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ON 2-GROUPS, ALL OF WHOSE FINITE SUBGROUPS ARE OF NILPOTENCY CLASS 2

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Abstract. We prove that if all finite subgroups of a 2-group G are of nilpotency class 2 then G is of nilpotency class 2.

Keywords: p-group, nilpotent group.

1. Introduction

The goal of this article is to prove the following result.

THEOREM. *If all finite subgroups of a* 2*-group G are of nilpotency class* 2 *then G is of nilpotency class* 2*.*

Obvious corollary of this theorem is that a 2-group is Abelian if and only if all of its finite subgroups are Abelian. Note that the analog even of the last assertion is not true for p-groups where $p > 2$, since for example all finite subgroups of Novikov-Adjan group (non locally finite free group of odd period) are cyclic [1]. On the other hand all finite subgroups of not nilpotent free Burnside groups of period 2^n for $n \geqslant 13$ can be embedded into direct product of dihedral groups of order 2^{n+1} [2], Theorem 2 and therefore this group is nilpotent of class n. Hence Theorem cannot be generalised for the case of 2-groups with finite subgroups of bounded nilpotency class. Nevertheless naturally arise

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QUESTIONS. 1. What is the maximal number n which guarantees nilpotency of every 2-group every finite subgroup of which is of nilpotency class n?

2. Is it true that a 2-group every finite subgroup of which has nilpotency class 3 is nilpotent?

2. Proof of Theorem

Let G be a 2-group all finite subgroups of which are of nilpotency class 2.

Lemma 1. *The order of the product of any two involutions of G is at most 4.*

Proof. If u, v are involutions from G, then $\langle u, v\rangle$ is a finite dihedral group and if 2^m is the order of uv, then nilpotency class of $\langle u, v\rangle$ is equal to m. The lemma is proved.

Lemma 2. *The group*

$$H = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^4 = (xz)^4 = (yz)^4 = \big((xy)^2 z\big)^4 = \\ = \big((xz)^2 y\big)^4 = \big((yz)^2 x\big)^4 = \big((xy)^2 (xz)^2\big)^4 = 1\rangle$$

is finite.

Proof. Calculations in GAP [3] using coset enumeration algorithm show that $|H| = 2^{19}$.

Lemma 3. *If a, b, c are involutions of G, then $\langle a, b, c\rangle$ is a finite subgroup and $[[a, b], c] = 1$.*

Proof. By Lemma 1 $\langle a, b, c\rangle$ is a homomorphic image of the group H from Lemma 2 and therefore finite. The lemma is proved.

Denote by I the (normal) subgroup, generated by all involutions of G.

Lemma 4. *The subgroup I is nilpotent of nilpotency class 2, in particular, it is locally finite.*

Proof. If a, b are involutions of G, then $[a, b]$ lies in the center $Z(I)$ of I by Lemma 3. Hence $I/Z(I)$ is Abelian. The lemma is proved.

Lemma 5. *If G is generated by three elements then it is finite.*

Proof. Suppose $G = \langle a, b, c\rangle$ and 2^m is the maximum of orders of a, b, c. Use induction on m. If $m \leqslant 1$ then $G \leqslant I$, therefore by Lemma 4 G is finite.

Let $m \geqslant 2$. If $\overline{X} = \{\overline{x_1}, \ldots, \overline{x_x}\}$ is a finite subgroup of G/I then by Schmidt's theorem the full preimage of $\overline{X}$ in G is locally finite being a finite extension of a locally finite group, hence the subgroup $X = \langle x_1, \ldots, x_s\rangle$ where $x_i I = \overline{x_i}$, $i = 1, \ldots, s$, is finite. By assumption, X is nilpotent of class 2, therefore subgroup $\overline{X} = XI/I$ which is isomorphic to $X/X \cap I$ is of nilpotency class 2.

So, G/I satisfies the assumption of Theorem and is generated by three elements aI, bI, cI. Besides, the maximum of their orders is equal to 2^{m-1}. By induction G/I is finite. Since I is locally finite by Lemma 4 and Schmidt's theorem and also finitely generated then G is finite. The lemma is proved.

Lemma 6. *The group G is nilpotent of class 2.*

Proof. Suppose $a, b, c \in G$ and $K = \langle a, b, c\rangle$. By Lemma 5 K is finite and hence nilpotent of class 2, thus $[[a, b], c] = 1$ and the lemma is proved.

Lemma 6 completes the proof of Theorem.

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