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CHARACTERIZATIONS OF SPACES WITH σ-LOCALLY COUNTABLE sn-NETWORKS

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Abstract. The concept of a σ-locally countable mapping is used for establishing relations between metric spaces and spaces with σ-locally countable sn-networks (cs^*-networks, weak bases).

Keywords: cs^*-network, sn-network, weak base, σ-locally countable mapping, sequence-covering mapping

1. Introduction

The general idea of establishing relations between topological spaces and metric spaces by means of various mappings is due to Alexandroff [1]. In [19], the concept of a σ-locally countable mapping was introduced and employed for establishing relations between metric spaces and spaces with σ-locally countable networks (k-networks, cs-networks, bases). The authors of [6, 7, 8, 12, 13, 15] ([3, 11, 12, 18, 20]) succeeded in studying spaces with various sn-networks (weak bases). In this paper, relations between metric spaces and spaces with σ-locally countable sn-networks (cs^*-networks, weak bases) are established by means of σ-locally countable mappings. It is also shown that σ-locally countable mappings provide an efficient tool for studying spaces with σ-locally countable collections.

Definition 1.1. Let $\mathcal{P}$ be a cover of a space X.

(1) The cover $\mathcal{P}$ is called a cs-network [10] if, for every sequence $\{x_n\}$ convergent to a point $x \in X$ and every neighborhood U of x, we have $\{x\} \cup \{x_n : n \geq m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$.

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(2) The cover $\mathcal{P}$ is called a cs^*-network [5] if, for every sequence $\{x_n\}$ convergent to a point $x \in X$ and every neighborhood U of x, we have $\{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ and some $P \in \mathcal{P}$.

Definition 1.2 ([4])**.** Let X be a space.

(1) Let $x \in P \subset X$. The set P is called a sequential neighborhood of x in X if, for every sequence $\{x_n\}$ convergent to x, we have $\{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$.

(2) Let $P \subset X$. The set P is called a sequentially open subset of X if P is a sequential neighborhood of x for every $x \in P$.

(3) The space X is called a sequential space if every sequentially open subset of X is open.

Definition 1.3. Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X such that, for each $x \in X$,

(1) $\mathcal{P}_x$ is a network of x in X, i.e., $x \in \cap\mathcal{P}_x$ and, given an open subset U of X with $x \in U$, we have $P \subset U$ for some $P \in \mathcal{P}_x$;

(2) if $P_1, P_2 \in \mathcal{P}_x$ then there exists $P \in \mathcal{P}_x$ such that $P \subset P_1 \cap P_2$.

Say that $\mathcal{P}$ is a weak base [2] for X if, for every $U \subset X$, U is open in X if and only if for each $x \in U$ there exists $P \in \mathcal{P}_x$ such that $P \subset U$.

Whenever the above holds, $\mathcal{P}_x$ is called a weak neighborhood base of x in X. If each $\mathcal{P}_x$, $x \in X$, is countable, the space X is called g-first countable. Say that $\mathcal{P}$ is an sn-network [12, 13] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x for every $x \in X$. If the latter holds, $\mathcal{P}_x$ is called an sn-network of x in X.

Definition 1.4. Let $f : X \to Y$ be a mapping.

(1) The mapping f is called a sequence-covering mapping [9] if each convergent sequence of Y is the image of some compact subset of X.

(2) The mapping f is called a 1-sequence-covering mapping [12] if, for each $y \in Y$, there is $x \in f^{-1}(y)$ such that for every sequence $\{y_n\}$ convergent to y in Y there exists a sequence $\{x_n\}$ converging to x in X with $x_n \in f^{-1}(y_n)$.

(3) The mapping f is weakly open [20] if there is a weak base $\mathcal{B} = \cup\{\mathcal{B}_y : y \in Y\}$ for Y and for each $y \in Y$ there is $x(y) \in f^{-1}(y)$ such that every open neighborhood U of $x(y)$ in X satisfies $B_y \subset f(U)$ for some $B_y \in \mathcal{B}_y$.

(4) The mapping f is called a σ-locally countable mapping [19] if there exists a base $\mathcal{B}$ for X such that $f(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$ is σ-locally countable in Y.

It is easy to check that every weakly open mapping is a quotient mapping.

In this paper, all spaces are regular T_1, and all mappings are continuous and surjective. The symbol $\mathbb{N}$ denotes the set of naturals.

2. Spaces with σ-locally countable cs^*-networks

Lemma 2.1 ([14])**.** *Let $\mathcal{P}$ be a point-countable cs^*-network for X. Given a convergent sequence $\{x\} \cup \{x_n\}$ in X, put $K = \{x\} \cup \{x_n : n \in \mathbb{N}\}$. If U is an open neighborhood of K then there exists a finite subcollection $\mathcal{F}$ of $\mathcal{P}$ subject to the following property denoted by $F(K, U)$:*

(i) $K \subset \cup\mathcal{F} \subset U$;

(ii) for each $P \in \mathcal{F}$ we have $P \cap K \neq \varnothing$; if P contains a subsequence of $\{x_n\}$ then $x \in P$.

Theorem 2.2. *A space X possesses a σ-locally countable cs^*-network if and only if X is a sequence-covering σ-locally countable image of a metric space.*

Proof. Sufficiency. Suppose that X is a sequence-covering σ-locally countable image of a metric space M under f. Let $\mathcal{B}$ be a base of M such that $f(\mathcal{B})$ is a σ-locally countable collection in X. Then it is easy to check that $f(\mathcal{B})$ is a cs^*-network for X.

Necessity. Let $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ be a σ-locally countable cs^*-network for X. We may assume that each $\mathcal{P}_n$ is closed under finite intersections and $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$. Let $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$. Put

$$M = \Big\{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x(\alpha) \text{ in } X\Big\}$$

and endow M with the subspace topology induced from the usual product topology of the discrete spaces A_n. Then M is a metric space. Define $f : M \to X$ by putting $f(\alpha) = x(\alpha)$.

1) The mapping f is continuous and surjective.

Since $\mathcal{P}$ is a point countable network of X, it is easy to check that f is continuous and surjective.

2) The mapping f is σ-locally countable.

For each $n \in \mathbb{N}$ and $\alpha_n \in A_n$ put

$$U(\alpha_1, \dots, \alpha_n) = \big\{\gamma \in M : \text{ the } i\,\text{th coordinate of } \gamma \text{ is } \alpha_i \text{ for } i \le n\big\}.$$

It is easy to check that $\{U(\alpha_1, \dots, \alpha_n) : n \in \mathbb{N}\}$ is a local neighborhood base of $\alpha = (\alpha_n)$. Put

$$\mathcal{U} = \{U(\alpha_1, \dots, \alpha_n) : \alpha_i \in A_i,\ i \le n,\ n \in \mathbb{N}\}.$$

Then $\mathcal{U}$ is a base of M. We claim that $f\big(U(\alpha_1, \dots, \alpha_n)\big) = \bigcap_{i\le n} P_{\alpha_i}$. Indeed, if $\beta = (\beta_i) \in U(\alpha_1, \dots, \alpha_n)$ then $f(\beta) = \bigcap_{i\in\mathbb{N}} P_{\beta_i} \subset \bigcap_{i\le n} P_{\alpha_i}$, $f\big(U(\alpha_1, \dots, \alpha_n)\big) \subset \bigcap_{i\le n} P_{\alpha_i}$. If $z \in \bigcap_{i\le n} P_{\alpha_i}$ then there is a subcollection $\{P_{\gamma_i} : i \in \mathbb{N}\}$ of $\mathcal{P}$ such that $\gamma_i = \alpha_i$ for $i \le n$, and $\{P_{\gamma_i} : i \in \mathbb{N}\}$ forms a network of z in X. Put $\gamma = (\gamma_i)$. Then $\gamma \in U(\alpha_1, \dots, \alpha_n)$ and $z = f(\gamma) \in f\big(U(\alpha_1, \dots, \alpha_n)\big)$; hence, $\bigcap_{i\le n} P_{\alpha_i} \subset f\big(U(\alpha_1, \dots, \alpha_n)\big)$. Therefore, $f(\mathcal{U})$ is a σ-locally countable collection of X and f is a σ-locally countable mapping.

3) The mapping f is sequence-covering.

Given a convergent sequence $x_n \to x$, without loss of generality we may assume that the terms of $\{x_n\}$ are different, and $x_n \ne x$ for all $n \in \mathbb{N}$. Put $K = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ and

$$\mathcal{P}_i(K) = \big\{\mathcal{F} \subset \mathcal{P}_i : \mathcal{F} \text{ satisfies } F(K, X)\big\} \quad \text{(see Lemma 2.1)}$$

for each $i \in \mathbb{N}$. Then $|\mathcal{P}_i(K)| \le \aleph_0$. Let $\mathcal{P}_i(K) = \{\mathcal{P}_{ij} : j \in \mathbb{N}\}$ and put

$$\mathcal{P}'_n = \Big\{P \in \bigwedge_{i,j\le n} \mathcal{P}_{ij} : P \cap K \ne \varnothing\Big\} \quad (n \in \mathbb{N}).$$

Then $\mathcal{P}'_n \subset \mathcal{P}_n$ and $\mathcal{P}'_n$ satisfies $F(K, X)$. Hence there is a finite subset B_n of A_n such that $\mathcal{P}'_n = \{P_\alpha : \alpha \in B_n\}$. Put

$$L = \Big\{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} B_n : P_{\alpha_{n+1}} \subset P_{\alpha_n},\ n \in \mathbb{N}\Big\}.$$

Then it is not difficult to check that L is closed in $\prod_{n\in\mathbb{N}} A_n$. Hence L is a compact subset of $\prod_{n\in\mathbb{N}} B_n$. Given $\alpha = (\alpha_n) \in L$, put

$$K(\alpha) = K \cap \big(\cap\{P_{\alpha_n} : n \in \mathbb{N}\}\big).$$

Since $\{K \cap P_{\alpha_n} : n \in \mathbb{N}\}$ is a descending sequence of nonempty closed subsets of the compact subset K of X, we have $K(\alpha) \neq \varnothing$.

Take $y \in K(\alpha)$ and show that $\{P_{\alpha_n} : n \in \mathbb{N}\}$ forms a network of y in X. Indeed, let U be an open neighborhood of y in X.

Consider the case $y = x$. Then $\{y\} \cup \{x_n : n \geq m\} \subset U$ for some $m \in \mathbb{N}$. Put $K_1 = \{y\} \cup \{x_n : n \geq m\}$, $K_2 = K \setminus K_1$. By Lemma 2.1 there is a subcollection $\mathcal{F}'$ of $\mathcal{P}$ subject to $F(K_1, U)$. Since K_2 is finite and $K_2 \subset X \setminus \{y\}$, there is a finite subcollection $\mathcal{F}''$ of $\mathcal{P}$ such that $K_2 \subset \cup\mathcal{F}'' \subset X \setminus \{y\}$ and $P \cap K_2 \neq \varnothing$ for each $P \in \mathcal{F}''$. Put $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$. Then $\mathcal{F}$ satisfies $F(K, X)$, $\mathcal{F} = \mathcal{P}_{ij}$ for some $i, j \in \mathbb{N}$. Put $m_0 = \max\{i, j\}$. Since $y \in P_{\alpha_{m_0}} \in \mathcal{P}_{m_0}$ and $y \notin \cup\mathcal{F}''$, we have $y \in P_{\alpha_{m_0}} \subset \cup\mathcal{F}' \subset U$. Therefore, $\{P_{\alpha_n} : n \in \mathbb{N}\}$ forms a network of y in X.

Consider the case $y \neq x$. Then $y \in P \subset U \setminus (K \setminus \{y\})$ for some $P \in \mathcal{P}$. By Lemma 2.1 there is a subcollection $\mathcal{F}'$ of $\mathcal{P}$ subject to $F(K \setminus \{y\}, X \setminus \{y\})$. Put $\mathcal{F} = \mathcal{F}' \cup P$. Then $\mathcal{F}$ satisfies $F(K, X)$, $\mathcal{F} = \mathcal{P}_{ij}$ for some $i, j \in \mathbb{N}$. Put $m_0 = \max\{i, j\}$. Since $y \in P_{\alpha_{m_0}} \in \mathcal{P}_{m_0}$ and $y \notin \cup\mathcal{F}'$, we have $y \in P_{\alpha_{m_0}} \subset P \subset U$.

Therefore, $\{P_{\alpha_n} : n \in \mathbb{N}\}$ forms a network of y in X, $\alpha = (\alpha_n) \in M$, and $f(\alpha) = y \in K$.

We thus have $L \subset M$ and $f(L) \subset K$. Conversely, if $y \in K$, then there exists $P'_n \in \mathcal{P}'_n$ such that $y \in P'_n$ for each $n \in \mathbb{N}$; hence, there is $\alpha_n \in B_n$ such that $P_{\alpha_n} = \cap\{P'_i : i \leq n\}$ for each $n \in \mathbb{N}$. Put $\alpha = (\alpha_n)$. Then $\alpha \in L$ and $y \in K \cap \big(\cap\{P_{\alpha_n} : n \in \mathbb{N}\}\big)$. By the above, $\{P_{\alpha_n} : n \in \mathbb{N}\}$ forms a network of y in X, $y = f(\alpha)$. Consequently, $K \subset f(L)$. Therefore, $f(L) = K$, f is a sequence-covering mapping.

Remark 2.3. The method of constructing the metric space M is due to V. I. Ponomarev and was first used for representing a specific nonmetric space as the continuous image of a 0-dimensional metric space [17].

3. Spaces with σ-locally countable sn-networks

Lemma 3.1 ([12])**.** *Let $f : X \to Y$ be a mapping. If $\{B_n : n \in \mathbb{N}\}$ is a decreasing network of some point x and each $f(B_n)$ is a sequential neighborhood of $f(x)$ in Y then, given a sequence $y_n \to f(x)$, there exist $x_n \in f^{-1}(y_n)$ such that $x_n \to x$ in X.*

Theorem 3.2. *A space X possesses a σ-locally countable sn-network if and only if X is a* 1*-sequence-covering σ-locally countable image of a metric space.*

Proof. Sufficiency. Suppose that X is a 1-sequence-covering σ-locally countable image of a metric space M under f. Let $\mathcal{B}$ be a base of M such that $f(\mathcal{B})$ is a σ-locally countable collection. For each $x \in X$ there exists $\alpha_x \in f^{-1}(x)$ satisfying Definition 1.4 (2). Put

$$\mathcal{P}_x = \{f(B) : \alpha_x \in B \in \mathcal{B}\}, \quad \mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}.$$

It is easy to check that $\mathcal{P}$ is an sn-network for X.

Necessity. Let $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ be a σ-locally countable sn-network for X and let $\mathcal{P}_x = \bigcup_{n\in\mathbb{N}} \mathcal{P}_{nx}$ be an sn-network of x, where $\mathcal{P}_{nx} \subset \mathcal{P}_n$. We may assume that $X \in \mathcal{P}_{nx} \subset \mathcal{P}_{(n+1)\,x}$. Let $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$. Put

$$M = \Big\{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x(\alpha) \text{ in } X\Big\}$$

and endow M with the subspace topology induced from the usual product topology of the discrete spaces A_n. Then M is a metric space. Define $f : M \to X$ by putting $f(\alpha) = x(\alpha)$. By the proof of Theorem 2.2 the mapping f is continuous, surjective, and σ-locally countable. We only need to prove that f is a 1-sequence-covering mapping. For each $x \in X$ let $\mathcal{P}_{ix} = \{P_{ij} : j \in \mathbb{N}\}$ and put $P_n = \bigcap_{i,\,j\le n} P_{ij}$ for all $n \in \mathbb{N}$. Then there exist $k(n) \in \mathbb{N}$ and $P'_n \in \mathcal{P}_{k(n)x}$ such that $P'_n \subset P_n$. Since $\mathcal{P}_{nx} \subset \mathcal{P}_{(n+1)\,x}$, we assume that $k(n) < k(n+1)$, $n \in \mathbb{N}$. Put

$$P_{\gamma_j} = \begin{cases} P_{jj}\,, & 1 \le j < k(1);\\ P'_n\,, & k(n) \le j < k(n+1),\ n \in \mathbb{N}. \end{cases}$$

Then $P_{\gamma_j} \in \mathcal{P}_{j\,x}$ and $\{P_{\gamma_j} : j \in \mathbb{N}\}$ forms a network of x. Put $\gamma = (\gamma_i)$. We have $\gamma \in M$ and $\gamma \in f^{-1}(x)$. For each $n \in \mathbb{N}$ put

$$U(\gamma_1, \dots, \gamma_n) = \big\{\alpha \in M : \text{ the } i\text{th coordinate of } \alpha \text{ is } \gamma_i \text{ for } i \le n\big\}.$$

It is easy to check that $\{U(\gamma_1, \dots, \gamma_n) : n \in \mathbb{N}\}$ is a local decreasing base of $\gamma = (\gamma_i)$ and $f\big(U(\gamma_1, \dots, \gamma_n)\big) = \bigcap_{i\le n} P_{\gamma_i}$. If $x_j \to x$ in X then, due to the fact that $f\big(U(\gamma_1, \dots, \gamma_n)\big)$ is a sequential neighborhood of x, by Lemma 3.1 there exist $\beta_j \in f^{-1}(x_j)$ such that $\beta_j \to \gamma$ in M. Therefore, f is a 1-sequence-covering mapping.

4. Spaces with σ-locally countable weak bases

Theorem 4.1. *A space X possesses a σ-locally countable weak base if and only if X is a σ-locally countable weakly open image of a metric space.*

Proof. Sufficiency. Suppose that X is a σ-locally countable weakly open image of a metric space M under f. Let $\mathcal{B}$ be a base of M such that $f(\mathcal{B})$ is a σ-locally countable collection in X. Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a weak base for X satisfying Definition 1.4 (3). For each $x \in X$ there exists $\alpha_x \in f^{-1}(x)$ satisfying Definition 1.4 (3). Put

$$\mathcal{P}'_x = \{f(B) : \alpha_x \in B \in \mathcal{B}\}, \quad \mathcal{P}' = \cup\{\mathcal{P}'_x : x \in X\}.$$

Consider an open subset U of X and a point $x \in U$. There exists $\alpha_x \in f^{-1}(x) \subset f^{-1}(U)$. Since $\mathcal{B}$ is a base of M, there is $B \in \mathcal{B}$ such that $\alpha_x \in B \subset f^{-1}(U)$ and hence $x \in f(B) \subset U$, $f(B) \in \mathcal{P}'_x$. Conversely, suppose that for each $x \in U$ we have $f(B) \subset U$ for some $f(B) \in \mathcal{P}'_x$. Since f is weakly open, there exists $P_x \in \mathcal{P}_x$ such that $P_x \subset f(B) \subset U$. Consequently, U is open in X. It is easy to check that $\mathcal{P}'$ satisfies (1), (2) of Definition 1.3. Therefore, $\mathcal{P}'$ is a σ-locally countable weak base for X.

Necessity. Let $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ be a σ-locally countable weak base for X and let $\mathcal{P}_x = \bigcup_{n\in\mathbb{N}} \mathcal{P}_{nx}$ be a local weak base of x, where $\mathcal{P}_{nx} \subset \mathcal{P}_n$. We may assume that $X \in \mathcal{P}_{nx} \subset \mathcal{P}_{(n+1)\,x}$. Let $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$. Put

$$M = \Big\{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x(\alpha) \text{ in } X\Big\}$$

and endow M with the subspace topology induced from the usual product topology of the discrete spaces A_n. Then M is a metric space. Define $f : M \to X$ by putting $f(\alpha) = x(\alpha)$. By the proof of Theorem 2.2 the mapping f is continuous, surjective, and σ-locally countable. We only need to prove that f is a weakly open mapping. For each $x \in X$ let $\mathcal{P}_{ix} = \{P_{ij} : j \in \mathbb{N}\}$ and put $P_n = \bigcap_{i,\,j\le n} P_{ij}$ for all $n \in \mathbb{N}$. Then there exist $k(n) \in \mathbb{N}$ and $P'_n \in \mathcal{P}_{k(n)\,x}$ such that $P'_n \subset P_n$. Since $\mathcal{P}_{nx} \subset \mathcal{P}_{(n+1)\,x}$, we assume that $k(n) < k(n+1)$, $n \in \mathbb{N}$. Put

$$P_{\gamma_j} = \begin{cases} P_{jj}\,, & 1 \le j < k(1); \\ P'_n\,, & k(n) \le j < k(n+1),\ n \in \mathbb{N}. \end{cases}$$

Then $P_{\gamma_j} \in \mathcal{P}_{j\,x}$ and $\{P_{\gamma_j} : j \in \mathbb{N}\}$ forms a network of x. Put $\gamma = (\gamma_i)$. Then $\gamma \in M$ and $\gamma \in f^{-1}(x)$. For each $n \in \mathbb{N}$ put

$$U(\gamma_1, \dots, \gamma_n) = \big\{\alpha \in M : \text{ the } i\text{th coordinate of } \alpha \text{ is } \gamma_i \text{ for } i \le n\big\}.$$

It is easy to check that $\{U(\gamma_1, \dots, \gamma_n) : n \in \mathbb{N}\}$ is a local decreasing base of $\gamma = (\gamma_i)$ and $f\big(U(\gamma_1, \dots, \gamma_n)\big) = \bigcap_{i\le n} P_{\gamma_i}$; hence, there exists $P_{nx} \in \mathcal{P}_x$ such that $P_{nx} \subset f\big(U(\gamma_1, \dots, \gamma_n)\big)$ for each $n \in \mathbb{N}$. Given an open neighborhood V of γ in M, there is $P_{nx} \in \mathcal{P}_x$ such that $P_{nx} \subset f\big(U(\gamma_1, \dots, \gamma_n)\big) \subset f(V)$ for some $n \in \mathbb{N}$. Therefore, f is a weakly open mapping.

Lemma 4.2 ([20])**.** *Let $f : X \to Y$ be a weakly open mapping. If X is first countable then f is a* 1*-sequence-covering mapping.*

Lemma 4.3 ([12])**.** *If $\mathcal{P}$ is an sn-network for a sequential space X then $\mathcal{P}$ is a weak base for X.*

Corollary 4.4. *Given a space X, the following are equivalent:*

(1) X is a σ-locally countable weakly open image of a metric space;

(2) X is a σ-locally countable 1*-sequence-covering quotient image of a metric space;*

(3) X is a sequential space and X possesses a σ-locally countable sn-network;

(4) X possesses a σ-locally countable weak base;

(5) X is a g-first countable space and X possesses a σ-locally countable cs-network.

Proof. (1)⇒(2) follows from Lemma 4.2.

(2)⇒(3) follows from Theorem 3.2 and Theorem 6.D.2 in [16].

(3)⇒(4) follows from Lemma 4.3.

(4)⇔(5) by Theorem 3.4 in [21].

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