S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 7, стр. 476–479 (2010) УДК 517.5
MSC 26D05, 30D15

SHORT NOTE ON BERNSTEIN'S INEQUALITY

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Abstract. The famous Bernstein's inequality estimates the absolute value of a polynomial's derivative on the unit circle via the maximum absolute value of that polynomial over the circle. In this paper, we prove an explicit formula for increment of a polynomial along a ray, which allows to replace the maximum of absolute value over the unit circle by the maximum through the vertices of an inscribed regular polygon. As a consequence, a new proof of a discrete variant of Bernstein's polynomial inequality is given.

Keywords: Polynomials, Bernstein's inequality, Growth.

1. Introduction

Let P_n be the class of polynomials $P(z) = \sum_{v=0}^{n} a_v z^v$ of degree at most n. A famous result of S. Bernstein (for ref. see [2]) states that if $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \le n \cdot \max_{|z|=1} |P(z)|. \tag{1}$$

The equality in (1) holds for $P(z) = \alpha z^n$, $\alpha \neq 0$.

In this short note we give a new proof of the following refinement of inequality (1) obtained earlier in ([3], [4]).

Theorem 1. *If $P \in P_n$, then for an arbitrary real φ*

$$\max_{|z|=1} |P'(z)| \le n \cdot \max_{1\le k\le 2n} |P(e^{i\varphi} e^{k\pi i/n})|. \tag{2}$$

The proof of (2) will be a mere corollary from the more fundamental equality presented at the following theorem.

Baba, S.A., Wani, A.L., Short note on Bernstein's Inequality.
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Received November, 19, 2010, published December, 22, 2010.

Theorem 2. *If $P \in P_n$, then for all real γ and $R > r \geq 0$,*

$$e^{i\gamma}\left\{P(Re^{i\theta}) - P(re^{i\theta})\right\} = \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k(R,r,\gamma)P(e^{i(\theta+(k\pi+\gamma)/n)}), \tag{3}$$

with coefficients

$$A_k(R,r,\gamma) = R^n - r^n + 2\sum_{j=1}^{n-1}(R^{n-j} - r^{n-j})\cos\left(j\frac{k\pi+\gamma}{n}\right) \tag{4}$$

satisfy the equality

$$\frac{1}{2n}\sum_{k=1}^{2n} A_k(R,r,\gamma) = R^n - r^n. \tag{5}$$

Besides, if $r \geq 1$ then $A_k(R,r,\gamma)$ are all real and non-negative.

Proof. Let $P(z) = \sum_{v=0}^{n} a_v z^n$. Using the equalities

$$\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k e^{mk\pi i/n} = \begin{cases} 1 & \text{if } m = (2s+1)n \\ \frac{-e^{m\pi i/n}}{2n}\cdot\frac{e^{2nm\pi i/n}-1}{-e^{m\pi i/n}-1} = 0 & \text{otherwise} \end{cases}, \tag{6}$$

$$\cos\left(j\frac{k\pi+\gamma}{n}\right) = \frac{1}{2}e^{ij(k\pi+\gamma)/n} + \frac{1}{2}e^{-ij(k\pi+\gamma)/n},$$

and substituting for $A_k(R,r,\gamma)$ and $P(e^{i(\theta+(k\pi+\gamma)/n)})$, we obtain

$$\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k(R,r,\gamma)P(e^{i(\theta+(k\pi+\gamma)/n)})$$

$$= (R^n - r^n)\frac{1}{2n}\sum_{k=1}^{2n}\sum_{v=0}^{n}(-1)^k a_v e^{iv(\theta+(k\pi+\gamma)/n)}$$

$$+\frac{1}{n}\sum_{k=1}^{2n}\sum_{j=1}^{n-1}\sum_{v=0}^{n}(-1)^k a_v(R^{n-j} - r^{n-j})\cos\left(j\frac{k\pi+\gamma}{n}\right)e^{iv(\theta+(k\pi+\gamma)/n)}$$

$$= (R^n - r^n)a_n e^{in\theta+i\gamma} + \frac{1}{2n}\sum_{j=1}^{n-1}\sum_{v=0}^{n}\sum_{k=1}^{2n} a_v(R^{n-j} - r^{n-j})e^{iv\theta+i(v+j)\gamma/n}e^{(n+v+j)k\pi i/n}$$

$$+\frac{1}{2n}\sum_{j=1}^{n-1}\sum_{v=0}^{n}\sum_{k=1}^{2n} a_v(R^{n-j} - r^{n-1})e^{iv\theta+i(v-j)\gamma/n}e^{(n+v-j)k\pi i/n}.$$

Since $1 \leq v+j \leq 2n-1$ we have $s = 0$ in (6), so

$$\frac{1}{2n}\sum_{k=1}^{2n} e^{(n+v+j)k\pi i/n} = \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k e^{(v+j)k\pi i/n} = \begin{cases} 1 & \text{if } v+j = n \\ 0 & \text{otherwise} \end{cases}.$$

Since $-n+1 \leq v-j \leq n-1$ we have by (6)

$$\frac{1}{2n}\sum_{k=1}^{2n} e^{(n+v-j)k\pi i/n} = \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k e^{(v-j)k\pi i/n} = 0.$$

Therefore,

$$\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k(R,r,\gamma)P(e^{i(\theta+(k\pi+\gamma)/n)})$$

$$= (R^n - r^n)a_n e^{in\theta+i\gamma} + \sum_{v=1}^{n-1}(R^v - r^v)a_v e^{iv\theta+i\gamma} = e^{i\gamma}\{P(Re^{i\theta}) - P(re^{i\theta})\}.$$

This proves (3).

In order to prove the identity (5), we may substitute $P(e^{i\theta}) = e^{in\theta}$ in (4).

The property $A_k(R, r, \gamma) \geq 0$ follows from a result of Rogosinski and Szego [1, p. 75], according to which for any real θ

$$\alpha_0 + 2\sum_{j=1}^{n} \alpha_j \cos(j\theta) \geq 0,$$

if $\alpha_n \geq 0$, $\alpha_{n-1} - 2\alpha_n \geq 0$ and $\alpha_{j-1} - 2\alpha_j + \alpha_{j+1} \geq 0$ for $1 \leq j \leq n-1$. Since in (4) we have $\alpha_j = R^{n-j} - r^{n-j}$ $(j = 0, 1, ..., n)$ the conditions for α_j are fulfilled. The theorem is proved.

Proof of Theorem 1. Let θ and φ be arbitrary real numbers, then choosing $\gamma = -n\theta + \varphi n$ in Theorem 2, we get

$$|P(Re^{i\theta}) - P(re^{i\theta})| \leq \frac{1}{2n}\left\{\sum_{k=1}^{2n} A_k(R, r, -n\theta + n\varphi)\right\} \max_{1\leq k\leq 2n} |P(e^{i\varphi}e^{k\pi i/n})|,$$

which in conjunction with (5), gives

$$|P(Re^{i\theta}) - P(re^{i\theta})| \leq (R^n - r^n) \max_{1\leq k\leq 2n} |P(e^{i\varphi}e^{k\pi i/n})|. \tag{7}$$

Dividing both sides of (7) by $R - r$ and letting $R \to r$, we get the desired inequality (2). The theorem is proved.

Following [3] we remark that in Theorem 1 the maximum in the $(n+m)$th roots of unity with $m < n$ does not suffice. Indeed, let $\varepsilon > 0$ and $\varphi = 0$. Consider the polynomial $P_\varepsilon(z) = z^n - \varepsilon z^{(n-m)} + \varepsilon$, $1 \leq m < n$. It is easy to check that $\max_{1\leq k\leq n+m} |P_\varepsilon(e^{2k\pi i/(n+m)})| \leq (1 + 4\varepsilon^2)^{1/2}$, whereas $\max_{|z|=1} P'_\varepsilon(z) = n + (n-m)\varepsilon$. Hence $\max_{|z|=1} P'_\varepsilon(z) > \max_{1\leq k\leq n+m} |P_\varepsilon(e^{2k\pi i/(n+m)})|$ for sufficiently small $\varepsilon > 0$. Also, by [4] the upper bound in Theorem 1 is attained if and only if $P(z) = \alpha z^n$, $\alpha \neq 0$.

References

[1] W. Rogosinski and G. Szego, *Über die Abschnitte von Potenzreihen die in einen Kreise beschränkt bleiben* , Math. Z., **28** (1928), 73–94.

[2] A.C. Schaeffer, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc., **47** (1941), 565–579.

[3] C.Frappier, Q.I. Rahmau, St. Ruscheweyh, *New inequalities for polynomials*, Trans. Amer. Math. Society, **288** (1985), 69–99.

[4] D. Dryanov, R. Fournier, *On discrete vatriant of Bernstein inequality*, J. Aprox. Theory, **136** (2005), 84–90.

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