend on the smoothness of the initial data.

We consider the diagonal matrix

$$\Upsilon(x,\lambda) = (e^{\lambda\Upsilon_j(x)+\mathcal{B}_j(x)}\delta_{ij})_{i,j=1,\dots,n},$$

where $\Upsilon_j(x) = \int\limits_0^x \frac{-1}{k_j(\xi)}\,d\xi$, $\mathcal{B}_j(x) = \int\limits_0^x \frac{a_{jj}(\xi)}{k_j(\xi)}\,d\xi$, δ_{ij} is the Kronecker symbol, λ is a complex parameter, and introduce the expression

$$X(\lambda) = I_0 + I_1\Upsilon(1,\lambda). \tag{12}$$

It is well known that $\det X(\lambda)=e^{\lambda\sum\limits_{i=p+1}^{n}\Upsilon_i+\sum\limits_{i=p+1}^{n}\mathcal{B}_i}\Delta(\lambda)$, where

$$\Upsilon_i = \Upsilon_i(1), \quad \mathcal{B}_i = \mathcal{B}_i(1), \quad i = 1,\dots,n, \tag{13}$$

$$\Delta(\lambda) = 1 + \sum_{k=1}^{M} E_k e^{-\lambda\beta_k}. \tag{14}$$

Here E_k are real numbers determined by entries of the matrices I_0 and I_1 and numbers $\mathcal{B}_i$; numbers $0 < \beta_1 < \dots < \beta_M$ are determined via Υ_i; $i = 1,\dots,n$.

Definition 2. *Boundary conditions* (8) *for the problem* (7) *are called B—regular if E_k=0 (k=1, ..., M) in* (14), *i.e.* $\Delta(\lambda) \equiv 1$.

The following theorem is proved in [7]:

Theorem 2. *Let $F(x,t) \equiv 0$, $\mathcal{A}(x)$, $\mathcal{K}(x) \in C^{k+2}[0,1]$. Then B–regularity of boundary conditions* (8) *is the necessary and sufficient condition for the problem* (7), (8) *to possess the increasing smoothness property of PSS $U(x,t)$ up to the order k for arbitrary initial data $U_0(x) \in C^1[0,1]$; $k = 0,1,\dots$. The solution $U(x,t)$ for $t > T(k)$ satisfies the estimate*

$$\|D^{\alpha,\beta}_{x,t}U(x,t)\|_{C[0,1]} \le K(t)\|U_0(x)\|_{L_2[0,1]}, \tag{15}$$

where $\alpha + \beta \le k$, the constant $K(t)$ is independent of the initial data and depends on the coefficients of the problem and t.

► We will describe the main steps of the proof, based on a study for $|\lambda| \to \infty$ asymptotic properties of function $\widetilde{U}(x,\lambda)$, which is the Laplace transform of the original problem. A similar approach to problems with splitted boundary conditions was used earlier in [8], [9]. We apply the Laplace transform by t to the system (7), (8) and obtain the following boundary value problem with the parameter λ:

$$\lambda\widetilde{U} - L_{\mathcal{A}}\widetilde{U} = U_0(x), \quad I_0\widetilde{U}(0,\lambda) + I_1\widetilde{U}(1,\lambda) = 0, \tag{16}$$

where $\widetilde{U}(x,\lambda)=\int_0^\infty U(x,t)e^{-\lambda t}\,dt$. It is shown in [10] that the solution $\widetilde{U}(x,\lambda)$ to problem (16) has the form

$$\widetilde{U}(x,\lambda)=-\int_0^1 G(x,\xi,\lambda)U_0(\xi)\,d\xi,$$

where $G(x,\xi,\lambda)$ is the Green function of the last problem.

If conditions of the theorem 2 are fulfilled, then the function $G(x,\xi,\lambda)$ in the domain of its analyticity by $|\lambda|\to\infty$ has the form

$$G(x,\xi,\lambda)=\sum_{n=0}^{k+2}\frac{G_n(x,\xi,\lambda)}{\lambda^n},$$

which allows to obtain the following presentation for $\tilde{U}(x,\lambda)$:

$$\tilde{U}(x,\lambda)=\sum_{n=0}^{k+1}\frac{\tilde{U}_n(x,\lambda)}{\lambda^n}+\frac{\tilde{U}_{k+2}(x,\lambda)}{\lambda^{k+2}}. \tag{17}$$

Due to the smoothness of coefficients $\mathcal{A}(x)$ and $\mathcal{K}(x)$, the inverse Laplace transform of the last summand in (17) is k times differentiable function by t for $t\geq 0$.

We consider the inverse Laplace transform of the first summand (of the wohl sum) in (17). In [7] it is shown that two cases are possible. If $\Delta(\lambda)\not\equiv 1$, then infinite number of eigenvalues of problem (16) lie in the strip parallel to the imaginary axis. Then $G_n(x,\xi,\lambda)$ $(n=0,...,k+1)$ are meromorphic in λ functions. If the zero-order compatibility conditions for the initial data are not fulfilled or the initial data has discontinuity of the first-order, then the inverse Laplace transform of the first summand in (17) has discontinuities on the infinite number of characteristic lines of the system (7). Moreover, the number of such characteristic lines is infinite on every set $\Pi\setminus\{[0,1]\times[0,t]\}$, $t>0$.

If $\Delta(\lambda)\equiv 1$, then the following estimate is true for eigenvalues λ of problem (16) as $|\lambda|\to\infty$:

$$Re\lambda\leq -q\ln|Im\lambda|,\quad q>0.$$

In this case $G_n(x,\xi,\lambda)$ $(n=0,...,k+1)$ are entire in λ functions of the exponential type. By properties of the Laplace transform, there exists a number $T(k)\geq 0$ such that the inverse Laplace transform of the first summand in (17) is infinitely differentiable by t function for large $t\geq T(k)$. The inverse Laplace transform of the function $\tilde{U}(x,\lambda)$ is the piecewise smooth solution $U(x,t)$ to the original homogeneous problem which satisfies the differential system (7) a.e. in Π. Consequently, if $t\geq T(k)$, the function $U(x,t)$ is k-times differentiable.◀

Remark 1. *B–regularity of the boundary conditions guarantees the existence of numbers $T(0)$, $T(1)$, $T(2)$,...,$T(k)$,... such that every PSS $U(x,t)$ becomes j-times continuously differentiable for $t>T(j)$ if coefficients of the system are sufficiently smooth $(j=0,1,...,k)$. Moreover, for $t>T(k)$ the estimate* (15) *is valid.*

Remark 2. *Let us point out one interesting property of B-regular conditions. For this purpose along with a homogeneous problem* (7), (8) *we will consider a problem*

$$U_t-L_{\mathcal{A}_d}U=0,\quad (x,t)\in\Pi, \tag{18}$$

$$I_0U(0,t)+I_1U(1,t)=0,\qquad U(x,0)=U_0(x),$$

where entries of the diagonal matrix $\mathcal{A}_d$ are the diagonal entries of matrix $\mathcal{A}$. The B-regularity of boundary conditions (8) *for problem* (7) *means that the spectral problem*

$$\lambda\widetilde{U} - L_{\mathcal{A}_d}\widetilde{U} = 0, \quad I_0\widetilde{U}(0,\lambda) + I_1\widetilde{U}(1,\lambda) = 0$$

does not have eigenvalues because the characteristic equation for them has the form

$$\Delta(\lambda) = 0, \quad \textit{where} \quad \Delta(\lambda) \equiv 1.$$

In this case there exist a number T^* such that for any initial data $U_0 \in C^1[0,1]$ the corresponding PSS $U(x,t)$ of problem (18) *will be equal to zero for $t > T^*$.* [7]

We note that for the system of two equations

$$\begin{gathered} u_t + k_1(x)u_x = a(x)u + b(x)v, \qquad v_t - k_2(x)v_x = c(x)u + d(x)v, \\ u|_{x=0} = \alpha v|_{x=0}, \qquad v|_{x=1} = \beta u|_{x=1}, \\ u|_{t=0} = u_0(x), \qquad v|_{t=0} = v_0(x), \end{gathered} \tag{19}$$

the determinant of matrix (12) $X(\lambda)$ has the form

$$\det X(\lambda) = e^{\lambda \int_0^1 \frac{dx}{k_2(x)} + \int_0^1 \frac{d(x)}{k_2(x)} dx}\Delta(\lambda),$$

where

$$\Delta(\lambda) = 1 - \alpha\beta e^{-\lambda\left(\int_0^1 \frac{dx}{k_1(x)} + \int_0^1 \frac{dx}{k_2(x)}\right) + \left(\int_0^1 \frac{a(x)}{k_1(x)}dx - \int_0^1 \frac{d(x)}{k_2(x)}dx\right)}.$$

Therefore boundary conditions in problem (19) are B-regular if and only if $\alpha\beta = 0$ that was illustrated in the introduction.

We will show that theorem 2 is valid not only for PSS but also for a wide class of solutions. For this purpose by analogy with the concept of the generalized solution for hyperbolic equation ([11]) we introduce the next definition of the generalized solution to homogeneous problem (7), (8).

Definition 3. *The function $U(x,t) \in C(L_2(0,1),[0,\infty))$ is called a generalized solution of homogeneous problem* (7), (8) *if there is a sequence of continuously differentiable solutions $U_k(x,t)$, $k = 1,2,...,$ of this problem such that for any $T > 0$*

$$||U_k(x,t) - U(x,t)||_{L_2(0,1)} \to 0 \quad \textit{uniformly in} \quad t \in [0,T].$$

For continuously differentiable solutions of considered problem the a priori estimate is known ([8])

$$||U(x,t)||_{L_2(0,1)} \le Ke^{At}||U_0(x)||_{L_2(0,1)}, \quad t \ge 0,$$

from which the solvability of an original problem in a class of introduced generalized solutions follows if the initial data $U_0(x) \in L_2(0,1)$.

Remark 3. *Due to the density in $L_2(0,1)$ of the infinitely differentiable on a piece $[0,1]$ functions with compact support and due to estimate* (15) *for continuously differentiable solutions of considered problem, it is obvious that the theorem 2 and the remarks 1, 2 remain true for introduced generalized solutions.*

It is shown in [12] that the increasing smoothness property also takes place for nonhomogeneous linear problems and some nonlinear hyperbolic systems as well. By the Duhamel method the following theorem is proved:

Theorem 3. *Let $\mathcal{A}(x)$ and $\mathcal{K}(x)$ be in $C^{k+2}[0,1]$. Then B–regularity of boundary conditions* (8) *is the necessary and sufficient condition for nonhomogeneous linear problem* (7), (8) *to possess the increasing smoothness property of PSS $U(x,t)$ up to the order k for arbitrary initial data $U_0(x) \in C^1[0,1]$ and for arbitrary $k+2$ times continuously differentiable in $\overline{\Pi}$ functions $F(x,t)$; $k \geq 0$. The solution $U(x,t)$ for $t > T(k)$ satisfies the estimate*

$$\|D^{\alpha,\beta}_{x,t}U(x,t)\|_{C[0,1]} \leq K(t)(\|U_0(x)\|_{L_2(0,1)} + \|F(x,t)\|_{C^k([0,1]\times[0,t])}),$$

where $\alpha + \beta \leq k$, the constant $K(t)$ is independent of the initial data and depends on the coefficients of the problem and t.

We consider the following nonlinear boundary value problem:

$$U_t + K(x)U_x = F(x,t,U), \qquad (x,t) \in \Pi, \tag{20}$$

$$u_i(0,t) = \alpha_i u_n(0,t), \qquad i = 1, ..., n-1,$$

$$u_n(1,t) = \sum_{i=1}^{n-1} \beta_i u_i(1,t), \qquad U(0,t) = U_0(x),$$

where

$$K(x) = \begin{pmatrix} k_1(x) & 0 & ... & 0 \\ 0 & k_1(x) & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & ... & k_1(x) & 0 \\ 0 & 0 & ... & k_2(x) \end{pmatrix},$$

and $k_1(x) > 0$, $k_2(x) < 0$. We suppose that the problem has a continuous solution $U(x,t)$. Using the method of integration along the characteristics the following theorem is proved in [12]:

Theorem 4. *Let $K(x)$ and $F(x,t,U)$ be k times continuously differentiable in all variables functions, $k \geq 1$. If $\alpha_1 = \alpha_2 = ... = \alpha_{n-1} = 0$ or $\beta_1 = \beta_2 = ... = \beta_{n-1} = 0$, then the problem* (20) *possesses the increasing smoothness property of continuous solutions $U(x,t)$ up to the order k.*

The boundary conditions with time delay for linear problem (7) was considered in [13]

$$\sum_{k=0}^{\mathfrak{m}} (A_k U(0, t-\tau_k) + B_k U(1, t-\tau_k)) + \sum_{k=1}^{\mathfrak{m}} (\int_0^{\tau_k} \sum_{r=0,1} \Phi^r_k(\xi) U(r, t-\xi)\, d\xi) = 0, \tag{21}$$

The delay times τ_k in boundary conditions (21) are fixed real numbers: $0 = \tau_0 < \tau_1 < ... < \tau_{\mathfrak{m}}$, $\mathfrak{m} \geq 0$. A_k and B_k are $n \times n$ real matrices, $k = 0, 1, ..., \mathfrak{m}$. The matrices A_0, B_0 are assumed to have the form

$$A_0 = \begin{pmatrix} 1 & . & 0 & \alpha_{1,p+1} & . & \alpha_{1,n} \\ . & . & . & . & . & . \\ 0 & . & 1 & \alpha_{p,p+1} & . & \alpha_{p,n} \\ 0 & 0 & 0 & \alpha_{p+1,p+1} & . & \alpha_{p+1,n} \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \alpha_{n,p+1} & . & \alpha_{n,n} \end{pmatrix}, \quad B_0 = \begin{pmatrix} \beta_{1,1} & . & \beta_{1,p} & 0 & 0 & 0 \\ . & . & . & . & . & . \\ \beta_{p,1} & . & \beta_{p,p} & 0 & 0 & 0 \\ \beta_{p+1,1} & . & \beta_{p+1,p} & 1 & . & 0 \\ . & . & . & . & . & . \\ \beta_{n,1} & . & \beta_{n,p} & 0 & . & 1 \end{pmatrix}.$$

The entries of matrices $\Phi_k^r(\xi)$ are the smooth functions on the corresponding intervals $[0,\tau_k]$ ($r = 0,1;\ k = 1,...,\mathfrak{m}$). The initial data $\bar{U}(x,t)$ is given on the set Γ

$$U(x,t)|_\Gamma = \bar{U}(x,t), \tag{22}$$

which guarantees the well-posedness of the initial-boundary value problem.

Hyperbolic systems with time delay in boundary conditions arise in mathematical models of countercurrent chemical reactor with recycle, when some substances return partly to the entrance of the reactor after exiting with some time delay which is needed for transportation of these substances (via tubes or mechanically, etc.). Usually, the proportionality coefficients indicate which part of the substance returns.

In [13] the existence of classical solution to (7), (21), (22) in the half-strip Π is proved and the linearization principle for the analysis of stability of stationary solutions to the nonlinear autonomous hyperbolic system ($U_t + \mathcal{K}(x)U_x = F(x,U)$) is justified.

We focus our attention on the homogeneous linear problem (7), (21), (22), i.e. $F(x,t) \equiv 0$. Under some relations on coefficients in the boundary conditions this problem also possesses the increasing smoothness property. The definition of this property is similar to the definition 1.

By analogy with the problem with splitted boundary conditions, we introduce the matrix

$$X(\lambda) = A_0 + \sum_{k=1}^{\mathfrak{m}} e^{-\lambda\tau_k} A_k + (B_0 + \sum_{k=1}^{\mathfrak{m}} e^{-\lambda\tau_k} B_k)\Upsilon(1,\lambda).$$

Its determinant has the form

$$\det X(\lambda) = e^{\lambda \sum\limits_{i=p+1}^{n} \Upsilon_i + \sum\limits_{i=p+1}^{n} \mathcal{B}_i} \Delta(\lambda). \tag{23}$$

Here Υ_i, $\mathcal{B}_i$ are determined in (13). Dirichlet polynomial $\Delta(\lambda)$ has the form

$$\Delta(\lambda) = 1 + \sum_{k=1}^{M} E_k e^{-\lambda\beta_k}, \tag{24}$$

in which the real numbers E_k are determined by entries of the matrices A_k, B_k ($k = 0,...,\mathfrak{m}$) and numbers $\mathcal{B}_i$ ($i = 1,...,n$). The numbers $0 < \beta_1 < ... < \beta_M$ are determined by the delay times τ_k ($k = 1,...,\mathfrak{m}$) and numbers Υ_i ($i = 1,...,n;$).

Definition 4. *Boundary conditions* (21) *for problem* (7), (21), (22) *are called P—regular if* $\Delta(\lambda) \equiv 1$, *i.e.* $E_k = 0$ *(*$k = 1,...,M$*) in* (24).

In [14] for the homogeneous problem ($F(x,t) \equiv 0$) (7), (21), (22) the following theorem is proved:

Theorem 5. *Let* $0{=}\tau_0{<}\tau_1{<}\ ...{<}\ \tau_m$ *be arbitrary fixed real numbers and boundary conditions* (21) *be P–regular. If* $\mathcal{A}(x)$, $\mathcal{K}(x) \in C^{k+2}[0,1]$, *then the considered problem possesses the increasing smoothness property of classical solutions* $U(x,t)$ *up to the order* k *for arbitrary initial data* $\bar{U}(x,t) \in C^1(\Gamma)$ *satisfying the compatibility conditions, and any functions* $\Phi_k^r(\xi) \in C^1[0,\tau_k]$ *(*$r = 0,1,\ k =$

$1, ..., \mathfrak{m}$), setting distributed delay; $k \geq 0$. The solution $U(x,t)$ for $t > T(k)$ satisfies the estimate

$$\|D_{x,t}^{\alpha,\beta} U(x,t)\|_{C[0,1]} \leq K(t) \|\bar{U}(x,t)\|_{L_2(\Gamma)},$$

where $\alpha + \beta \leq k$, the constant $K(t)$ is independent of the initial data and depends on the coefficients of the problem and t.

Remark 4. *The theorem 5 is formulated for classical solutions, and classical solutions necessarily exist if the compatibility conditions for $\bar{U}(x,t)$ are fulfilled. But the theorem stays valid also for piecewise smooth solutions which appear if initial data does not satisfy the compatibility conditions.*

Now we consider a system of two equations as an example

$$u_t + u_x = a(x)u + b(x)v, \quad v_t - v_x = c(x)u + d(x)v, \tag{25}$$

$$\begin{aligned} u(0,t) &= \alpha_1 v(0,t) + \beta_1 u(1,t) + \gamma_1 u(0,t-1) + \int_0^{\tau_1} f_1(\xi) u(0,t-\xi)\, d\xi, \\ v(1,t) &= \alpha_2 v(0,t) + \beta_2 u(1,t) + \gamma_2 v(1,t-1) + \int_0^{\tau_2} f_2(\xi) v(1,t-\xi)\, d\xi, \end{aligned} \tag{26}$$

$$(u,v)|_\Gamma = (\bar{u}, \bar{v}).$$

In this case we have

$$A_0 = \begin{pmatrix} 1 & -\alpha_1 \\ 0 & -\alpha_2 \end{pmatrix}, \quad B_0 = \begin{pmatrix} -\beta_1 & 0 \\ -\beta_2 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} -\gamma_1 & 0 \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma_2 \end{pmatrix},$$

$$\Upsilon(x,\lambda) = \begin{pmatrix} e^{-\lambda x + \int_0^x a(\xi)\, d\xi} & 0 \\ 0 & e^{\lambda x - \int_0^x d(\xi)\, d\xi} \end{pmatrix},$$

$$\det(A_0 + A_1 e^{-\lambda} + (B_0 + B_1 e^{-\lambda})\Upsilon(1,\lambda)) = e^{\lambda - \int_0^1 d(\xi)\, d\xi} \Delta(\lambda), \qquad \text{where}$$

$$\Delta(\lambda) = 1 - e^{-\lambda}(\gamma_1 + \gamma_2 + \beta_1 e^{\int_0^1 a(\xi)\, d\xi} + \alpha_2 e^{\int_0^1 d(\xi)\, d\xi}) + e^{-2\lambda}(\beta_1 \gamma_2 e^{\int_0^1 a(\xi)\, d\xi} + \alpha_2 \gamma_1 e^{\int_0^1 d(\xi)\, d\xi} + \gamma_1 \gamma_2 + (\alpha_2 \beta_1 - \alpha_1 \beta_2) e^{\int_0^1 (a(\xi) + d(\xi))\, d\xi}).$$

So, the boundary conditions (26) for problem (25) are P—regular if entries of matrices A_0, B_0, A_1, B_1 satisfy the relations

$$\gamma_1 + \gamma_2 + \beta_1 e^{\int_0^1 a(\xi)\, d\xi} + \alpha_2 e^{\int_0^1 d(\xi)\, d\xi} = 0,$$

$$\beta_1 \gamma_2 e^{\int_0^1 a(\xi)\, d\xi} + \alpha_2 \gamma_1 e^{\int_0^1 d(\xi)\, d\xi} + \gamma_1 \gamma_2 + (\alpha_2 \beta_1 - \alpha_1 \beta_2) e^{\int_0^1 (a(\xi) + d(\xi))\, d\xi} = 0.$$

3. Initial-boundary value problems for wave equation

The above results for hyperbolic systems allow to define a class of boundary conditions given on the lateral sides of Π for the wave equation. These conditions are necessary and sufficient conditions for every solution of

$$u_{tt} - a^2 u_{xx} = f(x,t), \qquad (x,t) \in \Pi, \quad (a > 0), \tag{27}$$

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x)$$

to be a function of $C^k[0,1]$ as t grows, if $u_0(x) \in C^3[0,1]$, $u_1(x) \in C^2[0,1]$, $f(x,t) \in C^\infty(\overline{\Pi})$, where k is an arbitrary natural number [15].

For the wave equation (27) on the lateral sides of Π we set the boundary conditions

$$u_t - \alpha(u_t - au_x)|_{x=1} = 0, \qquad u_t - au_x - \beta u|_{x=0} = 0 \tag{28}$$

or

$$u_t - \alpha(u_t + au_x)|_{x=0} = 0, \qquad u_t + au_x - \beta u|_{x=1} = 0. \tag{29}$$

By means of reduction of wave equation (27) to first-order hyperbolic system

$$u_t - au_x = v, \quad v_t + av_x = f(x,t)$$

or

$$u_t + au_x = v, \quad v_t - av_x = f(x,t)$$

in [15] it is shown that the initial-boundary value problems (27), (28) and (27), (29) are well-posed in the class of continuously differentiable functions if the initial data u_0, u_1 satisfies to the natural compatibility conditions. If the compatibility conditions are not fulfilled, then considered problems have piecewise smooth solutions for which, with use the theorem 3, the following theorem is proved:

Theorem 6. *Let $f(x,t)$ be $k+2$ times continuously differentiable in $\overline{\Pi}$ function; $k \geq 0$. Then $\alpha\beta = 0$ is the necessary and sufficient condition for the problems (27), (28) and (27), (29) to possess the increasing smoothness property of PSS $u(x,t)$ up to the order k for arbitrary initial data $u_0(x) \in C^3[0,1]$, $u_1(x) \in C^2[0,1]$. The solution $u(x,t)$ for $t > T(k)$ satisfies the estimate*

$$\|D^{\alpha,\beta}_{x,t} u(x,t)\|_{C[0,1]} \leq K(t)(\max(\|u_0(x)\|_{W^1_2(0,1)}, \|u_1(x)\|_{L_2(0,1)}) + ||f(x,t)||_{C^k([0,1]\times[0,t])}),$$

where $\alpha + \beta \leq k$, the constant $K(t)$ is independent of u_0, u_1, f and depends on the coefficients of the problem and t.

Remark 5. *If $f(x,t) \in C^\infty(\overline{\Pi})$ and $\alpha\beta = 0$, then the considered problems possess the increasing smoothness property of PSS up to any order k, $k \geq 0$.*

The hyperbolic equation

$$u_{tt} - u_{xx} + 2b(x)u_t + 2c(x)u_x + d(x)u = f(x,t) \tag{30}$$

can be written as a hyperbolic system of two equations

$$Z_t + KZ_x = A(x)Z + F(x,t),$$

where $Z = (u, (b-c)u + u_x + u_t))^T$, $F(x,t) = (0, f(x,t))^T$,

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A(x) = \begin{pmatrix} c-b & 1 \\ q & -(b+c) \end{pmatrix}, \quad q = (c-b)_x + b^2 - c^2 - d.$$

So, for the obtained system we have the initial-boundary value problem with splitted boundary conditions. By the properties of such systems, for the equation (30) the initial-boundary value problem in the half-strip Π with boundary conditions

$$u|_{x=0} = 0, \quad u_t + u_x - \gamma u|_{x=1} = 0, \quad \gamma \text{ is an arbitrary number,}$$

and the initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x)$$

is well-posed in the class of continuously differentiable on Π functions. By the theorem 3, this problem possesses the property of increasing smoothness up to the order k, determined by the smoothness of the function $f(x,t)$ and coefficients of the equation.

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