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SEMIGROUPS SATISFYING P-CONDITION AND TOPOLOGICAL SELF-SIMILAR SETS

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Abstract. We give a definition of self-similar sets, which works in Hausdorff topological spaces and prove topological version of Hutchinson theorem without the assumption of completeness of the space.

Keywords: self-similar sets, fractal, self-similar structure, Hausdorff topological space.

Classical definition of self-similar sets originally proposed by J. Hutchinson [3] (1981) was formulated for contraction maps in complete metric spaces. The attempts to extend this construction to topological spaces were made by many authors. A. Kamayama [4](1993) introduced and investigated self-similar symbolic spaces defined as quotient spaces of the abstract Cantor set I^∞, W. J. Charatonik and A. Dilks [1] (1994) studied four types of self-homeomorpfic spaces, L. Bartholdi, R. Grigorchuk and V. Nekrashevych [2] (2001) considered self-similar structures generated by finite systems of injective continuous maps. Our approach is based on defining a class of semigroups of injective continuous mappings of Hausdorff topological space X to itself, possessing the properties similar to those of semigroups of contractions in complete metric spaces. Resulting definition of self-similarity is close to the one used by J.Kigami [5](1993), but being based on semigroup action, it has certain advantages.

All the above-cited definitions of self-similarity structure on compact set K are intrinsic. But in the metric case, self-similar set usually arise in connection with the ambient space X, as attractors of discrete dynamical system, generated by Hutchinson operator in the hyperspace $C(X)$. We propose an analogue of this

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construction for topological self-similar structures. The condition (P) which we formulate allows to prove Hutchinson theorem for general Hausdorff topological spaces, while the classical proof of the theorem was essentially metric and based on the completeness of the space.

1. Notation. By $C(X)$ we denote the space of all nonempty compact subsets of topological space X with Vietoris topology. By I we denote a set $\{1,...,m\}$, and I^* is the set of all finite words $i_1i_2...i_n$ where $i_j \in I$. By I^∞ we denote the set of all infinite sequences $i_1i_2...i_n...$ where $i_j \in I$, endowed with a topology of infinite product of discrete finite sets. Thus the space I^∞ is compact; it is also called the abstract Cantor set on m symbols. The maps $\sigma_k : I^\infty \to I^\infty$ defined by $\sigma_k(i_1i_2...i_n...) = ki_1i_2...i_n...$ are continuous injections satisfying $I^\infty = \bigcup\limits_{k=1}^{m} \sigma_k(I^\infty)$. The sets $\sigma_{i_1}...\sigma_{i_n}(I^\infty)$ form a basis of topology on I^∞. By G_σ we denote the semigroup generated by $\{\sigma_1,...,\sigma_m\}$.

2. The (P) condition for semigroups.

Definition 1. *Let X be Hausdorff topological space, and G be a semigroup of injective continuous maps $g : X \to X$. We say G satisfies the (P) condition, if there is such a subset $P \subset X$, that*

1. for any $g \in G$, $g(P) \subset P$;

2. for any sequence $g_1, g_2, .. \in G$ the sequence $g_1 \cdot g_2 \cdot ... \cdot g_n(P)$ converges to a one-point set;

3. for any compact set $K \subset X$ there is such m, that for any sequence $g_1, g_2, .. \in G$ the set $g_1 \cdot g_2 \cdot ... \cdot g_m(K)$ is contained in P.

For compact X, such set P is X itself. If X is a complete metric space, then a finitely generated semigroup G of injective contraction mappings of X to X satisfies the (P) condition. If X is a linear space, possibly non-metrizable, then each semigroup generated by a finite number of homotheties $h_k(\mathbf{x}) = q_k(\mathbf{x} - \mathbf{a}_k) + \mathbf{a}_k$ of X with ratios $0 < q_k < 1$ also satisfies the (P) condition.

Definition 2. *A self-similar topological structure is a pair (K, G) where K is a compact topological space and G is a finitely generated semigroup of injective continuous mappings $g : K \to K$ satisfying (P) condition and such that $G(K) = K$.*

3. Hutchinson Theorem for topological self-similar structures.

Theorem 3. *Let X be a Hausdorff topological space. Let G be a finitely generated semigroup of injective continuous mappings $g : X \to X$ with generators $\mathcal{S} = \{S_1, ..., S_m\}$ satisfying (P) condition.*

Then the operator $T : C(X) \to C(X)$, defined by the equation $T(A) = \bigcup\limits_{i=1}^{m} S_i(A)$ has unique fixed point $K \in C(X)$, and for any compact $A \subset X$, the sequence $T^n(A)$ has a topological limit K.

Moreover, the set K is the only compact subset in X satisfying $G(K) = K$.

Proof. The (P) condition implies that for any sequence of indices $i_1, i_2, ...$ and for any k, $S_{i_1...i_{k+1}}(P) \subset S_{i_1...i_k}(P)$. So, for any $\mathbf{i} = i_1i_2... \in I^\infty$, the sets $S_{i_1...i_k}(P)$ satisfy the inclusions $S_{i_1}(P) \supset S_{i_1i_2}(P) \supset ... \supset S_{i_1...i_k}(P) \supset ...$, so that $\bigcap\limits_{p=1}^{\infty} S_{i_1...i_p}(P) = \lim\limits_{k\to\infty} S_{i_1...i_k}(P) = \{x\}$.

The last equality uniquely determines the index map $\pi(\mathbf{i}) = x$, compatible with the action of semigroups G_σ and G. Denote the set $\pi(I^\infty)$ by K. Since for any $\mathbf{i} \in I^\infty$ the point $\pi(x)$ lies in P, the set K is contained in P.

Take a point $\mathbf{i} \in I^\infty$ and $x = \pi(\mathbf{i})$ and consider the neighborhood $V(x)$ of the point x in X. There is a number k, that $S_{i_1...i_k}(P) \subset V(x)$. But $S_{i_1...i_k}(K) \subset S_{i_1...i_k}(P)$, therefore $\pi^{-1}(V(X)) \supset \sigma_{i_1...i_k}(I^\infty)$. The latter set is the neighborhood of the point $\mathbf{i}$, and this shows that the map π is continuous. Therefore the set K is compact as the continuous image of the compact space I^∞. Evidently,

$T(K) = T(\pi(I^\infty)) = \bigcup\limits_{i=1}^{m} S_i(\pi(I^\infty)) = \pi(\bigcup\limits_{i=1}^{m} \sigma_i(I^\infty)) = \pi(I^\infty) = K.$

For any $x \in K$ there is such a sequence of indices $i_1 i_2 i_3 ...$, that $\bigcap\limits_{p=1}^{\infty} S_{i_1...i_p}(K) = \{x\}$. By the (P) property, for any nonempty compact $A \subset X$, the sequence $S_{i_1...i_p}(A)$ converges to $\{x\}$. Therefore, $x \in \liminf\limits_{n\to\infty} T^n(A)$.

Let $x \notin K$ and suppose, that for some compact A, $x \in \limsup\limits_{n\to\infty} T^n(A)$. Then there is such sequence $\mathbf{w}^n = w_1^n w_2^n ... w_{k_n}^n$ in I^*, that for any neighborhood $V(x)$ of x there is such N, that for $n > N$, $S_{\mathbf{w}^n}(A) \cap V \neq \emptyset$. Since the set of indices is finite, we can perform the following diagonal algorithm:

Step 1. Since the set of indices $\{w_1^n\}$ is finite, one of these indices is repeated infinitely; take a subsequence (denoted in the same way — $\mathbf{w}^n$), for which all the indices w_1^n take the same value, which we'll denote by j_1.

Step p. Suppose we have chosen such sequence that for $k < p$, $w_k^n = w_k^k = j_k$ for $n \geq k$. The set of values taken by the indices w_p^n, for $n \geq p$, is finite; leaving first $p-1$ terms unchanged, and taking into account that the set of indices $\{w_p^n\}$ is finite, take a subsequence $\mathbf{w}^n$ (denoted in the same way as $\mathbf{w}^n$) for which $w_p^n = w_p^p = j_p$ while $n \geq p$.

Denote $j_1 j_2 ... = \mathbf{j}$. Let m be such number, that for any $g_1, ..., g_m \in G$, we have $g_1 ... g_m(A) \subset P$. Without loss of generality we may suppose that the words $\mathbf{w}^1, \mathbf{w}^2, ...$ are chosen so that the length of $\mathbf{w}^k$ is larger than $k + m$.

Then for any k, $S_{j_1...j_k}(P) \supset S_{\mathbf{w}^p}(A)$. According to (P) condition $\bigcap\limits_{p=1}^{\infty} S_{j_1...j_p}(P)$ is a one-point set; since $x \in \limsup\limits_{p\to\infty} S_{j_1...j_p}(X)$, this one-point set is $\{x\}$. Therefore $x \in K$. The contradiction shows that $\limsup\limits_{n\to\infty} T^n(A) = \liminf\limits_{n\to\infty} T^n(A) = K$. ■

4. An example of a semigroup G satisfying the (P) condition in non-metrizable topological space.

Let K be the middle-third Cantor set. Let L be the set of pairs $\{(x, y) : x, y, \in K,\ x < y\}$, endowed with the discrete topology. Let M be the sum of topological spaces K and $L \times [0, 1]$, the last being the product of a discrete space and unit segment.

Consider the equivalence relation R on M, for which the equivalence class of each $x \in K$ consists of all elements of the form $(x, y, 0) \in K \times [0, 1]$, for $y > x$ and of all elements of the form $(y, x, 1) \in L \times [0, 1]$ for $y < x$. For each point $(x, y, t) \in L \times [0, 1]$ it's class consists of this single point.

Let $\tilde{K} = M/R$, and let $\pi : M \to \tilde{K}$ be the canonical projection. The restriction of π to the set K and to each subset $\{x, y\} \times [0, 1]$ is an embedding. For each pair $(x, y) \in L$ denote $J_{xy} = \pi(\{x, y\} \times [0, 1])$.

We define the topology $\mathcal{T}$ on the space $\tilde{K}$ as follows: a subset $A \subset \tilde{K}$ is open if for any $(x, y) \in L$ the set $\pi^{-1}(A) \bigcap (\{x, y\} \times [0, 1])$ is open in $\{x, y\} \times [0, 1]$, $\pi^{-1}(A) \bigcap K$ is open in K and for each point $z \in \pi^{-1}(A) \bigcap K$ there exists a neighborhood V of the point z in K, contained in $\pi^{-1}(A) \bigcap K$, that for all pairs (x, y) for which both points x and y lie in $V(z)$, the set $\{x, y\} \times [0, 1]$ is contained in $\pi^{-1}(A)$.

Such a topology $\mathcal{T}$ on the space $\tilde{K}$ is weaker than the quotient topology of the space M/R, but the restrictions of $\pi : K \to \pi(K)$ and $\pi : \{x, y\} \times [0, 1] \to I_{xy}$ are homeomorphisms; so further we will identify K and $\pi(K)$, as well as $\{x, y\} \times [0, 1]$ and J_{xy}.

Since for every point $x \in K$ the set of segments J_{xy} with the endpoint x is infinite, in the topology $\mathcal{T}$ each of the points $z \in K$ does not have a countable fundamental system of neighborhoods, and K is a set of points in $\tilde{K}$, possessing non-metrizable neighborhoods. Therefore, *the topology $\mathcal{T}$ on the set $\tilde{K}$ is non-metrizable.*

For every injective mapping $f : K \to K$ define a mapping $\tilde{f} : \tilde{K} \to \tilde{K}$, putting $\tilde{f}(\pi(x)) = \pi(f(x))$ for $x \in K$ and $\tilde{f}(\pi((x, y, t))) = \pi((f(x), f(y), t))$ for the elements of the set $\pi(L \times [0, 1])$.

Let $S_1(x) = \frac{x}{3}$ and $S_2(x) = \frac{x}{3} + \frac{2}{3}$, be the mappings determining the self-similar structure on K.

The semigroup $\widetilde{G}$, generated by the maps $\tilde{S}_1$ and $\tilde{S}_2$, satisfies the (P) condition. First, for each element $\tilde{g} \in \widetilde{G}$, $\tilde{g} = \tilde{S}_{i_1} \tilde{S}_{i_2} ... \tilde{S}_{i_k}$, it's action on $\tilde{K}$ is defined by the equality $\tilde{g}(\pi(z)) = \pi \cdot g(z)$, if $z \in \pi(K)$, where $g = S_1 ... S_{i_k}$, and by $\tilde{g}(\pi(x, y, t)) = \pi((g(x), g(y), t))$, for all other points.

Let $g_1, ..., g_n, ...$ be some sequence of elements of the semigroup G. There is a point $z \in K$ for which $\lim\limits_{n\to\infty} g_1 ... g_n(K) = \{z\}$. We show that $\lim\limits_{n\to\infty} \tilde{g}_1 ... \tilde{g}_n(\tilde{K}) = \{z\}$.

Let U be the neighborhood of the point $\pi(z)$ in $\tilde{K}$. There is such a neighborhood V of the point z in K, that $V \subset U$ and for each $x,\ y \in V$ $J_{xy} \subset U$. If $g_1 ... g_n(K) \subset V$, then $\tilde{g}_1 ... \tilde{g}_n(\tilde{K}) \subset U$. Therefore $\lim\limits_{n\to\infty} g_1 .. g_n(\tilde{K}) = \{z\}$. This shows that the semigroup $\widetilde{G}$ satisfies the (P) condition.

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