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ON PROJECTIVE MODELS OF THURSTON GEOMETRIES, SOME RELEVANT NOTES ON NIL ORBIFOLDS AND MANIFOLDS

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Abstract. A unified method is given for describing 3-orbifolds and 3-manifolds in terms of a projective interpretation of Thurston geometries. As an application, a new characterization is given for **Nil**-orbifolds and manifolds arising as fiber spaces over Euclidean 2-orbifolds with prescribed fundamental groups. (Theorems 3.1-3.2, see also Figure 1).

Keywords: projective spheres, Thurston geometries, **Nil**-orbifolds and manifolds.

1. Projective models

The eight homogeneous Thurston 3-geometries

$$\mathbf{E}^3,\ \mathbf{S}^3,\ \mathbf{H}^3,\ \mathbf{S}^2\times\mathbf{R},\ \mathbf{H}^2\times\mathbf{R},\ \widetilde{\mathbf{SL_2R}},\ \mathbf{Nil},\ \mathbf{Sol} \tag{1}$$

have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, \boldsymbol{V}_4, \mathbb{R})$, where a symmetric projective polarity $({}_*)$ and scalar product $\langle\, ,\, \rangle$

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$$(2) \qquad \begin{gathered} ({}_*) : \boldsymbol{V}_4 \to \mathbf{V}^4,\ \boldsymbol{u} \to \boldsymbol{u};\\ \langle\ ,\ \rangle : \boldsymbol{V}_4 \times \boldsymbol{V}_4 \to \mathbf{R} :\\ \langle \boldsymbol{u}, \boldsymbol{v}\rangle = (\boldsymbol{u}_* \boldsymbol{v}) = \langle \boldsymbol{b}^i u_i, \boldsymbol{b}^j v_j \rangle := (u_i b^{ir} \mathbf{a}_r \boldsymbol{b}^j v_j) = u_i b^{ir} \delta_r^j v_j = u_i b^{ij} v_j \end{gathered}$$

are given by the Table below (with Einstein's sum convention for the same upper and lower indices from $\{0, 1,\ 2, 3\}$). See also [3] for more details. In (2), the dual basis pair $\{\mathbf{a}_r\} \subset \mathbf{V}^4$, $\{\boldsymbol{b}^j\} \subset \boldsymbol{V}_4$, where $\mathbf{a}_r \boldsymbol{b}^j = \delta_r^j$ is the Kronecker symbol, and the polarity matrix (b^{ij})

$$(3) \qquad \boldsymbol{b}_*^i := b^{ir} \mathbf{a}_r \text{ with } \ b^{ir} = b^{ri} = \langle \boldsymbol{b}^i, \boldsymbol{b}^r \rangle$$

are defined up to projective freedom.

More precisely, $\mathbf{y} = c\mathbf{x}$ with $0 < c \in \mathbb{R}$ (or $c \in \mathbb{R} \setminus \{0\}$) defines a point $(\mathbf{x}) = (\mathbf{y})$ of $\mathcal{PS}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$ are identified). Also, $\boldsymbol{v} = \boldsymbol{u}\frac{1}{c}$ defines a plane $(\boldsymbol{u}) = (\boldsymbol{v})$ of $\mathcal{PS}^3$ (or that of $\mathcal{P}^3$). Thus $0 = \mathbf{x}\boldsymbol{u} = \mathbf{y}\boldsymbol{v}$ defines the incidence of the point $(\mathbf{x}) = (\mathbf{y})$ and the plane $(\boldsymbol{u}) = (\boldsymbol{v})$. In what follows, we denote it by $(\mathbf{x})\mathrm{I}(\boldsymbol{u})$.

Analogously, the matrices (b^{ir}) and $(c \cdot b^{ir})$ with $0 < c \in \mathbb{R}$ (or with $c \in \mathbb{R} \setminus \{0\}$) defines the same polarity and scalar product in $\mathcal{PS}^3$ (or in $\mathcal{P}^3$). The signature of the scalar product, i.e. the signs in the diagonal form of the quadratic form

$$(4) \qquad u_i b^{ij} u_j \ \text{ with Einstein's sum convention, } i, j \in \{0, 1, 2, 3\}$$

and other invariant elements in the Table define the corresponding projective metric space $\mathbf{X}$ with its transformation group $\mathbf{G}$. The later consists of linear transforms (collineations), leaving the polarity $({}_*)$ (or the scalar product $\langle\ ,\ \rangle$) invariant.

Such a linear transform $\boldsymbol{\tau}[\mathbf{T} \to (t_i^j);\ \boldsymbol{T} \to (T_s^r) = (t_i^j)^{-1}] \in \mathbf{G}$ acts on the points (vectors) by $\mathbf{T}$ and on the planes (forms) by $\boldsymbol{T}$, as well:

$$(5) \qquad \begin{gathered} \boldsymbol{\tau} : \mathbf{x} \to \mathbf{y} = \mathbf{x}\mathbf{T};\ x^i \mathbf{a}_i \to y^j \mathbf{a}_j = x^i(\mathbf{a}_i \mathbf{T}) = x^i t_i^j \mathbf{a}_j;\ \text{i.e. } y^j = x^i t_i^j;\\ \boldsymbol{u} \to \boldsymbol{v} = \boldsymbol{T}\boldsymbol{u};\ \boldsymbol{b}^r u_r \to \boldsymbol{b}^s v_s = (\boldsymbol{T}\boldsymbol{b}^r) u_r = \boldsymbol{b}^s T_s^r u_r;\ \text{i.e. } v_s = T_s^r u_r;\\ \text{so that } (\mathbf{y}\boldsymbol{v}) = (y^j \mathbf{a}_j \boldsymbol{b}^s v_s) = (x^i t_i^j \mathbf{a}_j \boldsymbol{b}^s T_s^r u_r) = (x^i t_i^j \delta_j^s T_s^r u_r) =\\ = (x^i t_i^j T_j^r u_r) = (x^i \delta_i^r u_r) = (x^i u_i) = (\mathbf{x}\boldsymbol{u}). \end{gathered}$$

That means that the form value $(\mathbf{x}\boldsymbol{u}) = (\mathbf{y}\boldsymbol{v})$ is unchanged under $\boldsymbol{\tau}$, especially the incidence $(\mathbf{x})\mathrm{I}(\boldsymbol{u})$ implies the incidence of the $\boldsymbol{\tau}$-images: $(\mathbf{y})\mathrm{I}(\boldsymbol{v})$.

We note that such a transform $\boldsymbol{\tau}$ is a similarity (moreover, an isometry), if and only if it keeps the polarity $({}_*)$. Furthermore, it is a product of reflections, i.e. involutory transforms. The matrices $(t_i^j) \sim (c \cdot t_i^j)$ and $(T_s^r) \sim (T_s^r \cdot \frac{1}{c})$, again are determined up to projective freedom $0 < c \in \mathbb{R}$ for $\mathcal{PS}^3$ (or $c \in \mathbb{R} \setminus 0$ for $\mathcal{P}^3$). For the sake of convenience, the identities $Det(t_i^j) = Det(T_s^r) = 1$ are assumed.

We emphasize advantages of using $\mathcal{PS}^3$ with the above mentioned polarity. Certain simply connected pieces of $\mathcal{PS}^3$ can be used to construct 3-manifolds and 3-orbifolds in all the eight Thurston geometries. The polarity $({}_*)$ defines the orthogonality of planes $(\boldsymbol{u}), (\boldsymbol{v})$ by $(\boldsymbol{u}_*)\mathrm{I}(\boldsymbol{v}) \Leftrightarrow (\boldsymbol{u}_* \boldsymbol{v}) = 0 = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ even in degenerate cases, i.e. in $\mathbf{E}^3$, $\mathbf{S}^2 \times \mathbf{R}$, $\mathbf{H}^2 \times \mathbf{R}$, $\mathbf{Nil}$, $\mathbf{Sol}$.

2. A strategy for explicit metric realization of orbifolds and manifolds

Now we suggest a strategy for an explicit geometric realization of 3-orbifolds and 3-manifolds given by an "adequate combinatorial way". This latter one means that our orbifold $\mathcal{O}$ can be given in the projective 3-sphere $\mathcal{PS}^3$ by a coordinate simplex with a dual basis pair $\{\mathbf{a}_i\} \subset \boldsymbol{V}^4$, $\{\boldsymbol{b}^j\} \subset \boldsymbol{V}_4$ for simplex vertices $A_0A_1A_2A_3$ and opposite faces $\beta^0, \beta^1, \beta^2, \beta^3$. We assume that the orbifold fundamental group $\mathbf{\Gamma}(\mathcal{O})$ is finitely generated and is given by some linear transforms $\boldsymbol{\gamma}_k$ (expressed in $\{\mathbf{a}_i\}$ and so in $\{\boldsymbol{b}^j\}$ on the base of piece-wise linear topology), pairing "bent" faces $\boldsymbol{\gamma}_k : \gamma_k^{-1} \to \gamma_k$ of certain fundamental polyhedron $\mathcal{F}_{\mathbf{\Gamma}} \sim \mathcal{O}$, $(k = 1, 2, \dots, g$ is numbering the generators of $\mathbf{\Gamma}$). Following [4], we represent the barycentric simplex orbits of $\mathbf{\Gamma}$ in $\mathcal{F}_{\mathbf{\Gamma}}$ by D-symbols, or by a group-graph with coloured edges for generators of $\mathbf{\Gamma}$.

We allowed that the generator matrices $\boldsymbol{\gamma}_k$ contain some unknown parameters depending on the geometry of $\mathcal{F}_{\mathbf{\Gamma}}$, as well as of simplexes which fit $\mathcal{F}_{\mathbf{\Gamma}}$ in various way (see the concrete examples in [4, 3, 5, 6]). Some of these parameters will be expressed in terms of the other ones from certain matrix equations. These equations are consequences of the defining relations for the group $\mathbf{\Gamma}$. Then, to every generator $\boldsymbol{\gamma}_k[\mathbf{C} \to (c_i^j),\ \boldsymbol{C} \to (C_s^r) = (c_i^j)^{-1}]$ we look for an invariant polarity $({}_*)$ with a matrix (b^{ir}) in (1.3), the same for each $k = 1, 2, \dots, g$. As a result, we obtain a system of linear equations for (b^{ir})

$$C_s^r b^{st} C_t^u = c_k \cdot b^{ru}, \text{or equivalently } c_i^j b^{iv} c_v^w = \overline{c}_k \cdot b^{jw} \tag{6}$$

with certain c_k and $\overline{c}_k$ for $\boldsymbol{\gamma}_k$ depending on k.

In some particular cases the only possible solution for (b^{ir}) is zero matrix; then a splitting effect may occur along some spherical ($\mathbf{S}^2$) or Euclidean ($\mathbf{E}^2$) 2-dimensional suborbifolds. We note that the solution of the Thurston geometrical conjecture, given by G. Perelman ([10], [2]) is based of the another strategy (so called Ricci flow technique, initiated by R. Hamilton [9]).

A non-zero matrix (b^{ir}) with an "isometry group" $\mathbf{\Gamma}(\boldsymbol{\gamma}_k)$, satisfying all defining relations, will determine a 3-geometry above, realizing our orbifold $\mathcal{O}$ started with.

For some delicate phenomena and precisions see e.g. [4, 6]. For instance, for non-compact orbifolds and manifolds the above mentioned approach holds for the interior of $\mathcal{O}$. In this case, certain lower dimensional topological changes can be done on the "boundary" of $\mathcal{O}$. Sometimes, a boundary point can be changed by a boundary segment, or a plane with a distinguished point of segment.

3. On the projective interpretation of Nil orbifolds and manifolds

The **Nil** geometry of the Heisenberg group can be projectively (affinely) interpreted [5, 7] by the "right translations" on the points

$$(1; a, b, c) \to (1; a, b, c) \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x + a, y + b, z + bx + c). \tag{7}$$

Table for the eight homogeneous geometries

Space $\mathbf{X}$	Signature of polarity $\Pi(_*)$ or scalar product $\langle\ ,\ \rangle$ in $\boldsymbol{V}_4$	Domain of proper points of $\mathbf{X}$ in $\mathcal{PS}^3$ $(\mathbf{V}^4(\mathbf{R}), \boldsymbol{V}_4)$	The group G = Isom $\mathbf{X}$ as a special transformation group of $\mathcal{PS}^3$
$\mathbf{S}^3$	$(++++)$	$\mathcal{PS}^3$	Coll. $\mathcal{PS}^3$ preserving $\Pi(_*)$
$\mathbf{H}^3$	$(-+++)$	$\{(\mathbf{x})\in\mathcal{P}^3 : \langle\mathbf{x},\mathbf{x}\rangle < 0\}$	Coll. $\mathcal{P}^3$ preserving $\Pi(_*)$
$\widetilde{\mathbf{SL}_2\mathbf{R}}$	$(--++)$ with skew line fibering	Universal covering of $\mathcal{H} := \{[\mathrm{x}] \in\mathcal{PS}^3 : \langle\mathbf{x},\mathbf{x}\rangle < 0\}$ by fibering transformations	Coll. $\mathcal{PS}^3$ preserving $\Pi(_*)$ and fibres
$\mathbf{E}^3$	$(0+++)$	$\mathcal{A}^3 = \mathcal{P}^3\setminus\{\omega^\infty\}$ where $\omega^\infty := (\boldsymbol{b}^0)$, $\boldsymbol{b}^0_* = \mathbf{0}$	Coll. $\mathcal{P}^3$ preserving $\Pi(_*)$ generated by plane reflections
$\mathbf{S}^2\times\mathbf{R}$	$(0+++)$ with O-line bundle fibering	$\mathcal{A}^3\setminus\{O\}$ O is a fixed origin	G is generated by plane reflections and sphere inversions, leaving invariant the O-concentric 2-spheres of $\Pi(_*)$
$\mathbf{H}^2\times\mathbf{R}$	$(0-++)$ with O-line bundle fibering	$\mathcal{C}^+ = \{X \in \mathcal{A}^3 : \langle\overrightarrow{OX}, \overrightarrow{OX}\rangle < 0$, half cone$\}$ by fibering	G is generated by plane reflections and hyperboloid inversions, leaving invariant the O concentric half-hyperboloids in the half cone $\mathcal{C}^+$ by $\Pi(_*)$
Sol	$(0-++)$ with parallel plane fibering	$\mathcal{A}^3$ with parallel plane fibering	Coll. $\mathcal{P}^3$ preserving $\Pi(_*)$ and the parallel plane fibering
Nil	$(0\ 0\ \pm0\ +)$ with parallel line bundle fibering	$\mathcal{A}^3$ with a distinguished parallel plane pencil along each fibre line	Coll. of $\mathcal{P}^3$ (•) preserving $\Pi(_*)$ and the line bundle with the plane pencil along each fibre line

(•) more precisely: these are conjugate to quadratic mappings by a fixed transform $x' = x,\ y' = y,\ z' = z - \frac{1}{2}xy$

The set $\mathbf{L}$ of translations is non-commutative. The matrices with $(x, y) = (0, 0)$ belong to the center $\mathbf{K}(z)$ of $\mathbf{L}$.

Here the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0)$,$E_1^\infty(\mathbf{e}_1)$,$E_2^\infty(\mathbf{e}_2)$, $E_3^\infty(\mathbf{e}_3)$ is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. The dual system $\{(\boldsymbol{e}^i)\}$ describes the simplex planes, especially the plane at infinity $(\boldsymbol{e}^0) = E_1^\infty E_2^\infty E_3^\infty$. Thus **Nil** can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well [5], [12].

The coordinate differentials $\mathrm{d}\overline{x}$, $\mathrm{d}\overline{y}$, $\mathrm{d}\overline{z}$ at the origin $(1; 0, 0, 0)$ can be "pull back" to an arbitrary point $(1; x, y, z)$ of **Nil** by the inverse matrix of (7). Thus, the transform

$$(0; \mathrm{d}\overline{x}, \mathrm{d}\overline{y}, \mathrm{d}\overline{z}) = (0; \mathrm{d}x, \mathrm{d}y, \mathrm{d}z) \begin{pmatrix} 1 & -x & -y & -z+xy \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

$$\text{and} \quad (\mathrm{d}s)^2 = (\mathrm{d}\overline{x})^2 + (\mathrm{d}\overline{y})^2 + (\mathrm{d}\overline{z})^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (-\mathrm{d}y \cdot x + \mathrm{d}z)^2$$

is an infinitesimal arclength-square transform. That is, the Riemann metric in **Nil** can be defined as usual [11, 13, 12].

It is an important observation [11] *that a rotation trough angle* ω *about the* z*-axis in the origin*, as isometry of **Nil**, keeping invariant the Riemann metric everywhere, will be a quadratic mapping in x, y to z-image $\overline{z}$ as follows:

$$\begin{gathered} \mathbf{r}(O,\omega) : (1; x, y, z) \to (1; \overline{x}, \overline{y}, \overline{z}); \\ \overline{x} = x\cos\omega - y\sin\omega, \quad \overline{y} = x\sin\omega - y\cos\omega, \\ \overline{z} = z - \frac{1}{2}xy + \frac{1}{4}(x^2 - y^2)\sin 2\omega + \frac{1}{2}xy\cos 2\omega. \end{gathered} \tag{9}$$

This rotation formula, however, is conjugate by the quadratic mapping

$$x \to x' = x, \quad y \to y' = y, \quad z \to z' = z - \frac{1}{2}xy \quad \text{to}$$

$$(1; x', y', z') \to (1; x', y', z') \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x", y", z"), \tag{10}$$

$$\text{with} \quad x" \to \overline{x} = x", \quad y" \to \overline{y} = y", \quad z" \to \overline{z} = z" + \frac{1}{2}x"y",$$

i.e. to the linear rotation formula. This quadratic conjugacy modifies Nil-translations in (7), as well. Thus we have verified the essential part of the following theorem.

Theorem 3.1 (**Classification**). **1**. *Any group of Nil isometries, containing a* 3*-dimensional translation lattice, is conjugate by the quadratic mapping in (10) to an affine group of the affine space* $\mathbf{A}^3$ *whose projection onto the* (x, y) *plane is an isometry group of* $\mathbf{E}^2$*. Such an affine group preserves a plane - point polarity of signature* $(0, 0, \pm 0, +)$.

2. *The reflection across the* y *axis*

$$(1; x, y, z) \to (1; -x, y, -z), \tag{11}$$

preserve the Riemann metric in (8), and its conjugates by the the same isometries as in **1.** *are also* **Nil***-isometries. There does not exist an orientation reversing* **Nil***-isometry given by (8).*

The following result can be obtained by an explicit algorithmic procedure given in section 2. Compare with ([1], [11], [13]).

Theorem 3.2. *For an arbitrary plane crystallographic group of* $\mathbf{E}^2$ *there exist infinitely many different* **Nil***-orbifolds with fundamental groups whose projections coincide with the starting plane group.*

As illustration, let the $\mathbf{E}^2$-group $\mathbf{244} = \mathbf{p4}$ be sketched here [7]. For the 4-rotations $\overline{s}_1$ (about $(0,0)$) and $\overline{s}_2$ (about $(\frac{1}{2},\frac{1}{2})$) of $\mathbf{244}$ we get

$$(12)\qquad \mathbf{s}_1: \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{s}_2: \begin{pmatrix} 1 & 1 & 0 & b \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & -1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{s} = \mathbf{s}_1\mathbf{s}_2: \begin{pmatrix} 1 & 1 & 0 & b+a \\ 0 & -1 & 0 & -\frac{1}{2} \\ 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

as **Nil**-extensions. The "screw motion" $\mathbf{s}_1$ maps the screw axis $(1;0,0,z)$ to $(1;0,0,a+z)$. Similarly, $\mathbf{s}_2$ maps $(1;\frac{1}{2},\frac{1}{2},z)$ to $(1;\frac{1}{2},\frac{1}{2},b+z)$. Their product $\mathbf{s} = \mathbf{s}_1\mathbf{s}_2$ maps $(1;\frac{1}{2},0,z)$ to $(1;\frac{1}{2},0,b+a-\frac{1}{4}+z)$. Thus $\overline{\mathbf{s}}$ is a half-turn in $\mathbf{E}^2$ about $(\frac{1}{2},0)$, but the screw component of $\mathbf{s}$ in **Nil** is not additive, as expected for.

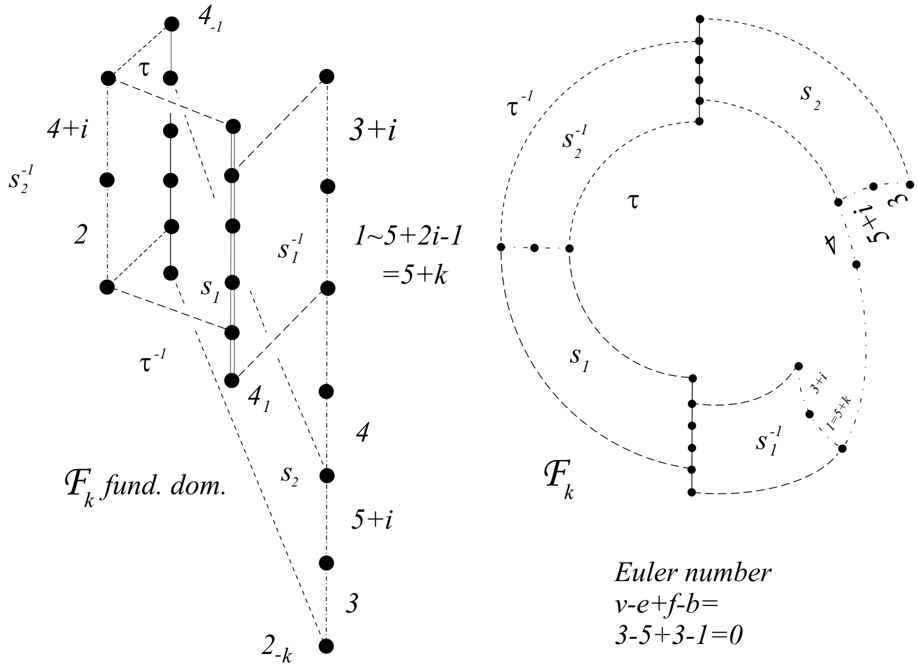

FIGURE 1. An infinite series of **Nil** manifolds $\mathbf{2_k 4_{-1} 4_1} \Leftrightarrow \mathbf{2_{-k} 4_1 4_{-1}}$ $(k = 1,3,5,7\dots)$ fibred over $\mathbf{E}^2$ plane group $\mathbf{p4} = \mathbf{244}$ by Schlegel diagrams of their fundamental domains.

Considering $\mathbf{s}_1^4,\ \mathbf{s}_2^4,\ \mathbf{s}^2$ from formula (12), we get

$$(13)\qquad 4a \equiv 4b \equiv 2b + 2a - \frac{1}{2} \equiv 0 \ (\mathrm{mod}\ \tau)$$

as Frobenius-type congruences to the fibre translation generator $\boldsymbol{\tau}(\tau)$ in the z-direction. Expressed otherwise:

$$
\begin{aligned}
& a = \frac{\tau}{4} \cdot a', \ b = \frac{\tau}{4} \cdot b' \Rightarrow \frac{\tau}{4} \cdot (a' + b') - \frac{1}{4} = \frac{\tau}{2} \cdot \lambda \\
& \Leftrightarrow \tau(a' + b' - 2\lambda) = 1 \text{ with integers } \lambda, a', b' \in \mathbb{Z} \\
& \text{for } \mathbf{s}_1, \ \mathbf{s}_2 \text{ and } \mathbf{s} \text{ and with (rational now) } \tau \in \mathbb{Q}.
\end{aligned}
\tag{14}
$$

Of course, some solutions lead to geometrically equivalent (equivariant) orbifolds and manifolds. See [7] for corresponding examples. We extract some special cases.

We start with the case $(\lambda, a', b') = (0, 0, 0), (1, 1, 1)$ having no solutions for (14). These leads to $\mathbf{E}^3$ space groups $\Gamma = \mathbf{p4}, \mathbf{p4_1}$, orbifold and manifold, respectively.

The group for $(\lambda, a', b') = (1, 0, 0), \ \tau = -\frac{1}{2}$ can be denoted by $\mathbf{2_1 44}$ and $\mathcal{O} = \mathbf{Nil}/\mathbf{2_1 44}$. The solution $(-1, 0, 0), \ \tau = \frac{1}{2}$ will be equivariant to this.

The case $(2\mu, 0, 0), \ \tau = -\frac{1}{2\mu}, \ \mu \in \mathbb{N}$ (natural numbers) lead to infinitely many non-equivariant orbifolds. Dunbar's paper [1] counts only one (for $\mu = 1$ with simply linked $2-, 4-, 4-$circles).

And, finally the case $(-k, 1, -1), \ \tau = \frac{1}{2k}, \ k = 2i - 1 \ (i = 1, 2, \dots)$ lead to infinitely many (non-homeomorphic) compact **Nil**-manifolds whose fundamental groups can be denoted by $\mathbf{2_{-k} 4_1 4_{-1}}$ (or equivalently by $\mathbf{2_k 4_{-1} 4_1}$) with first homology group $\mathbb{Z}_2 \times \mathbb{Z}_{8k}$ (see Figure 1).

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