THE INCREASING SMOOTHNESS PROPERTY
OF SOLUTIONS TO SOME HYPERBOLIC PROBLEMS
IN TWO INDEPENDENT VARIABLES

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ABSTRACT. The initial-boundary problems for first-order hyperbolic systems and for the wave equation are considered in the half-strip $\Pi = \{(x, t) : 0 < x < 1, t > 0\}$. Boundary conditions which guarantee the increasing of smoothness of the solutions to the considered problems as $t$ grows are formulated.

Keywords: first-order hyperbolic systems on the plane, wave equation, initial-boundary problems, increasing smoothness of the solutions.

Dedicated to the memory of professor T.I. Zelenyak

1. Introduction

Mixed problems for hyperbolic systems in two independent variables arise in the mathematical modeling of physical and chemical processes in connection with the phenomena of warm-and mass-transfer. An extensive literature (see the references in [1]–[6]) is devoted to studying the qualitative properties of solutions to these problems (existence of global (in $t$) solutions, existence of periodic solutions, propagation of discontinuities, bifurcation of solutions, stability of solutions, and so on). This article is a survey of the results established by the authors, concerning the increasing smoothness properties of solutions to some mixed problems for hyperbolic systems in two variables $x, t$. Full proofs of the results stated below one can found in the references to this article.

Lyulko N.A. The increasing smoothness property of solutions to some hyperbolic problems in two independent variables.

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The work is supported by RFFI (grant 09-02-00221), Presidium of the Russian Academy of Sciences (Program 2, Project 121), the Russian Federal Agency for Education (Project 2.1.1.4918).

It is known that the smoothness of a solution to the Cauchy problem for the simplest hyperbolic system of two equations with constant coefficients

\[ u_t + u_x = au + bv, \quad v_t - v_x = cu + dv, \]

\[ u|_{t=0} = u_0(x), \quad v|_{t=0} = v_0(x) \]
is not higher than the smoothness of the initial data \( u_0 \) and \( v_0 \). If the point \( x = x_0 \) is the jump discontinuity of the initial data \( u_0 \) and \( v_0 \), this discontinuity will persist for function \( u(x,t) \) along the characteristic \( t = x - x_0 \) and for the function \( v(x,t) \) along the characteristic \( t = -x + x_0 \) of system, starting at that point. This discontinuity will exist for any period of time \( t \). A similar situation takes place for the Cauchy problem for the wave equation

\[ u_{tt} - u_{xx} = 0, \quad u|_{t=0} = u_0(x), \quad u|_{t=0} = u_1(x). \]
The discontinuity of the initial data \( u_0 \) and \( u_1 \) at \( x = x_0 \) for the solution \( u(x,t) \) propagates along both mentioned characteristics simultaneously.

For initial-boundary value problems in the half-strip \( \Pi = \{(x,t) : 0 < x < 1, t > 0 \} \) it is possible to set boundary conditions on lateral sides of \( \Pi \) such that the smoothness of solutions to the corresponding boundary problems increases as \( t \) grows. Here we give an example, which belongs to T.I.Zelenyak, that illustrate the increasing smoothness of solutions to the hyperbolic problem

\[ u_t + u_x = 2v, \quad v_t - v_x = 0, \quad (x,t) \in \Pi, \]

\[ u|_{x=0} = \alpha v|_{x=0}, \quad v|_{x=1} = \beta u|_{x=1}, \]

\[ u|_{t=0} = u_0(x), \quad v|_{t=0} = \frac{dv_0(x)}{dx}, \]

where \( u_0(x), \frac{dv_0(x)}{dx} \in C^1[0,1]; \) \( \alpha, \beta \) are constants.

The general solution to the system (1) has the form

\[ u(x,t) = \varphi(t + x) + \psi(t - x), \quad v(x,t) = \frac{d}{dx}\varphi(\xi)|_{\xi=t+x}, \]

where \( \varphi(\xi) \in C^2[0,\infty), \psi(\xi) \in C^1[-1,\infty) \) are arbitrary functions.

We consider the case \( \alpha \beta = 0 \). Let \( \alpha = 0, \beta = 1 \). Substituting (4) into (2), we obtain the relations for \( \xi \geq 0 \)

\[ \frac{d}{dx}\varphi(\xi + 1) = \varphi(\xi + 1) + \psi(\xi - 1), \quad \varphi(\xi) + \psi(\xi) = 0. \]

From (3) we have for \( 0 \leq \xi \leq 1 \)

\[ \varphi(\xi) = v_0(\xi), \quad \psi(-\xi) = u_0(\xi) - v_0(\xi). \]

One can see from (5) that the smoothness of functions \( \varphi(\xi) \) and \( \psi(\xi) \) increases by 1 over every interval with respect to \( \xi \) of the length 2. By (4), for solutions to the problem (1)-(3) the smoothness increases in \( t \) over intervals of the length 2.

The case \( \alpha \neq 0, \beta = 0 \) is treated similarly. For \( \alpha = \beta = 0 \), obviously, \( v(x,t) \equiv 0 \) for \( t + x > 1 \) and \( u(x,t) \equiv 0 \) for \( t - x > 1 \).

Let \( \alpha = \beta = 1 \), then from boundary conditions (2), by virtue of (4), we have the relations for \( \xi \geq -1 \)

\[ \psi(\xi + 2) = \psi(\xi), \quad \frac{d}{dx}\varphi(\xi + 2) - \varphi(\xi + 2) = \psi(\xi), \]
which imply that the function \( \psi(\xi) \) is 2-periodic. From (4) and (6) we obtain that the smoothness of solutions to (1)-(3) is not higher than the smoothness of the initial data.

So, the above example shows that in the case of mixed problem (1)-(3) the increasing of the smoothness of solutions \( u(x,t) \), \( v(x,t) \) up to any order \( k, k \in \mathbb{N} \), of the coefficients in boundary conditions (2) satisfy \( \alpha\beta = 0 \).

In this work we consider the well-posed initial-boundary value problems for first-order hyperbolic systems and for the wave equation in the half-strip \( \Pi = \{(x,t) : 0 < x < 1, t > 0 \} \). We formulate the boundary conditions which guarantee the increasing smoothness property of the solutions to the considered problems as \( t \) grows.

Henceforth, the containment of \( F(x,t) \) in \( C_{x,t}^{k,m}(\Pi) \) is understood as follows: \( F(x,t) \in C_{x,t}^{k,m}(\Pi_T) \) for any \( T > 0 \), where \( \Pi_T = \{(x,t) : 0 \leq x \leq 1, 0 \leq t \leq T \} \). We denote by \( K \) and \( A \) the constants depending on the coefficients of problem and independent of \( t \) and \( U_0(x) \).

2. INITIAL-BOUNDRY VALUE PROBLEMS FOR HYPERBOLIC SYSTEMS

In the half-strip \( \Pi = \{(x,t) : 0 < x < 1, t > 0 \} \) we consider the following problem:

\[
U_t - Lu = F(x,t), \quad (x,t) \in \Pi,
\]

\[
I_0 U(0,t) + I_1 U(1,t) = 0, \quad U(x,0) = U_0(x).
\]

Here \( U(x,t) = [u_1(x,t), ..., u_n(x,t)]^T \) is an \( n \)-dimensional vector of unknown functions, \( F(x,t) \) is the \( n \)-dimensional vector of right hand part,

\[
L = -K(x)U_x + A(x)U, \quad A(x) = (a_{ij}(x))_{i,j=1,...,n},
\]

\( K(x) \) is the diagonal matrix with entries \( k_i(x) \neq k_j(x) (i \neq j) \), the first \( p \) of them are positive, and the rest \( n-p \) of them are negative, moreover \( 1 \leq p < n, n \geq 2 \).

The boundary conditions are splitted, i.e. the matrices \( I_0 \) and \( I_1 \) have the form

\[
I_0 = \begin{pmatrix}
1 & 0 & \alpha_{1,p+1} & \alpha_{1,n} \\
0 & 1 & \alpha_{p,p+1} & \alpha_{p,n} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad I_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\beta_{p+1,1} & \beta_{p+1,p} & 1 & 0 \\
\beta_{n,1} & \beta_{n,p} & 0 & 1
\end{pmatrix}.
\]

If \( K(x) \in C^1[0,1] \), then through each point \( (x_0,t_0) \in \Pi \) pass \( n \) various characteristics \( x = \varphi_i(t;x_0,t_0) \) of system (7) defined by the equations

\[
\frac{d\varphi_i}{dt} = k_i(\varphi_i), \quad \varphi_i|_{t=t_0} = x_0, \quad i = 1,...,n.
\]

Following [1], [7], we denote \( \chi_i(x_0,t_0) = \text{inf}\{t : (\varphi_i(t;x_0,t_0),t) \in \Pi\} \), then, obviously, \( 0 \leq \chi_i(x_0,t_0) \leq t_0 \) and if \( \chi_i(x_0,t_0) > 0 \), then \( \varphi_i(\chi_i(x_0,t_0);x_0,t_0) \) will be equal to 0 or 1. We consider the following sets for any \( i, 1 \leq i \leq n \):

\[
\Pi_i = \{(x,t) \in \Pi : \chi_i(x,t) = 0 \},
\]

\[
\Pi_i^0 = \{(x,t) \in \Pi : \chi_i(x,t) > 0 \quad \text{and} \quad \varphi_i(\chi_i(x,t);x,t) = 0 \},
\]

\[
\Pi_i^1 = \{(x,t) \in \Pi : \chi_i(x,t) > 0 \quad \text{and} \quad \varphi_i(\chi_i(x,t);x,t) = 1 \}.
\]
Let has a unique continuously differentiable solution (8).

If the compatibility conditions (10), (11) are fulfilled, then the piecewise smooth existence in

At the points where the derivatives exist, the function is discontinuous on the set of characteristics of system (7) which is at most a countable.

We integrate the i-th equation of system (7) along the corresponding characteristic . From (8) we obtain the following system of integral equations:

If it is a solution of integral system (9). A necessary condition for the continuity of a solution in the half-strip is the fulfillment of the zero-order compatibility conditions (10), then in [1] the question of well-posedness of problem (7), (8) in spaces of continuous, continuously differentiable and integrable functions is investigated.

Let be a solution of problem (7), (8). If the compatibility conditions (10), (11) are fulfilled, then

Theorem 1. Let . Then problem (7), (8) has a unique continuously differentiable solution in the half-strip if it satisfies the estimate

where is a classical solution of problem (7), (8), and the initial data of problem and its derivatives are

Obviously, , . If the compatibility conditions (10), (11) are not fulfilled, then PSS of problem and its derivatives are discontinuous on the set of characteristics of system (7) which is at most a countable.

Furthermore, if the compatibility conditions (10), (11) are not fulfilled, then the piecewise smooth solution is a classical solution of problem (7), (8).
Definition 1. We say that the problem (7), (8) possesses the increasing smoothness property up to the order \( k \) if there exists a number \( T(k) > 0 \) such that every PSS \( U(x, t) \) to the problem (7), (8) is \( k \) times continuously differentiable for \( t > T(k) \).

In [7] a class of the so-called \( B \)-regular boundary conditions was described. The mentioned conditions are necessary and sufficient for the linear homogeneous problem (7), (8) to possess the increasing smoothness property up to the order \( k \), which is determined by the smoothness of the matrices \( K(x) \) and \( A(x) \) and does not depend on the smoothness of the initial data.

We consider the diagonal matrix

\[
\Upsilon(x, \lambda) = (e^{\lambda \Upsilon_j(x)} + B_j(x) \delta_{ij})_{i,j=1,...,n},
\]

where \( \Upsilon_j(x) = \int_0^x \frac{1}{s_j(x)} \, ds \), \( B_j(x) = \int_0^x \frac{a_{ij}(t)}{\xi_i(t)} \, d\xi \), \( \delta_{ij} \) is the Kronecker symbol, \( \lambda \) is a complex parameter, and introduce the expression

\[
X(\lambda) = I_0 + I_1 \Upsilon(1, \lambda).
\]

It is well known that \( \det X(\lambda) = e^{\lambda \sum_{i=p+1}^\lambda \Upsilon_i + \sum_{i=p+1}^\lambda B_i} \Delta(\lambda) \), where

\[
\Upsilon_i = \Upsilon_i(1), \quad B_i = B_i(1), \quad i = 1, ..., n,
\]

\[
\Delta(\lambda) = 1 + \sum_{k=1}^M E_k e^{-\lambda \beta_k}.
\]

Here \( E_k \) are real numbers determined by entries of the matrices \( I_0 \) and \( I_1 \) and numbers \( \beta_i \); numbers \( 0 < \beta_1 < ... < \beta_M \) are determined via \( \Upsilon_i; \ i = 1, ..., n \).

Definition 2. Boundary conditions (8) for the problem (7) are called \( B \)-regular if \( E_k = 0 \) \((k=1,...,M)\) in (14), i.e. \( \Delta(\lambda) \equiv 1 \).

The following theorem is proved in [7]:

Theorem 2. Let \( F(x,t) \equiv 0, A(x), K(x) \in C^{k+2}[0,1] \). Then \( B \)-regularity of boundary conditions (8) is the necessary and sufficient condition for the problem (7), (8) to possess the increasing smoothness property of PSS \( U(x,t) \) up to the order \( k \) for arbitrary initial data \( U_0(x) \in C^1[0,1] \), \( k = 0, 1, ... \). The solution \( U(x,t) \) for \( t > T(k) \) satisfies the estimate

\[
\|D^\alpha_\lambda U(x, t)\|_{C[0,1]} \leq K(t)\|U_0(x)\|_{L^2[0,1]},
\]

where \( \alpha + \beta \leq k \), the constant \( K(t) \) is independent of the initial data and depends on the coefficients of the problem and \( t \).

We will describe the main steps of the proof, based on a study for \(|\lambda| \to \infty\) asymptotic properties of function \( \tilde{U}(x, \lambda) \), which is the Laplace transform of the original problem. A similar approach to problems with splitted boundary conditions was used earlier in [8], [9]. We apply the Laplace transform by \( t \) to the system (7), (8) and obtain the following boundary value problem with the parameter \( \lambda \):

\[
\lambda \tilde{U} - L_A \tilde{U} = U_0(x), \quad I_0 \tilde{U}(0, \lambda) + I_1 \tilde{U}(1, \lambda) = 0,
\]
where $\tilde{U}(x, \lambda) = \int_0^\infty U(x, t) e^{-\lambda t} dt$. It is shown in [10] that the solution $\tilde{U}(x, \lambda)$ to problem (16) has the form

$$\tilde{U}(x, \lambda) = -\int_0^1 G(x, \xi, \lambda) U_0(\xi) \, d\xi,$$

where $G(x, \xi, \lambda)$ is the Green function of the last problem.

If conditions of the theorem 2 are fulfilled, then the function $G(x, \xi, \lambda)$ in the domain of its analyticity by $|\lambda| \to \infty$ has the form

$$G(x, \xi, \lambda) = \sum_{n=0}^{k+2} \frac{G_n(x, \xi, \lambda)}{\lambda^n},$$

which allows to obtain the following presentation for $\tilde{U}(x, \lambda)$:

$$\tilde{U}(x, \lambda) = \sum_{n=0}^{k+1} \frac{\tilde{U}_n(x, \lambda)}{\lambda^n} + \frac{\tilde{U}_{k+2}(x, \lambda)}{\lambda^{k+2}}. \quad (17)$$

Due to the smoothness of coefficients $\mathcal{A}(x)$ and $\mathcal{K}(x)$, the inverse Laplace transform of the last summand in (17) is $k$ times differentiable function by $t$ for $t \geq 0$.

We consider the inverse Laplace transform of the first summand (of the whole sum) in (17). In [7] it is shown that two cases are possible. If $\Delta(\lambda) \neq 1$, then infinite number of eigenvalues of problem (16) lie in the strip parallel to the imaginary axis. Then $G_n(x, \xi, \lambda) \, (n = 0, ..., k + 1)$ are meromorphic in $\lambda$ functions. If the zero-order compatibility conditions for the initial data are not fulfilled or the initial data has discontinuity of the first-order, then the inverse Laplace transform of the first summand in (17) has discontinuities on the infinite number of characteristic lines of the system (7). Moreover, the number of such characteristic lines is infinite on every set $\Pi \setminus \{0, 1 \times [0, t] \}, \, t > 0$.

If $\Delta(\lambda) \equiv 1$, then the following estimate is true for eigenvalues $\lambda$ of problem (16) as $|\lambda| \to \infty$:

$$\text{Re}\lambda \leq -q \ln |\text{Im}\lambda|, \quad q > 0.$$ 

In this case $G_n(x, \xi, \lambda) \, (n = 0, ..., k + 1)$ are entire in $\lambda$ functions of the exponential type. By properties of the Laplace transform, there exists a number $T(k) \geq 0$ such that the inverse Laplace transform of the first summand in (17) is infinitely differentiable by $t$ function for large $t \geq T(k)$. The inverse Laplace transform of the function $\tilde{U}(x, \lambda)$ is the piecewise smooth solution $U(x, t)$ to the original homogeneous problem which satisfies the differential system (7) a.e. in $\Pi$. Consequently, if $t \geq T(k)$, the function $U(x, t)$ is $k$-times differentiable. $\blacksquare$

**Remark 1.** B-regularity of the boundary conditions guarantees the existence of numbers $T(0), T(1), T(2), ..., T(k), ...$ such that every PSS $U(x, t)$ becomes $j$-times continuously differentiable for $t > T(j)$ if coefficients of the system are sufficiently smooth ($j = 0, 1, ..., k$). Moreover, for $t > T(k)$ the estimate (15) is valid.

**Remark 2.** Let us point out one interesting property of B-regular conditions. For this purpose along with a homogeneous problem (7), (8) we will consider a problem

$$U_t - L_\mathcal{A} U = 0, \quad (x, t) \in \Pi,$$

$$I_0 U(0, t) + I_1 U(1, t) = 0, \quad U(x, 0) = U_0(x), \quad (18)$$

where $U_t$ is the first order partial derivative with respect to $t$. The problem (18) has solutions for the initial data $U_0(x)$, which satisfy the compatibility conditions $I_0 U(0, t) + I_1 U(1, t) = 0$.
The function the corresponding PSS that was illustrated in the introduction. By the Duhamel method the following theorem is proved:

If there is a sequence of continuously differentiable solutions of considered problem such that for any initial data $U_0 \in C^1[0, 1]$ means that the spectral problem

$$
\lambda \vec{U} - L_{A_0} \vec{U} = 0, \quad I_0 \vec{U}(0, \lambda) + I_1 \vec{U}(1, \lambda) = 0
$$

does not have eigenvalues because the characteristic equation for them has the form

$$
\Delta(\lambda) = 0, \quad \text{where} \quad \Delta(\lambda) \equiv 1.
$$

In this case there exist a number $T^*$ such that for any initial data $U_0 \in C^1[0, 1]$ the corresponding PSS $U(x, t)$ of problem (18) will be equal to zero for $t > T^*$.

We note that for the system of two equations

$$
\begin{align*}
    u_t + k_1(x)u_x &= a(x)u + b(x)v, \\
    v_t - k_2(x)v_x &= c(x)u + d(x)v,
\end{align*}
$$

(19)

the determinant of matrix (12) $X(\lambda)$ has the form

$$
\det X(\lambda) = e^{\lambda \int_0^1 \frac{\partial g}{\partial x} + \int_0^1 \frac{\partial f}{\partial x} \ dx} \Delta(\lambda),
$$

where

$$
\Delta(\lambda) = 1 - \alpha \beta e^{-\lambda \int_0^1 \frac{\partial g}{\partial x} + \int_0^1 \frac{\partial f}{\partial x} \ dx} \left( f^1_0 \frac{\partial g}{\partial x} + f^1_0 \frac{\partial f}{\partial x} \right).
$$

Therefore boundary conditions in problem (19) are B-regular if and only if $\alpha \beta = 0$ that was illustrated in the introduction.

We will show that theorem 2 is valid not only for PSS but also for a wide class of solutions. For this purpose by analogy with the concept of the generalized solution for hyperbolic equation ( [11]) we introduce the next definition of the generalized solution to homogeneous problem (7), (8).

Definition 3. The function $U(x, t) \in C(L_2(0, 1), [0, \infty))$ is called a generalized solution of homogeneous problem (7), (8) if there is a sequence of continuously differentiable solutions $U_k(x, t), \ k = 1, 2, \ldots$, of this problem such that for any $T > 0$

$$
||U_k(x, t) - U(x, t)||_{L_2(0, 1)} \to 0 \quad \text{uniformly in} \quad t \in [0, T].
$$

For continuously differentiable solutions of considered problem the a priori estimate is known ( [8])

$$
||U(x, t)||_{L_2(0, 1)} \leq Ke^{At}||U_0(x)||_{L_2(0, 1)}, \quad t \geq 0,
$$

from which the solvability of an original problem in a class of introduced generalized solutions follows if the initial data $U_0(x) \in L_2(0, 1)$.

Remark 3. Due to the density in $L_2(0, 1)$ of the infinitely differentiable on a piece [0, 1] functions with compact support and due to estimate (15) for continuously differentiable solutions of considered problem, it is obvious that the theorem 2 and the remarks 1, 2 remain true for introduced generalized solutions.

It is shown in [12] that the increasing smoothness property also takes place for nonhomogeneous linear problems and some nonlinear hyperbolic systems as well. By the Duhamel method the following theorem is proved:
Theorem 3. Let $A(x)$ and $K(x)$ be in $C^{k+2}[0,1]$. Then $B$-regularity of boundary conditions (8) is the necessary and sufficient condition for nonhomogeneous linear problem (7) to possess the increasing smoothness property of PSS $U(x,t)$ up to the order $k$ for arbitrary initial data $U_0(x) \in C^1[0,1]$ and for arbitrary $k+2$ times continuously differentiable in $\Pi$ functions $F(x,t)$; $k \geq 0$. The solution $U(x,t)$ for $t > T$ satisfies the estimate

$$
\|D_x^{\alpha,\beta} U(x,t)\|_{C([0,1])} \leq K(t)(\|U_0(x)\|_{L_2(0,1)} + \|F(x,t)\|_{C^2([0,1] \times [0,t])}),
$$

where $\alpha + \beta \leq k$, the constant $K(t)$ is independent of the initial data and depends on the coefficients of the problem and $t$.

We consider the following nonlinear boundary value problem:

$$(20) \quad U_t + K(x)U_x = F(x,t,U), \quad (x,t) \in \Pi,$$

$$u_i(0,t) = \alpha_i u_i(0,t), \quad i = 1, ..., n - 1,$$

$$u_n(1,t) = \sum_{i=1}^{n-1} \beta_i u_i(1,t), \quad U(0,t) = U_0(x),$$

where

$$K(x) = \begin{pmatrix} k_1(x) & 0 & \ldots & 0 \\ 0 & k_1(x) & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & k_2(x) \end{pmatrix},$$

and $k_1(x) > 0$, $k_2(x) < 0$. We suppose that the problem has a continuous solution $U(x,t)$. Using the method of integration along the characteristics the following theorem is proved in [12]:

Theorem 4. Let $K(x)$ and $F(x,t,U)$ be $k$ times continuously differentiable in all variables functions, $k \geq 1$. If $\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0$ or $\beta_1 = \beta_2 = \ldots = \beta_{n-1} = 0$, then the problem (20) possesses the increasing smoothness property of continuous solutions $U(x,t)$ up to the order $k$.

The boundary conditions with time delay for linear problem (7) was considered in [13]

$$(21) \quad \sum_{k=0}^{m} (A_k U(0,t - \tau_k) + B_k U(1,t - \tau_k)) + \sum_{k=1}^{m} \left( \int_{r=0}^{\tau_k} \Phi_n^{(r)}(\xi) U(r,t - \xi) \, d\xi \right) = 0,$$

The delay times $\tau_k$ in boundary conditions (21) are fixed real numbers: $0 = \tau_0 < \tau_1 < \ldots < \tau_m$, $m \geq 0$. $A_k$ and $B_k$ are $n \times n$ real matrices, $k = 0, 1, \ldots, m$. The matrices $A_0, B_0$ are assumed to have the form

$$A_0 = \begin{pmatrix} 1 & 0 & \alpha_{1,p+1} & \alpha_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \alpha_{p,p+1} & \alpha_{p,n} \\ 0 & 0 & \alpha_{p+1,p+1} & \alpha_{p+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \alpha_{n,p+1} & \alpha_{n,n} \end{pmatrix}, \quad B_0 = \begin{pmatrix} \beta_{1,1} & \ldots & \beta_{1,p} & 0 & 0 \\ \beta_{p,1} & \ldots & \beta_{p,p} & 0 & 0 \\ \beta_{p+1,1} & \ldots & \beta_{p+1,p} & 1 & 0 \\ \beta_{n,1} & \ldots & \beta_{n,p} & 0 & 1 \end{pmatrix}.$$
The entries of matrices $\Phi^r_k(\xi)$ are the smooth functions on the corresponding intervals $[0, \tau_k]$ ($r = 0, 1; k = 1, ..., m$). The initial data $\bar{U}(x, t)$ is given on the set $\Gamma$

\begin{equation}
U(x, t)|_\Gamma = \bar{U}(x, t),
\end{equation}

which guarantees the well-posedness of the initial-boundary value problem.

Hyperbolic systems with time delay in boundary conditions arise in mathematical models of countercurrent chemical reactor with recycle, when some substances return partly to the entrance of the reactor after exiting with some time delay which is needed for transportation of these substances (via tubes or mechanically, etc.). Usually, the proportionality coefficients indicate which part of the substance returns.

In [13] the existence of classical solution to (7), (21), (22) in the half-strip $\Pi$ is proved and the linearization principle for the analysis of stability of stationary solutions to the nonlinear autonomous hyperbolic system $(U_t + K(x)U_x = F(x, U))$ is justified.

We focus our attention on the homogeneous linear problem (7), (21), (22), i.e. $F(x, t) \equiv 0$. Under some relations on coefficients in the boundary conditions this problem also possesses the increasing smoothness property. The definition of this property is similar to the definition 1.

By analogy with the problem with splitted boundary conditions, we introduce the matrix

\[ X(\lambda) = A_0 + \sum_{k=1}^m e^{-\lambda \tau_k} A_k + (B_0 + \sum_{k=1}^m e^{-\lambda \tau_k} B_k) \Upsilon(1, \lambda). \]

Its determinant has the form

\begin{equation}
\det X(\lambda) = e^{\lambda \sum_{i=p+1}^n \Upsilon_i + \sum_{i=p+1}^n B_i} \Delta(\lambda).
\end{equation}

Here $\Upsilon_i, B_i$ are determined in (13). Dirichlet polynomial $\Delta(\lambda)$ has the form

\begin{equation}
\Delta(\lambda) = 1 + \sum_{k=1}^M E_k e^{-\lambda \beta_k},
\end{equation}

in which the real numbers $E_k$ are determined by entries of the matrices $A_k, B_k$ ($k = 0, ..., m$) and numbers $B_i$ ($i = 1, ..., n$). The numbers $0 < \beta_1 < ... < \beta_M$ are determined by the delay times $\tau_k$ ($k = 1, ..., m$) and numbers $\Upsilon_i$ ($i = 1, ..., n$).

**Definition 4.** Boundary conditions (21) for problem (7), (21), (22) are called P-regular if $\Delta(\lambda) \equiv 1$, i.e. $E_k = 0$ ($k = 1, ..., M$) in (24).

In [14] for the homogeneous problem $(F(x, t) \equiv 0)$ (7), (21), (22) the following theorem is proved:

**Theorem 5.** Let $0 = \tau_0 < \tau_1 < ... < \tau_m$ be arbitrary fixed real numbers and boundary conditions (21) be P-regular. If $A(x), K(x) \in C^{k+2}[0, 1]$, then the considered problem possesses the increasing smoothness property of classical solutions $U(x, t)$ up to the order $k$ for arbitrary initial data $\bar{U}(x, t) \in C^1(\Gamma)$ satisfying the compatibility conditions, and any functions $\Phi^r_k(\xi) \in C^1[0, \tau_k]$ ($r = 0, 1, k = ...$).
Theorem 5 is formulated for classical solutions, and classical initial data does not satisfy the compatibility conditions. But the theorem stays valid also for piecewise smooth solutions which appear if \( u \) conditions given on the lateral sides of matrices \( u \). So, the boundary conditions (26) for problem (25) are regular if entries of 
\[ \Delta(\lambda) = 1 - e^{-\lambda}(\gamma_1 + \gamma_2 + \beta_1 e^{f_0 a(\xi)} d\xi + \alpha_2 e^{f_0 a(\xi)} d\xi) + e^{-2\lambda}(\beta_1 \gamma_2 e^{f_0 a(\xi)} d\xi + \alpha_2 \gamma_1 e^{f_0 a(\xi)} d\xi + \gamma_1 \gamma_2 + (\alpha_2 \beta_1 - \alpha_1 \beta_2) e^{f_0 a(\xi)} d\xi) d\xi, \]
So, the boundary conditions (26) for problem (25) are regular if entries of matrices \( A_0, B_0, A_1, B_1 \) satisfy the relations 
\[ \gamma_1 + \gamma_2 + \beta_1 e^{f_0 a(\xi)} d\xi + \alpha_2 e^{f_0 a(\xi)} d\xi = 0, \]
\[ \beta_1 \gamma_2 e^{f_0 a(\xi)} d\xi + \alpha_2 \gamma_1 e^{f_0 a(\xi)} d\xi + \gamma_1 \gamma_2 + (\alpha_2 \beta_1 - \alpha_1 \beta_2) e^{f_0 a(\xi)} d\xi = 0. \]

3. Initial-boundary value problems for wave equation

The above results for hyperbolic systems allow to define a class of boundary conditions given on the lateral sides of \( \Pi \) for the wave equation. These conditions are necessary and sufficient conditions for every solution of 
\[ u_{tt} - a^2 u_{xx} = f(x, t), \quad (x, t) \in \Pi, \quad (a > 0), \]
\[ u|_{t=0} = u_0(x), \quad u|_{t=0} = u_1(x) \]
to be a function of \( C^k[0, 1] \) as \( t \) grows, if \( u_0(x) \in C^3[0, 1], u_1(x) \in C^2[0, 1], f(x, t) \in C^\infty(\Pi) \), where \( k \) is an arbitrary natural number [15].
For the wave equation (27) on the lateral sides of II we set the boundary conditions
\begin{align}
(28) \quad & u_t - \alpha(u_t - au_x)\big|_{x=1} = 0, \quad u_t - au_x - \beta u\big|_{x=0} = 0 \\
(29) \quad & u_t - \alpha(u_t + au_x)\big|_{x=0} = 0, \quad u_t + au_x - \beta u\big|_{x=1} = 0.
\end{align}

By means of reduction of wave equation (27) to first-order hyperbolic system
\[
\begin{align*}
\alpha u_t &= v, \\
\beta u_t &= f(x, t)
\end{align*}
\]

in [15] it is shown that the initial-boundary value problems (27), (28) and (27), (29) are well-posed in the class of continuously differentiable functions if the initial data \( u_0, u_1 \) satisfies to the natural compatibility conditions. If the compatibility conditions are not fulfilled, then considered problems have piecewise smooth solutions for which, with use the theorem 3, the following theorem is proved:

**Theorem 6.** Let \( f(x, t) \) be \( k + 2 \) times continuously differentiable in \( \Pi \) function; \( k \geq 0 \). Then \( \alpha \beta = 0 \) is the necessary and sufficient condition for the problems (27), (28) and (27), (29) to possess the increasing smoothness property of PSS \( u(x, t) \) up to the order \( k \) for arbitrary initial data \( u_0(x) \in C^3[0, 1], u_1(x) \in C^2[0, 1] \). The solution \( u(x, t) \) for \( t > T(k) \) satisfies the estimate
\[
\|D^\alpha_{x,t} u(x, t)\|_{C[0, 1]} \leq K(t)(\max(\|u_0(x)\|_{W^2_1(0, 1)}, \|u_1(x)\|_{L^2(0, 1)}) + \|f(x, t)\|_{C^k([0, 1] \times [0, 1])}),
\]
where \( \alpha + \beta \leq k \), the constant \( K(t) \) is independent of \( u_0, u_1, f \) and depends on the coefficients of the problem and \( t \).

**Remark 5.** If \( f(x, t) \in C^\infty(\Pi) \) and \( \alpha \beta = 0 \), then the considered problems possess the increasing smoothness property of PSS up to any order \( k \), \( k \geq 0 \).

The hyperbolic equation
\[
(30) \quad u_{tt} - u_{xx} + 2b(x)u_t + 2c(x) u_x + d(x) u = f(x, t)
\]
can be written as a hyperbolic system of two equations
\[
Z_t + KZ_x = A(x)Z + F(x, t),
\]
where
\[
Z = (u, (b - c)u + u_x + u_t)^T, \quad F(x, t) = (0, f(x, t))^T,
\]
\[
K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A(x) = \begin{pmatrix} c - b & 1 \\ q & -(b + c) \end{pmatrix}, \quad q = (c - b)x + b^2 - c^2 - d.
\]

So, for the obtained system we have the initial-boundary value problem with splitted boundary conditions. By the properties of such systems, for the equation (30) the initial-boundary value problem in the half-strip II with boundary conditions
\[
\begin{align*}
& u\big|_{x=0} = 0, \quad u_t + u_x - \gamma u\big|_{x=1} = 0, \quad \gamma \text{ is an arbitrary number},
\end{align*}
\]
and the initial data
\[
\begin{align*}
& u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x)
\end{align*}
\]
is well-posed in the class of continuously differentiable on \( \Pi \) functions. By the theorem 3, this problem possesses the property of increasing smoothness up to the order \( k \), determined by the smoothness of the function \( f(x, t) \) and coefficients of the equation.
References


