S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 8, стр. 127–158 (2011) УДК 517.958:531.3-1

MSC 35D05, 93B18, 74F10

SMALL PERTURBATIONS OF TWO-PHASE THERMOFLUID IN PORES: LINEARIZATION PROCEDURE AND EQUATIONS OF ISOTHERMAL MICROSTRUCTURE

S.A. SAZHENKOV AND E.V. SAZHENKOVA

ABSTRACT. We consider the most general dynamical model describing the joint motion of a heat-conductive elastic porous body and a two-phase heat-conductive Newtonian viscous compressible fluid. We assume that the fluid fills in the whole porous space. Since the fluid in pores and the solid composing the pore space are distinguished and at the same time we think of pores that they have very small diameters but their aggregate capacity is significant with the respect to the entire fluid-solid bulk, the considered model corresponds to microstructure.

In the present article the linearization procedure is fulfilled on a natural rest state by means of the classical formalism. On the base of the obtained linearized model, a simplified isothermal formulation is set up and the existence and uniqueness theory is built for it. The proofs are based on the classical methods in the theory of evolutionary partial differential equations.

Keywords: elastic solid, two-phase compressible viscous fluid, Rakhmatulin's scheme, linearization, existence and uniqueness theory, generalized solutions.

SAZHENKOV, S.A. AND SAZHENKOVA, E.V., SMALL PERTURBATIONS OF TWO-PHASE THERMOFLUID IN PORES: LINEARIZATION PROCEDURE AND EQUATIONS OF ISOTHERMAL MICROSTRUCTURE.

The work was partially supported by the Russian Foundation for Basic Research (grant no. 10-01-00447) and the Federal special-purpose program "Scientific and pedagogical-scientific brainpower of innovative Russia", 2009–2013 (contract no. 02.740.11.0617).

Received November, 01, 2010, Published July, 11, 2011.

Introduction

We are interested in proposing a mathematical description for small perturbations in the thermomechanical system consisting of interacting elastic porous body and two-phase viscous compressible fluid in pores. In our study we aim to integrate purely mechanical, thermodynamical, and heat transfer effects altogether under one umbrella. Within such unified approach, a thermoconductive elastic body may be named a *thermoelastic solid*. Also we can use the term *thermofluids*, which has been introduced recently for a subject that analyzes systems and processes involved in energy, various forms of energy, and transfer of energy in fluids [1, Sec. I.1].

The basic mathematical concept of reciprocal motion of thermoelastic solids and two-phase viscous thermofluids incorporates the classical conservation laws of continuum mechanics for all three phases, namely, for the two fluid phases in the liquid region (pores) and the elastic phase in the solid region (the porous skeleton), the first and the second laws of thermodynamics, a set of state laws specifying individual thermomechanical behavior of the components of media, and a set of laws defining interphase forces and interphase heat exchange in the two-phase fluid. This concept is precisely formulated in Sec. 1. It is quite universal and spans a large variety of different phenomena in nature and technology. (A rather general relevant observation may be found, for example, in [1, Secs. I.5–I.7, VI] and [2, Foreword; Ch. 1, Secs. 1,2].) At the same time this model is very complex and highly nonlinear. Therefore, some physically reasonable simplifications are necessary in view of further applications to natural problems and in engineering.

In investigation of small perturbations it is suitable to simplify the fundamental nonlinear model by implementing the classical formalism of linearization about a so-called natural rest state [3, Sec. V.7]. In the present paper the linearization procedure and some minor technical transformations are worked out in Secs. 2 and 3, and the resulting linearized formulation, named Model L_1, is outlined in Sec. 4. Derivation of this model is the first main result of the article.

Next in Sec. 5 we proceed with further reduction and set up the isothermal linearized Model IT_0. Also in Sec. 5 we give the notion of generalized solutions of initial-boundary value problem for Model IT_0.

In Secs. 6–8 following the lines of the well-known theory of generalized solutions of equations of mathematical physics, Galerkin's approximations are introduced and systematically studied, and the energy identity and inequality are established for both Galerkin's approximate solutions and any possible generalized solutions, which eventually leads to the second main result of this article – Theorem 1 on existence and uniqueness of solutions to Model IT_0.

In Appendix after Sec. 8 for convenience of reading we aggregate the list of the most frequently used notation in the article.

Since the fluid in pores and the solid composing the pore space are distinguished in both Models L_1 and IT_0, and at the same time we think of pores that they have very small diameters but their aggregate capacity is significant with the respect to the entire fluid-solid bulk, the obtained generalized solutions describe dynamics of the thermomechanical system on the micro-scale level. Models L_1 and IT_0 are the backbones for the analysis of small perturbations on macroscopic levels. More precisely, starting from the results of the present article, it becomes quite possible to build up effective models of small perturbations on macro-scales implementing homogenization procedures like in the recent works [4, 5, 6] or in the earlier works [7, 8].

1. The fundamental nonlinear formulation

1.1. Equations of the model. We suppose that at a moment $t \geq 0$ the fluid component fills in a sub-domain $\Omega_f(t) \subset \Omega$, where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is the volume occupied by the whole thermomechanical system "two-phase compressible fluid – elastic body", and obeys to the system of dynamical equations of the two-phase mixture of viscous compressible fluids, with absence of phase transitions, and with the reciprocal deformation of phases being described by Rakhmatulin's scheme. This system consists of the balance of mass equations

$$\frac{\partial(\rho_1^0 \alpha_1)}{\partial t} + \mathrm{div}_x(\rho_1^0 \alpha_1 \boldsymbol{v}_1) = 0, \tag{1.1a}$$

$$\frac{\partial(\rho_2^0 \alpha_2)}{\partial t} + \mathrm{div}_x(\rho_2^0 \alpha_2 \boldsymbol{v}_2) = 0, \tag{1.1b}$$

$$\alpha_1 + \alpha_2 = 1, \quad \alpha_1, \alpha_2 \geq 0, \tag{1.1c}$$

the balance of momentum equations

$$\rho_1^0 \alpha_1 \Big(\frac{\partial \boldsymbol{v}_1}{\partial t} + (\boldsymbol{v}_1 \cdot \nabla_x) \boldsymbol{v}_1 \Big) = -\alpha_1 \nabla_x p_1 + \mathrm{div}_x(\alpha_1 \boldsymbol{\tau}_1) + \alpha_1 \rho_1^0 \boldsymbol{g} - \boldsymbol{F}_{12}, \tag{1.1d}$$

$$\rho_2^0 \alpha_2 \Big(\frac{\partial \boldsymbol{v}_2}{\partial t} + (\boldsymbol{v}_2 \cdot \nabla_x) \boldsymbol{v}_2 \Big) = -\alpha_2 \nabla_x p_2 + \mathrm{div}_x(\alpha_2 \boldsymbol{\tau}_2) + \alpha_2 \rho_2^0 \boldsymbol{g} + \boldsymbol{F}_{12}, \tag{1.1e}$$

the balance of energy equations

$$\begin{aligned} \rho_1^0 \alpha_1 \Big(\frac{\partial U_1}{\partial t} + (\boldsymbol{v}_1 \cdot \nabla_x) U_1 \Big) = \frac{\alpha_1 p_1}{\rho_1^0} \Big(\frac{\partial \rho_1^0}{\partial t} + (\boldsymbol{v}_1 \cdot \nabla_x) \rho_1^0 \Big) + \boldsymbol{F}_{12} \cdot \boldsymbol{v}_1 - Q_{12} \\ + \alpha_1 \boldsymbol{\tau}_1 : \mathbb{D}(x, \boldsymbol{v}_1) - \mathrm{div}_x(\alpha_1 \boldsymbol{q}_1) + \alpha_1 \Psi_1, \end{aligned} \tag{1.1f}$$

$$\begin{aligned} \rho_2^0 \alpha_2 \Big(\frac{\partial U_2}{\partial t} + (\boldsymbol{v}_2 \cdot \nabla_x) U_2 \Big) = \frac{\alpha_2 p_2}{\rho_2^0} \Big(\frac{\partial \rho_2^0}{\partial t} + (\boldsymbol{v}_2 \cdot \nabla_x) \rho_2^0 \Big) - \boldsymbol{F}_{12} \cdot \boldsymbol{v}_2 + Q_{12} \\ + \alpha_2 \boldsymbol{\tau}_2 : \mathbb{D}(x, \boldsymbol{v}_2) - \mathrm{div}_x(\alpha_2 \boldsymbol{q}_2) + \alpha_2 \Psi_2, \end{aligned} \tag{1.1g}$$

the thermodynamical state equations of the phases

$$p_1 = p_1(\rho_1^0, \theta_1), \quad p_2 = p_2(\rho_2^0, \theta_2), \tag{1.1h}$$

$$U_1 = U_1(\rho_1^0, \theta_1), \quad U_2 = U_2(\rho_2^0, \theta_2), \tag{1.1i}$$

subjected to the condition of Rakhmatulin's scheme that the pressure is the same in the both liquid phases at any point [2, Ch. 1, Sec.1], [9]:

$$p_1(\rho_1^0, \theta_1) = p_2(\rho_2^0, \theta_2) = p; \tag{1.1j}$$

the Navier–Stokes law of viscous tensions $\boldsymbol{\tau}_i$

$$\boldsymbol{\tau}_1 = \Big(\nu_1 - \frac{2}{3} \mu_1 \Big) (\mathrm{div}_x \boldsymbol{v}_1) \mathbb{I} + 2\mu_1 \mathbb{D}(x, \boldsymbol{v}_1), \tag{1.1k}$$

$$\boldsymbol{\tau}_2 = \Big(\nu_2 - \frac{2}{3} \mu_2 \Big) (\mathrm{div}_x \boldsymbol{v}_2) \mathbb{I} + 2\mu_2 \mathbb{D}(x, \boldsymbol{v}_2), \tag{1.1l}$$

the Fourier law of the heat fluxes

$$\boldsymbol{q}_1 = -\kappa_1 \nabla_x \theta_1 + \frac{\beta_1}{\theta_*}(\boldsymbol{v}_1 - \boldsymbol{v}_2), \tag{1.1m}$$

$$\boldsymbol{q}_2 = -\kappa_2 \nabla_x \theta_2 + \frac{\beta_2}{\theta_*}(\boldsymbol{v}_1 - \boldsymbol{v}_2), \tag{1.1n}$$

the law of the interphase force

$$\boldsymbol{F}_{12} = -\frac{\beta_1}{\theta_*^2}\nabla_x \theta_1 - \frac{\beta_2}{\theta_*^2}\nabla_x \theta_2 + K_F(\boldsymbol{v}_1 - \boldsymbol{v}_2), \tag{1.1o}$$

and the law of the interphase heat exchange

$$Q_{12} = K_Q(\theta_1 - \theta_2). \tag{1.1p}$$

We suppose that at a moment $t \geq 0$ the solid component fills in a sub-domain $\Omega_s(t) = \Omega \setminus \overline{\Omega_f(t)}$ and satisfies the system of thermoelasticity equations [10], which consists of the equation of balance of mass

$$\rho = \rho_s(1 - \mathrm{div}_x \boldsymbol{w}), \tag{1.1q}$$

the balance of momentum equation

$$\rho_s \frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \mathrm{div}_x \mathbb{P}_s + \rho_s \boldsymbol{g}, \tag{1.1r}$$

the balance of energy equation

$$\rho_s c_v \frac{\partial \theta}{\partial t} = -\mathrm{div}_x \boldsymbol{q} - \gamma_s \eta \theta_* \frac{\partial}{\partial t}\mathrm{div}_x \boldsymbol{w} + \Psi_s, \tag{1.1s}$$

the Duhamel–Neumann law of elastic tensions

$$\mathbb{P}_s = \Big[-p_* - \gamma_s \eta(\theta - \theta_*) + \Big(\eta - \frac{2}{3}\lambda\Big)\mathrm{div}_x \boldsymbol{w}\Big]\mathbb{I} + 2\lambda \mathbb{D}(x, \boldsymbol{w}), \tag{1.1t}$$

and the Fourier law for heat flux

$$\boldsymbol{q} = -\kappa_s \nabla_x \theta. \tag{1.1u}$$

On the interface $\Gamma(t) := \partial\Omega_f(t) \cap \partial\Omega_s(t)$ we impose the no-slip conditions

$$\boldsymbol{v}_1 = \frac{d\boldsymbol{w}}{dt}, \quad \boldsymbol{v}_2 = \frac{d\boldsymbol{w}}{dt}, \quad \boldsymbol{x} \in \Gamma(t), \tag{1.2a}$$

the conditions of local thermodynamical equilibrium

$$\theta_1 = \theta, \quad \theta_2 = \theta, \quad \boldsymbol{x} \in \Gamma(t), \tag{1.2b}$$

and the conditions of continuity of the normal tension

$$-\alpha_1 p_1 \boldsymbol{n} - \alpha_2 p_2 \boldsymbol{n} + \alpha_1 \boldsymbol{\tau}_1 \cdot \boldsymbol{n} + \alpha_2 \boldsymbol{\tau}_2 \cdot \boldsymbol{n} = \mathbb{P}_s \cdot \boldsymbol{n}, \quad \boldsymbol{x} \in \Gamma(t), \tag{1.2c}$$

and the normal heat flux

$$\alpha_1 \boldsymbol{q}_1 \cdot \boldsymbol{n} + \alpha_2 \boldsymbol{q}_2 \cdot \boldsymbol{n} = \boldsymbol{q} \cdot \boldsymbol{n}, \quad \boldsymbol{x} \in \Gamma(t). \tag{1.2d}$$

In equations (1.2c)–(1.2d) and further by $\boldsymbol{n}$ the unit vector to the interface $\Gamma(t)$ is denoted. We suppose that $\boldsymbol{n}$ is pointing into $\Omega_f(t)$. On the left-hand sides of equations (1.2a)–(1.2d) for the functions $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, θ_1, etc, the one-side limiting values on $\Gamma(t)$, attained from inside $\Omega_f(t)$, are written. Correspondingly, on the right-hand sides of equations (1.2a)–(1.2d) for the functions $d\boldsymbol{w}/dt$, θ, etc, the one-side limiting values on $\Gamma(t)$, attained from inside $\Omega_s(t)$, are written. Equations (1.2c)–(1.2d) are the standard Rankine–Hugoniot conditions on the contact discontinuity surface $\Gamma(t)$

between the elastic solid and the "unified mixed" (one-phase) fluid. The latter is defined as the superposition of the two compressible pure fluid phases [11, Sec. 15].

Equations (1.1)–(1.2) constitute the original nonlinear model.

In (1.1)–(1.2) the differential operator $\mathbb{D}(x, \cdot)$ denotes the symmetric part of the gradient, i.e., $\mathbb{D}(x, \cdot) = (1/2)(\nabla_x + \nabla_x^t)$, and the rest notation for differential operators is the commonly accepted one. Physical quantities involved in (1.1)–(1.2) are as follows:

ρ_1^0 and ρ_2^0 are the *genuine* densities of the matter in the fluid phases, i.e., ρ_i^0 is the mass of the i-th phase in the unit volume of the i-th phase;

α_1 and α_2 are the volumetric saturations of the fluid phases;

$\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the velocities of the fluid phases; correspondingly further by $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ we denote the displacements in the fluid phases;

p_1 and p_2 are the pressure distributions in the fluid phases, they are equal according to Rakhmatulin's scheme;

$\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are the tensors of viscous stresses in the fluid phases;

$\boldsymbol{g}$ is the density of distributed mass forces, namely, the acceleration of free fall;

$\boldsymbol{F}_{12}$ is the interphase force in the fluid, consisting of the interphase friction force $K_F(\boldsymbol{v}_1 - \boldsymbol{v}_2)$ and the thermophoretic forces $-(\beta_1/\theta_*^2)\nabla_x\theta_1$ and $-(\beta_2/\theta_*^2)\nabla_x\theta_2$ $(\beta_1, \beta_2 = \text{const} > 0)$;

θ_1 and θ_2 are absolute temperatures in the fluid phases;

θ_* is a temperature rather close to both θ_1 and θ_2 in line with the hypothesis of small non-equilibrium (see the relevant discussion in [2, Ch. 1, Sec. 1, Formulas (1.1.73)]), therefore and due to the forthcoming linearization procedure we equate it to the constant temperature at a *natural* rest state of the whole thermomechanical system, on which the linearization will be fulfilled;

U_1 and U_2 are the specific internal energies in the fluid phases;

Q_{12} is the intensity of contact heat exchange between the fluid phases;

$\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are the heat fluxes in the fluid phases;

Ψ_1 and Ψ_2 are volumetric densities of external heat application in the fluid phases;

ν_1 and ν_2 are the bulk viscosity coefficients in the fluid phases;

μ_1 and μ_2 are the shear viscosity coefficients in the fluid phases;

κ_1 and κ_2 are the heat conduction coefficients in the fluid phases;

K_F is the coefficient of interphase friction;

K_Q is the coefficient of interphase heat exchange intensity;

ρ is the density of the (elastic) solid phase;

ρ_s is the mean constant density of the solid phase at the natural rest state;

$\boldsymbol{w}$ is the displacement of the solid phase;

$\mathbb{P}_s$ is the stress tensor in the solid state;

c_v is the coefficient of heat capacity at constant volume in the solid phase;

θ is the absolute temperature of the solid phase;

$\boldsymbol{q}$ is the heat flux in the solid phase;

γ_s is the thermal extension coefficient in the solid phase;

Ψ_s is a volumetric density of external heat application in the solid phase;

p_* is the mean constant pressure at the natural rest state in the solid phase;

η and λ are the bulk and shear elastic modules in the solid state;

and κ_s is the heat conduction coefficient in the solid phase.

1.2. **Some explanatory remarks and simplifying assumptions.** The coefficients ν_1, ν_2, μ_1, μ_2, κ_1, κ_2, K_F, K_Q, c_v, γ_s, η, λ, and κ_s are assumed to be given and positive and additionally, in line with thermodynamics fundamentals, we have $\nu_1 - (2/3)\mu_1 > 0$, $\nu_2 - (2/3)\mu_2 > 0$, and $\eta - (2/3)\lambda > 0$.

The starting point of the linear framework, which will be used further, consists of the concept of a *natural (equilibrium) rest state* of the considered thermomechanical system. Following the lines of [11, Sec. 10] we say that the state of the system is *natural*, if in both fluid and solid components deformations are absent, i.e., $\boldsymbol{w} = \boldsymbol{w}_1 = \boldsymbol{w}_2 = 0$, temperatures are the same and constant, $\theta = \theta_1 = \theta_2 = \theta_*$, and the pressure is constant, $p_1 = p_2 = p_*$. In view of the forthcoming linearization procedure, the following simplifying assumptions are quite reasonable:

1) all the coefficients ν_1, ν_2, μ_1, μ_2, κ_1, κ_2, K_F, K_Q, c_v, γ_s, η, λ, and κ_s are constant with values corresponding to the natural rest state, as given above;

2) the thermodynamical state laws of the fluid phases are linear, i.e., (1.1h) and (1.1i) have the forms

$$p_1 = p_* + c_{\rho 1}(\rho_1^0 - \rho_{1f}^0) + c_{\theta 1}(\theta_1 - \theta_*), \tag{1.3a}$$

$$p_2 = p_* + c_{\rho 2}(\rho_2^0 - \rho_{2f}^0) + c_{\theta 2}(\theta_2 - \theta_*), \tag{1.3b}$$

$$U_1 = -c_{p1}(\rho_1^0 - \rho_{1f}^0) + c_{v1}(\theta_1 - \theta_*), \tag{1.3c}$$

$$U_2 = -c_{p2}(\rho_2^0 - \rho_{2f}^0) + c_{v2}(\theta_2 - \theta_*), \tag{1.3d}$$

where ρ_{1f}^0, ρ_{2f}^0, $c_{\rho 1}$, $c_{\rho 2}$, $c_{\theta 1}$, $c_{\theta 2}$, c_{p1}, c_{p2}, c_{v1}, and c_{v2} are given positive constants. All of these constants refer to the natural rest state. The quantities ρ_{1f}^0 and ρ_{2f}^0 are the genuine mean densities, $c_{\rho 1}$ and $c_{\rho 2}$ are the squares of the speeds of sound, $c_{\theta 1}$ and $c_{\theta 2}$ are the coefficients of thermal extension, c_{p1} and c_{p2} are the coefficients of heat capacity at the constant temperature, and c_{v1} and c_{v2} are the coefficients of heat capacity at the constant volume. Equations (1.3) manifest the assumption about *physical linearity* of the fluid phases [11, Sec. 10].

Remark 1. *We would like to finish this inaugural section by the following brief observation. Rakhmatulin's scheme is widely accepted in theory of multi-phase fluids but also there are some other approaches to describe interaction between phases. For example, in filtration theory the Laplace law* $p_2 - p_1 = p_c(\boldsymbol{x}, \alpha_1)$ *is widely used for two-phase fluids in pores* [12]. *Here* p_c *is a given function called capillary pressure. Another example of constitutive law of interaction between phases can be found in Landau's theory of superfluids. In* [13], *describing superfluid states of liquid helium, Landau proposed that pressure depends on the square of the relative velocity of the components:* $p \sim \rho_n|\boldsymbol{v}_n - \boldsymbol{v}_s|^2$, *where* $\boldsymbol{v}_n$ *and* $\boldsymbol{v}_s$ *are the velocities of the "normal" and "superfluent" components and* ρ_n *is the density of the "normal" component of liquid helium. Of course, these two constitutive laws drastically differ from Rakhmatulin's scheme. Their investigation is a subject of a special interest and lies beyond the framework of the present paper.*

2. The linearization procedure

By Ω_f, Ω_s, and Γ_0 denote the fluid and solid domains at the natural rest state and the interface between them, respectively.

We derive the linearized equations from the fundamental model (1.1)–(1.3), using the classical linearization formalism [3, 14]. First of all, using the standard approach

[14, Ch. 7] we assume that perturbations of Γ_0 are rather small and therefore the boundary conditions on the actual interface $\Gamma(t)$ are shifted onto Γ_0 along the normal $\boldsymbol{n}$ of Γ_0. Hence the unknown functions are to be determined as solutions of the linearized equations in the given and fixed domains Ω_f and Ω_s. Next, from (1.1a) and (1.1b) it follows that

$$m_1\frac{\partial\rho_1^0}{\partial t}+\rho_{1f}^0\frac{\partial\alpha_1}{\partial t}+m_1\rho_{1f}^0\mathrm{div}_x\boldsymbol{v}_1=0,\quad m_2\frac{\partial\rho_2^0}{\partial t}+\rho_{2f}^0\frac{\partial\alpha_2}{\partial t}+m_2\rho_{2f}^0\mathrm{div}_x\boldsymbol{v}_2=0,$$

where m_1 and m_2 are mean constant volumetric saturations of the fluid phases at the natural rest state.

Integrating with respect to t with the proper choice of the constants of integration, we arrive at the linearized balance of mass equations

$$\text{(2.1a)}\quad m_1(\rho_1^0-\rho_{1f}^0)+\rho_{1f}^0(\alpha_1-m_1)+m_1\rho_{1f}^0\mathrm{div}_x\boldsymbol{w}_1=0,\quad(\boldsymbol{x},t)\in\Omega_f\times(0,T),$$
$$\text{(2.1b)}\quad m_2(\rho_2^0-\rho_{2f}^0)+\rho_{2f}^0(\alpha_2-m_2)+m_2\rho_{2f}^0\mathrm{div}_x\boldsymbol{w}_2=0,\quad(\boldsymbol{x},t)\in\Omega_f\times(0,T).$$

Substitute (1.1k) and (1.3a) into (1.1d), and (1.1l) and (1.3b) into (1.1e), to get

$$\rho_1^0\alpha_1\Big(\frac{\partial\boldsymbol{v}_1}{\partial t}+(\boldsymbol{v}_1\cdot\nabla_x)\boldsymbol{v}_1\Big)=-c_{\rho1}\alpha_1\nabla_x(\rho_1^0-\rho_{1f}^0)-c_{\theta1}\alpha_1\nabla_x(\theta_1-\theta_*)$$
$$+\,\mathrm{div}_x\Big[\alpha_1\Big(\nu_1-\frac{2}{3}\mu_1\Big)(\mathrm{div}_x\boldsymbol{v}_1)\mathbb{I}+2\alpha_1\mu_1\mathbb{D}(x,\boldsymbol{v}_1)\Big]+\alpha_1\rho_1^0\boldsymbol{g}-\boldsymbol{F}_{12},$$

$$\rho_2^0\alpha_2\Big(\frac{\partial\boldsymbol{v}_2}{\partial t}+(\boldsymbol{v}_2\cdot\nabla_x)\boldsymbol{v}_2\Big)=-c_{\rho2}\alpha_2\nabla_x(\rho_2^0-\rho_{2f}^0)-c_{\theta2}\alpha_2\nabla_x(\theta_2-\theta_*)$$
$$+\,\mathrm{div}_x\Big[\alpha_2\Big(\nu_2-\frac{2}{3}\mu_2\Big)(\mathrm{div}_x\boldsymbol{v}_2)\mathbb{I}+2\alpha_2\mu_2\mathbb{D}(x,\boldsymbol{v}_2)\Big]+\alpha_2\rho_2^0\boldsymbol{g}+\boldsymbol{F}_{12}.$$

Linearizing these equations, we end up with the following ones:

$$\text{(2.1c)}\quad \rho_{1f}^0m_1\frac{\partial\boldsymbol{v}_1}{\partial t}=-c_{\rho1}m_1\nabla_x(\rho_1^0-\rho_{1f}^0)-c_{\theta1}m_1\nabla_x(\theta_1-\theta_*)$$
$$+\,\mathrm{div}_x\Big[m_1\Big(\nu_1-\frac{2}{3}\mu_1\Big)(\mathrm{div}_x\boldsymbol{v}_1)\mathbb{I}+2m_1\mu_1\mathbb{D}(x,\boldsymbol{v}_1)\Big]+m_1\rho_{1f}^0\boldsymbol{g}-\boldsymbol{F}_{12},$$
$$(\boldsymbol{x},t)\in\Omega_f\times(0,T),$$

$$\text{(2.1d)}\quad \rho_{2f}^0m_2\frac{\partial\boldsymbol{v}_2}{\partial t}=-c_{\rho2}m_2\nabla_x(\rho_2^0-\rho_{2f}^0)-c_{\theta2}m_2\nabla_x(\theta_2-\theta_*)$$
$$+\,\mathrm{div}_x\Big[m_2\Big(\nu_2-\frac{2}{3}\mu_2\Big)(\mathrm{div}_x\boldsymbol{v}_2)\mathbb{I}+2m_2\mu_2\mathbb{D}(x,\boldsymbol{v}_2)\Big]+m_2\rho_{2f}^0\boldsymbol{g}+\boldsymbol{F}_{12},$$
$$(\boldsymbol{x},t)\in\Omega_f\times(0,T).$$

Here we notice that $\boldsymbol{F}_{12}$ is linear originally and therefore preserves its form.

Now substitute (1.3c), (1.1k), (1.1m), and (1.1o) into (1.1f), and also (1.3d), (1.1l), (1.1n), and (1.1o) into (1.1g) to get

$$\rho_1^0\alpha_1\Big\{-c_{p1}\frac{\partial(\rho_1^0-\rho_{1f}^0)}{\partial t}+c_{v1}\frac{\partial(\theta_1-\theta_*)}{\partial t}+(\boldsymbol{v}_1\cdot\nabla_x)\big[-c_{p1}(\rho_1^0-\rho_{1f}^0)+c_{v1}(\theta_1-\theta_*)\big]\Big\}=$$
$$\frac{\alpha_1p_1}{\rho_1^0}\Big[\frac{\partial(\rho_1^0-\rho_{1f}^0)}{\partial t}+(\boldsymbol{v}_1\cdot\nabla_x)(\rho_1^0-\rho_{1f}^0)\Big]-\frac{\beta_1}{\theta_*^2}\nabla_x\theta_1\cdot\boldsymbol{v}_1-\frac{\beta_2}{\theta_*^2}\nabla_x\theta_2\cdot\boldsymbol{v}_1+K_F(\boldsymbol{v}_1-\boldsymbol{v}_2)\cdot\boldsymbol{v}_1$$
$$-Q_{12}+\alpha_1\Big[\Big(\nu_1-\frac{2}{3}\mu_1\Big)(\mathrm{div}_x\boldsymbol{v}_1)\mathbb{I}+2\mu_1\mathbb{D}(x,\boldsymbol{v}_1)\Big]:\mathbb{D}(x,\boldsymbol{v}_1)$$
$$+\mathrm{div}_x(\alpha_1\kappa_1\nabla_x\theta_1)-\mathrm{div}_x\Big[\frac{\alpha_1\beta_1}{\theta^*}(\boldsymbol{v}_1-\boldsymbol{v}_2)\Big]+\alpha_1\Psi_1,$$

$$\rho_2^0\alpha_2\Big\{-c_{p2}\frac{\partial(\rho_2^0-\rho_{2f}^0)}{\partial t}+c_{v2}\frac{\partial(\theta_2-\theta_*)}{\partial t}+(\boldsymbol{v}_2\cdot\nabla_x)\big[-c_{p2}(\rho_2^0-\rho_{2f}^0)+c_{v2}(\theta_2-\theta_*)\big]\Big\}=$$
$$\frac{\alpha_2p_2}{\rho_2^0}\Big[\frac{\partial(\rho_2^0-\rho_{2f}^0)}{\partial t}+(\boldsymbol{v}_2\cdot\nabla_x)(\rho_2^0-\rho_{2f}^0)\Big]+\frac{\beta_1}{\theta_*^2}\nabla_x\theta_1\cdot\boldsymbol{v}_2+\frac{\beta_2}{\theta_*^2}\nabla_x\theta_2\cdot\boldsymbol{v}_2-K_F(\boldsymbol{v}_1-\boldsymbol{v}_2)\cdot\boldsymbol{v}_2$$
$$+Q_{12}+\alpha_2\Big[\Big(\nu_2-\frac{2}{3}\mu_2\Big)(\mathrm{div}_x\boldsymbol{v}_2)\mathbb{I}+2\mu_2\mathbb{D}(x,\boldsymbol{v}_2)\Big]:\mathbb{D}(x,\boldsymbol{v}_2)$$
$$+\mathrm{div}_x(\alpha_2\kappa_2\nabla_x\theta_2)-\mathrm{div}_x\Big[\frac{\alpha_2\beta_2}{\theta^*}(\boldsymbol{v}_1-\boldsymbol{v}_2)\Big]+\alpha_2\Psi_2.$$

Upon linearization these equations become

$$\text{(2.1e)}\quad \rho_{1f}^0m_1c_{v1}\frac{\partial(\theta_1-\theta_*)}{\partial t}=\rho_{1f}^0m_1c_{p1}\frac{\partial(\rho_1^0-\rho_{1f}^0)}{\partial t}+\frac{m_1p_*}{\rho_{1f}^0}\frac{\partial(\rho_1^0-\rho_{1f}^0)}{\partial t}-Q_{12}$$
$$+\mathrm{div}_x\Big[m_1\kappa_1\nabla_x\theta_1-\frac{m_1\beta_1}{\theta_*}(\boldsymbol{v}_1-\boldsymbol{v}_2)\Big]+m_1\Psi_1,\quad(\boldsymbol{x},t)\in\Omega_f\times(0,T)$$

and

$$\text{(2.1f)}\quad \rho_{2f}^0m_2c_{v2}\frac{\partial(\theta_2-\theta_*)}{\partial t}=\rho_{2f}^0m_2c_{p2}\frac{\partial(\rho_2^0-\rho_{2f}^0)}{\partial t}+\frac{m_2p_*}{\rho_{2f}^0}\frac{\partial(\rho_2^0-\rho_{2f}^0)}{\partial t}+Q_{12}$$
$$+\mathrm{div}_x\Big[m_2\kappa_2\nabla_x\theta_2-\frac{m_2\beta_2}{\theta_*}(\boldsymbol{v}_1-\boldsymbol{v}_2)\Big]+m_2\Psi_2,\quad(\boldsymbol{x},t)\in\Omega_f\times(0,T).$$

Here we notice that Q_{12} is originally linear and therefore preserves its form.

Equations (1.1q)–(1.1u) are the result of the linearization process readily, so, in order to complete our procedure, we are left to derive the equations on the contact discontinuity Γ_0. On the strength of (1.1k)–(1.1n), (1.1t), (1.1u), (1.3a), and (1.3b), by the standard procedure (see in [14, Ch. 7] or [3, Sec. II.3], for example), from (1.2) we deduce the following linearized relations:

$$\text{(2.2a)}\qquad \boldsymbol{v}_1=\frac{\partial\boldsymbol{w}}{\partial t},\quad \boldsymbol{v}_2=\frac{\partial\boldsymbol{w}}{\partial t},\quad(\boldsymbol{x},t)\in\Gamma_0\times(0,T),$$

$$\text{(2.2b)}\qquad \theta_1=\theta,\quad\theta_2=\theta,\quad(\boldsymbol{x},t)\in\Gamma_0\times(0,T),$$

(2.2c)
$$m_1\Big[-\big(c_{\rho 1}(\rho_1^0-\rho_{1f}^0)+c_{\theta 1}(\theta_1-\theta_*)\big)\boldsymbol{n}+\Big(\nu_1-\frac{2}{3}\mu_1\Big)(\mathrm{div}_x\boldsymbol{v}_1)\boldsymbol{n}+2\mu_1\mathbb{D}(x,\boldsymbol{v}_1)\boldsymbol{n}\Big]$$
$$+m_2\Big[-\big(c_{\rho 2}(\rho_2^0-\rho_{2f}^0)+c_{\theta 2}(\theta_2-\theta_*)\big)\boldsymbol{n}+\Big(\nu_2-\frac{2}{3}\mu_2\Big)(\mathrm{div}_x\boldsymbol{v}_2)\boldsymbol{n}+2\mu_2\mathbb{D}(x,\boldsymbol{v}_2)\boldsymbol{n}\Big]$$
$$=-\gamma_s\eta(\theta-\theta_*)\boldsymbol{n}+\Big(\eta-\frac{2}{3}\lambda\Big)(\mathrm{div}_x\boldsymbol{w})\boldsymbol{n}+2\lambda\mathbb{D}(x,\boldsymbol{w})\boldsymbol{n},\quad (\boldsymbol{x},t)\in\Gamma_0\times(0,T),$$

(2.2d)
$$m_1\Big(-\kappa_1\nabla_x\theta_1+\frac{\beta_1}{\theta_*}(\boldsymbol{v}_1-\boldsymbol{v}_2)\Big)\cdot\boldsymbol{n}+m_2\Big(-\kappa_2\nabla_x\theta_2+\frac{\beta_2}{\theta_*}(\boldsymbol{v}_1-\boldsymbol{v}_2)\Big)\cdot\boldsymbol{n}=-\kappa_s\nabla_x\theta\cdot\boldsymbol{n},$$
$$(\boldsymbol{x},t)\in\Gamma_0\times(0,T),$$

System (1.1c), (1.1j), (1.1o)–(1.1u), (1.3), (2.1), and (2.2) constitute the resulting fundamental linearized model L_0. In the three-dimensional case this system contains the 34 scalar equations and 4 relations on interface Γ_0 for the 34 unknown scalar functions ρ_1^0, ρ_2^0, α_1, α_2, w_1^i, w_2^i, θ_1, θ_2, F_{12}^i, Q_{12}, ρ, $\boldsymbol{w}^i$, p_1, p_2, U_1, U_2, $\mathbb{P}_s^{ij}$, θ, q^i and therefore is closed. (Here we notice that the tensor $\mathbb{P}_s^{ij}$ is symmetric.)

Clearly, this system admits some evident simplifying transformations and consequently can be wrapped to a more compact form.

3. Some technical transformations

We solve (2.1a) and (2.1b) for ρ_1^0 and ρ_2^0, respectively, substitute these functions and ρ into the other equations of model L_0, and fulfill a set of simple technical transformations.

We have

$$\rho_1^0=\rho_{1f}^0\Big(2-\frac{\alpha_1}{m_1}-\mathrm{div}_x\boldsymbol{w}_1\Big),\quad (\boldsymbol{x},t)\in\Omega_f\times(0,T), \tag{3.1a}$$
$$\rho_2^0=\rho_{2f}^0\Big(2-\frac{\alpha_2}{m_2}-\mathrm{div}_x\boldsymbol{w}_2\Big),\quad (\boldsymbol{x},t)\in\Omega_f\times(0,T). \tag{3.1b}$$

Substituting these equations into (2.1c) and (2.1d) we get

(3.2a)
$$\rho_{1f}^0 m_1\frac{\partial\boldsymbol{v}_1}{\partial t}=c_{\rho 1}\rho_{1f}^0(\nabla_x(\alpha_1-m_1)+m_1\nabla_x\mathrm{div}_x\boldsymbol{w}_1)-c_{\theta 1}m_1\nabla_x(\theta_1-\theta_*)$$
$$+\mathrm{div}_x\Big[m_1\Big(\nu_1-\frac{2}{3}\mu_1\Big)(\mathrm{div}_x\boldsymbol{v}_1)\mathbb{I}+2m_1\mu_1\mathbb{D}(x,\boldsymbol{v}_1)\Big]+m_1\rho_{1f}^0\boldsymbol{g}-\boldsymbol{F}_{12},$$
$$(\boldsymbol{x},t)\in\Omega_f\times(0,T),$$

(3.2b)
$$\rho_{2f}^0 m_2\frac{\partial\boldsymbol{v}_2}{\partial t}=c_{\rho 2}\rho_{2f}^0(\nabla_x(\alpha_2-m_2)+m_2\nabla_x\mathrm{div}_x\boldsymbol{w}_2)-c_{\theta 2}m_2\nabla_x(\theta_2-\theta_*)$$
$$+\mathrm{div}_x\Big[m_2\Big(\nu_2-\frac{2}{3}\mu_2\Big)(\mathrm{div}_x\boldsymbol{v}_2)\mathbb{I}+2m_2\mu_2\mathbb{D}(x,\boldsymbol{v}_2)\Big]+m_2\rho_{2f}^0\boldsymbol{g}+\boldsymbol{F}_{12},$$
$$(\boldsymbol{x},t)\in\Omega_f\times(0,T).$$

Substituting (3.1) into (2.1e) and (2.1f) we arrive at

(3.2c)
$$\rho^0_{1f} m_1 c_{v1} \frac{\partial \theta_1}{\partial t} = -(\rho^0_{1f})^2 c_{p1} \Big(\frac{\partial \alpha_1}{\partial t} + m_1 \mathrm{div}_x \boldsymbol{v}_1 \Big) - p_* \frac{\partial \alpha_1}{\partial t} - m_1 p_* \mathrm{div}_x \boldsymbol{v}_1 - Q_{12} + \mathrm{div}_x \Big[m_1 \kappa_1 \nabla_x \theta_1 - \frac{m_1 \beta_1}{\theta_*} (\boldsymbol{v}_1 - \boldsymbol{v}_2) \Big] + m_1 \Psi_1, \quad (\boldsymbol{x}, t) \in \Omega_f \times (0,T),$$

(3.2d)
$$\rho^0_{2f} m_2 c_{v2} \frac{\partial \theta_2}{\partial t} = -(\rho^0_{2f})^2 c_{p2} \Big(\frac{\partial \alpha_2}{\partial t} + m_2 \mathrm{div}_x \boldsymbol{v}_2 \Big) - p_* \frac{\partial \alpha_2}{\partial t} - m_2 p_* \mathrm{div}_x \boldsymbol{v}_2 + Q_{12} + \mathrm{div}_x \Big[m_2 \kappa_2 \nabla_x \theta_2 - \frac{m_2 \beta_2}{\theta_*} (\boldsymbol{v}_1 - \boldsymbol{v}_2) \Big] + m_2 \Psi_2, \quad (\boldsymbol{x}, t) \in \Omega_f \times (0,T).$$

Inserting (3.1a) and (3.1b) into (1.3a) and (1.3b), we get

$$p_1 = p_* + c_{\rho 1} \rho^0_{1f} \Big(1 - \frac{\alpha_1}{m_1} - \mathrm{div}_x \boldsymbol{w}_1 \Big) + c_{\theta 1} (\theta_1 - \theta_*), \tag{3.2e}$$

$$p_2 = p_* + c_{\rho 2} \rho^0_{2f} \Big(1 - \frac{\alpha_2}{m_2} - \mathrm{div}_x \boldsymbol{w}_2 \Big) + c_{\theta 2} (\theta_2 - \theta_*). \tag{3.2f}$$

Substituting (1.1q) and (1.1t) into (1.1r) we obtain

$$\rho_s \frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \mathrm{div}_x \Big\{ \Big[-p_* - \gamma_s \eta (\theta - \theta_*) + \Big(\eta - \frac{2}{3} \lambda \Big) \mathrm{div}_x \boldsymbol{w} \Big] \mathbb{I} + 2\lambda \mathbb{D}(x, \boldsymbol{w}) \Big\} + \rho_s \boldsymbol{g}. \tag{3.2g}$$

Also, substituting (1.1u) into (1.1s) we establish

$$\rho_s c_v \frac{\partial \theta}{\partial t} = \mathrm{div}_x (\kappa_s \nabla_x \theta) - \gamma_s \eta \theta_* \frac{\partial}{\partial t} \mathrm{div}_x \boldsymbol{w} + \Psi_s. \tag{3.2h}$$

Let us consider the interface conditions now. Integrate (2.2a) with respect to t to get

$$\boldsymbol{w}_1 = \boldsymbol{w}, \quad \boldsymbol{w}_2 = \boldsymbol{w}, \quad \boldsymbol{x} \in \Gamma_0, \quad t > 0. \tag{3.3a}$$

Further, we preserve conditions (2.2b) as they are. Next, we substitute (3.1a) and (3.1b) into (2.2c) and arrive at

(3.3b)
$$\Big[c_{\rho 1} \rho^0_{1f} (\alpha_1 - m_1) + m_1 c_{\rho 1} \rho^0_{1f} \mathrm{div}_x \boldsymbol{w}_1 - m_1 c_{\theta 1} (\theta_1 - \theta_*) \Big] \boldsymbol{n} + m_1 \Big(\nu_1 - \frac{2}{3} \mu_1 \Big) (\mathrm{div}_x \boldsymbol{v}_1) \boldsymbol{n} + 2 m_1 \mu_1 \mathbb{D}(x, \boldsymbol{v}_1) \boldsymbol{n} + \Big[c_{\rho 2} \rho^0_{2f} (\alpha_2 - m_2) + m_2 c_{\rho 2} \rho^0_{2f} \mathrm{div}_x \boldsymbol{w}_2 - m_2 c_{\theta 2} (\theta_2 - \theta_*) \Big] \boldsymbol{n} + m_2 \Big(\nu_2 - \frac{2}{3} \mu_2 \Big) (\mathrm{div}_x \boldsymbol{v}_2) \boldsymbol{n} + 2 m_2 \mu_2 \mathbb{D}(x, \boldsymbol{v}_2) \boldsymbol{n} = - \gamma_s \eta (\theta - \theta_*) \boldsymbol{n} + \Big(\eta - \frac{2}{3} \lambda \Big) (\mathrm{div}_x \boldsymbol{w}) \boldsymbol{n} + 2 \lambda \mathbb{D}(x, \boldsymbol{w}) \boldsymbol{n}, \quad (\boldsymbol{x}, t) \in \Gamma_0 \times (0,T),$$

Finally, we rewrite (2.2d) with account of (2.2a):

$$-m_1 \kappa_1 \nabla_x \theta_1 \cdot \boldsymbol{n} - m_2 \kappa_2 \nabla_x \theta_2 \cdot \boldsymbol{n} = -\kappa_s \nabla_x \theta \cdot \boldsymbol{n}, \quad (\boldsymbol{x}, t) \in \Gamma_0 \times (0,T). \tag{3.3c}$$

Now let us notice that α_1 and α_2 can be represented explicitly right out in the model, just like it was done for ρ^0_1 and ρ^0_2 in formulas (3.1).

Substituting (1.1c) into (3.2e) and (3.2f) and using Rakhmatulin's law (1.1j) we get

$$c_{\rho 1}\rho^0_{1f}\Big(1-\frac{\alpha_1}{m_1}-\mathrm{div}_x\boldsymbol{w}_1\Big)+c_{\theta 1}(\theta_1-\theta_*) = c_{\rho 2}\rho^0_{2f}\Big(1-\frac{1-\alpha_1}{m_2}-\mathrm{div}_x\boldsymbol{w}_2\Big)+c_{\theta 2}(\theta_2-\theta_*).$$

Solve this equation for α_1 to derive

$$\alpha_1 = \frac{m_1 m_2}{m_2 c_{\rho 1}\rho^0_{1f} + m_1 c_{\rho 2}\rho^0_{2f}}\Big[c_{\rho 1}\rho^0_{1f}(1-\mathrm{div}_x\boldsymbol{w}_1) - c_{\rho 2}\rho^0_{2f}(1-\mathrm{div}_x\boldsymbol{w}_2) + c_{\theta 1}(\theta_1-\theta_*) - c_{\theta 2}(\theta_2-\theta_*) + \frac{c_{\rho 2}\rho^0_{2f}}{m_2}\Big]. \tag{3.4a}$$

Analogously,

$$\alpha_2 = \frac{m_1 m_2}{m_2 c_{\rho 1}\rho^0_{1f} + m_1 c_{\rho 2}\rho^0_{2f}}\Big[c_{\rho 2}\rho^0_{2f}(1-\mathrm{div}_x\boldsymbol{w}_2) - c_{\rho 1}\rho^0_{1f}(1-\mathrm{div}_x\boldsymbol{w}_1) + c_{\theta 2}(\theta_2-\theta_*) - c_{\theta 1}(\theta_1-\theta_*) + \frac{c_{\rho 1}\rho^0_{1f}}{m_1}\Big]. \tag{3.4b}$$

Finishing this section let us substitute $\partial\boldsymbol{w}_1/\partial t$ and $\partial\boldsymbol{w}_2/\partial t$ on the places of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ throughout. In the next section we summarize the previous considerations in the form of the most general linearized model.

4. Model of joint small perturbations of a thermoelastic body and a two-phase heat-conducting viscous fluid

For simplicity of notation, denote

$$\bar\theta := m_1 m_2\big[c_{\theta 2}(\theta_2-\theta_*) - c_{\theta 1}(\theta_1-\theta_*)\big], \tag{4.1a}$$

$$\alpha^0_i := \frac{m_i}{c_{\rho i}\rho^0_{if}}\,\frac{c_{\rho 1}\rho^0_{1f}c_{\rho 2}\rho^0_{2f}}{m_1 c_{\rho 2}\rho^0_{2f} + m_2 c_{\rho 1}\rho^0_{1f}}, \quad i=1,2. \tag{4.1b}$$

Remark that $\alpha^0_1+\alpha^0_2=1$.

Model $\mathbf{L}_1$. In a bounded domain Ω, divided into two disjoint components Ω_f and Ω_s and the immovable interface $\Gamma_0=\partial\Omega_f\cap\partial\Omega_s$ between them, it is necessary to find the following:

(1) the temperatures θ_1 and θ_2 in the fluid phases, the displacement fields $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ in the fluid phases, the temperature θ in the solid phase, and the displacement field $\boldsymbol{w}$ in the solid phase, satisfying the following equations in Ω_f and Ω_s and relations on Γ_0:

(4.2a)

$$\rho^0_{if}m_i\frac{\partial^2\boldsymbol{w}_i}{\partial t^2} = c_{\rho i}\rho^0_{if}\alpha^0_i\nabla_x\Big(m_1\mathrm{div}_x\boldsymbol{w}_1+m_2\mathrm{div}_x\boldsymbol{w}_2+(-1)^i\frac{\bar\theta}{m_i}\Big)-c_{\theta i}m_i\nabla_x(\theta_i-\theta_*) + \mathrm{div}_x\Big[m_i\Big(\nu_i-\frac{2}{3}\mu_i\Big)\Big(\mathrm{div}_x\frac{\partial\boldsymbol{w}_i}{\partial t}\Big)\mathbb{I}+2m_i\mu_i\mathbb{D}\Big(x,\frac{\partial\boldsymbol{w}_i}{\partial t}\Big)\Big] + m_i\rho^0_{if}\boldsymbol{g}+(-1)^i\boldsymbol{F}_{12}, (\boldsymbol{x},t)\in\Omega_f\times(0,T), \quad i=1,2,$$

where

$$\boldsymbol{F}_{12} = -\frac{\beta_1}{\theta_*^2}\nabla_x\theta_1 - \frac{\beta_2}{\theta_*^2}\nabla_x\theta_2 + K_F\frac{\partial}{\partial t}(\boldsymbol{w}_1-\boldsymbol{w}_2);$$

(4.2b)

$$\rho_{if}^0 m_i c_{vi}\frac{\partial\theta_i}{\partial t} = -\Big[(\rho_{if}^0)^2 c_{pi} + p_*\Big]\alpha_i^0\frac{\partial}{\partial t}\Big(m_1\mathrm{div}_x\boldsymbol{w}_1 + m_2\mathrm{div}_x\boldsymbol{w}_2 + (-1)^i\frac{\bar\theta}{m_i}\Big)$$
$$+\,\mathrm{div}_x\Big[m_i\kappa_i\nabla_x\theta_i - \frac{m_i\beta_i}{\theta_*}\frac{\partial}{\partial t}(\boldsymbol{w}_1-\boldsymbol{w}_2)\Big] + (-1)^i Q_{12} + m_i\Psi_i,$$
$$(\boldsymbol{x},t)\in\Omega_f\times(0,T),\quad i=1,2,$$

where $Q_{12} = K_Q(\theta_1-\theta_2)$;

(4.2c) $$\rho_s\frac{\partial^2\boldsymbol{w}}{\partial t^2} = -\gamma_s\eta\nabla_x(\theta-\theta_*) + \Big(\eta-\frac{2}{3}\lambda\Big)\nabla_x\mathrm{div}_x\boldsymbol{w} + \mathrm{div}_x(2\lambda\mathbb{D}(x,\boldsymbol{w})) + \rho_s\boldsymbol{g},$$
$$(\boldsymbol{x},t)\in\Omega_s\times(0,T);$$

(4.2d) $$\rho_s c_v\frac{\partial\theta}{\partial t} = \mathrm{div}_x(\kappa_s\nabla_x\theta) - \gamma_s\eta\theta_*\frac{\partial}{\partial t}\mathrm{div}_x\boldsymbol{w} + \Psi_s,\quad (\boldsymbol{x},t)\in\Omega_s\times(0,T);$$

(4.2e) $$\boldsymbol{w}_1 = \boldsymbol{w},\quad \boldsymbol{w}_2 = \boldsymbol{w},\quad (\boldsymbol{x},t)\in\Gamma_0\times(0,T);$$

(4.2f) $$\theta_1 = \theta,\quad \theta_2 = \theta,\quad (\boldsymbol{x},t)\in\Gamma_0\times(0,T);$$

(4.2g) $$\Big\{c_{\rho 1}\rho_{1f}^0\alpha_1^0\Big(m_1\mathrm{div}_x\boldsymbol{w}_1 + m_2\mathrm{div}_x\boldsymbol{w}_2 - \frac{\bar\theta}{m_1}\Big) - m_1c_{\theta 1}(\theta_1-\theta_*)\Big\}\boldsymbol{n}$$
$$+\,m_1\Big(\nu_1-\frac{2}{3}\mu_1\Big)\Big(\mathrm{div}_x\frac{\partial\boldsymbol{w}_1}{\partial t}\Big)\boldsymbol{n} + 2m_1\mu_1\mathbb{D}\Big(x,\frac{\partial\boldsymbol{w}_1}{\partial t}\Big)\boldsymbol{n}$$
$$+\,\Big\{c_{\rho 2}\rho_{2f}^0\alpha_2^0\Big(m_1\mathrm{div}_x\boldsymbol{w}_1 + m_2\mathrm{div}_x\boldsymbol{w}_2 + \frac{\bar\theta}{m_2}\Big) - m_2c_{\theta 2}(\theta_2-\theta_*)\Big\}\boldsymbol{n}$$
$$+\,m_2\Big(\nu_2-\frac{2}{3}\mu_2\Big)\Big(\mathrm{div}_x\frac{\partial\boldsymbol{w}_2}{\partial t}\Big)\boldsymbol{n} + 2m_2\mu_2\mathbb{D}(x,\frac{\partial\boldsymbol{w}_2}{\partial t})\boldsymbol{n}$$
$$= -\gamma_s\eta(\theta-\theta_*)\boldsymbol{n} + \Big(\eta-\frac{2}{3}\lambda\Big)(\mathrm{div}_x\boldsymbol{w})\boldsymbol{n} + 2\lambda\mathbb{D}(x,\boldsymbol{w})\boldsymbol{n},\quad (\boldsymbol{x},t)\in\Gamma_0\times(0,T);$$

(4.2h) $$-m_1\kappa_1\nabla_x\theta_1\cdot\boldsymbol{n} - m_2\kappa_2\nabla_x\theta_2\cdot\boldsymbol{n} = -\kappa_s\nabla_x\theta\cdot\boldsymbol{n},\quad (\boldsymbol{x},t)\in\Gamma_0\times(0,T);$$

(2) after this, the volumetric saturations of the fluid phases α_i; the genuine densities ρ_i^0, $(i=1,2)$, the pressure p, the specific energies U_i, the tensors of viscous stresses $\boldsymbol{\tau}_i$, the heat fluxes $\boldsymbol{q}_i$, and $\boldsymbol{q}_s$ from the explicit representations (3.4), (3.1a)–(3.1b), (3.2e), (1.3c), (1.1k)–(1.1n), and (1.1u), respectively.

We endow this model by imposing initial data for $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}$, $\partial\boldsymbol{w}_1/\partial t$, $\partial\boldsymbol{w}_2/\partial t$, $\partial\boldsymbol{w}/\partial t$, θ_1, θ_2, and θ, and boundary conditions on $\partial\Omega$ for $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}$, θ_1, θ_2, and θ.

5. Isothermal case

5.1. **Formulation of the isothermal model.** Let us set the formulation for the isothermal case. We assume that

$$\theta_1(\boldsymbol{x},t) = \theta_2(\boldsymbol{x},t) = \theta_*\quad\text{for } (\boldsymbol{x},t)\in\Omega_f\times(0,T),$$
$$\theta(\boldsymbol{x},t) = \theta_*\quad\text{for } (\boldsymbol{x},t)\in\Omega_s\times(0,T).$$

Also, in the isothermal case the coefficients β_1 and β_2 in the thermophoretic force are assumed to be zero.

Model IT$_0$. In a bounded domain Ω, divided into two disjoint components Ω_f and Ω_s and the immovable interface $\Gamma_0 = \partial\Omega_f \cap \partial\Omega_s$ between them, it is necessary to find the following:

(1) the displacement fields $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ in the liquid phases, the density ρ and the displacement field $\boldsymbol{w}$ in the solid phase, satisfying the following equations in Ω_f and Ω_s and relations on Γ_0:

(5.1a)
$$\rho^0_{if} m_i \frac{\partial^2 \boldsymbol{w}_i}{\partial t^2} = c_{\rho i}\rho^0_{if}\alpha^0_i \nabla_x (m_1 \mathrm{div}_x \boldsymbol{w}_1 + m_2 \mathrm{div}_x \boldsymbol{w}_2) + \mathrm{div}_x \Big[m_i \Big(\nu_i - \frac{2}{3}\mu_i\Big)\Big(\mathrm{div}_x \frac{\partial \boldsymbol{w}_i}{\partial t}\Big)\mathbb{I} + 2 m_i \mu_i \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_i}{\partial t}\Big)\Big] + m_i \rho^0_{if}\boldsymbol{g} + (-1)^i \boldsymbol{F}_{12}, \quad (\boldsymbol{x},t) \in \Omega_f \times (0,T), \quad i = 1,2,$$

where $\boldsymbol{F}_{12} = K_F(\partial(\boldsymbol{w}_1 - \boldsymbol{w}_2)/\partial t)$;

(5.1b)
$$\rho_s \frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \Big(\eta - \frac{2}{3}\lambda\Big)\nabla_x \mathrm{div}_x \boldsymbol{w} + \mathrm{div}_x(2\lambda \mathbb{D}(x,\boldsymbol{w})) + \rho_s \boldsymbol{g}, \quad (\boldsymbol{x},t) \in \Omega_s \times (0,T);$$

(5.1c)
$$\boldsymbol{w}_1 = \boldsymbol{w}, \quad \boldsymbol{w}_2 = \boldsymbol{w}, \quad (\boldsymbol{x},t) \in \Gamma_0 \times (0,T);$$

(5.1d)
$$(c_{\rho 1}\rho^0_{1f}\alpha^0_1 + c_{\rho 2}\rho^0_{2f}\alpha^0_2)(m_1 \mathrm{div}_x \boldsymbol{w}_1 + m_2 \mathrm{div}_x \boldsymbol{w}_2)\boldsymbol{n} + m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big)\Big(\mathrm{div}_x \frac{\partial \boldsymbol{w}_1}{\partial t}\Big)\boldsymbol{n} + m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big)\Big(\mathrm{div}_x \frac{\partial \boldsymbol{w}_2}{\partial t}\Big)\boldsymbol{n} + 2m_1\mu_1 \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_1}{\partial t}\Big)\boldsymbol{n} + 2m_2\mu_2 \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_2}{\partial t}\Big)\boldsymbol{n} = \Big(\eta - \frac{2}{3}\lambda\Big)(\mathrm{div}_x \boldsymbol{w})\boldsymbol{n} + 2\lambda \mathbb{D}(x,\boldsymbol{w})\boldsymbol{n}, \quad (\boldsymbol{x},t) \in \Gamma_0 \times (0,T);$$

(2) after this, the volumetric saturations of the liquid phases α_i from the formulas

(5.1e)
$$\alpha_1 = \frac{m_1 m_2}{m_2 c_{\rho 1}\rho^0_{1f} + m_1 c_{\rho 2}\rho^0_{2f}} \Big[c_{\rho 1}\rho^0_{1f}(1 - \mathrm{div}_x \boldsymbol{w}_1) - c_{\rho 2}\rho^0_{2f}(1 - \mathrm{div}_x \boldsymbol{w}_2) + \frac{c_{\rho 2}\rho^0_{2f}}{m_2}\Big], \quad (\boldsymbol{x},t) \in \Omega_f \times (0,T),$$

(5.1f)
$$\alpha_2 = \frac{m_1 m_2}{m_2 c_{\rho 1}\rho^0_{1f} + m_1 c_{\rho 2}\rho^0_{2f}} \Big[c_{\rho 2}\rho^0_{2f}(1 - \mathrm{div}_x \boldsymbol{w}_2) - c_{\rho 1}\rho^0_{1f}(1 - \mathrm{div}_x \boldsymbol{w}_1) + \frac{c_{\rho 1}\rho^0_{1f}}{m_1}\Big], \quad (\boldsymbol{x},t) \in \Omega_f \times (0,T),$$

(remark that $\alpha_1 + \alpha_2 = 1$), the genuine densities ρ^0_i, $(i = 1,2)$ from the formulas

(5.1g)
$$\rho^0_i = \rho^0_{if}\Big[1 - \frac{\alpha^0_i}{m_i}(m_1 \mathrm{div}_x \boldsymbol{w}_1 + m_2 \mathrm{div}_x \boldsymbol{w}_2)\Big], \quad (\boldsymbol{x},t) \in \Omega_f \times (0,T), \quad i = 1,2,$$

and the pressure p from the formula

(5.1h)
$$p = p_* - c_{\rho i}\rho^0_{if}\frac{\alpha^0_i}{m_i}(m_1 \mathrm{div}_x \boldsymbol{w}_1 + m_2 \mathrm{div}_x \boldsymbol{w}_2), \quad (\boldsymbol{x},t) \in \Omega_f \times (0,T), \quad i = 1,2.$$

The model is closed by the homogeneous conditions on $\partial\Omega \times (0,T)$:

(5.1i)
$$\boldsymbol{w}_1 = \boldsymbol{w}_2 = 0 \quad \text{on } (\partial\Omega_f \cap \partial\Omega) \times (0,T), \quad \boldsymbol{w} = 0 \quad \text{on } (\partial\Omega_s \cap \partial\Omega) \times (0,T),$$

and initial data

(5.1j) $\boldsymbol{w}_1|_{t=0} = \boldsymbol{w}_1^0, \quad \boldsymbol{w}_2|_{t=0} = \boldsymbol{w}_2^0 \quad \text{for } \boldsymbol{x} \in \Omega_f, \quad \boldsymbol{w}|_{t=0} = \boldsymbol{w}^0 \quad \text{for } \boldsymbol{x} \in \Omega_s,$

(5.1k) $\dfrac{\partial \boldsymbol{w}_1}{\partial t}\Big|_{t=0} = \boldsymbol{v}_1^0, \quad \dfrac{\partial \boldsymbol{w}_2}{\partial t}\Big|_{t=0} = \boldsymbol{v}_2^0 \quad \text{for } \boldsymbol{x} \in \Omega_f, \quad \dfrac{\partial \boldsymbol{w}}{\partial t}\Big|_{t=0} = \boldsymbol{v}^0 \quad \text{for } \boldsymbol{x} \in \Omega_s.$

The given functions meet the requirements

(5.1l) $\boldsymbol{w}_1^0,\, \boldsymbol{w}_2^0 \in H^1(\Omega_f), \quad \boldsymbol{w} \in H^1(\Omega_s), \quad \boldsymbol{v}_1^0,\, \boldsymbol{v}_2^0 \in L^2(\Omega_f), \quad \boldsymbol{g} \in L^2(\Omega\times(0,T)).$

5.2. **Notion of generalized solutions of Model IT$_0$.** Let us substitute the vector-functions $\boldsymbol{w}_i$ in equations (5.1a) by the new thought vector-functions

$$\boldsymbol{w}_c := \frac{m_1\rho_{1f}^0}{\rho_f^0}\boldsymbol{w}_1 + \frac{m_2\rho_{2f}^0}{\rho_f^0}\boldsymbol{w}_2, \tag{5.2a}$$

$$\boldsymbol{w}_r := \boldsymbol{w}_2 - \boldsymbol{w}_1, \tag{5.2b}$$

which are the mean velocity of the two-phase fluid and the relative velocity of the fluid phases, respectively. In (5.2a)–(5.2b) and further by ρ_f^0 we denote the mean density of the two-phase fluid, namely,

$$\rho_f^0 := m_1\rho_{1f}^0 + m_2\rho_{2f}^0.$$

Then the system of equations (5.1a) is equivalent to the following one:

$$\begin{aligned}
\text{(5.3a)}\quad \rho_f^0\frac{\partial^2\boldsymbol{w}_c}{\partial t^2} &= \alpha_c^0\nabla_x\mathrm{div}_x\boldsymbol{w}_c + \Big[\Big(\nu_1 - \frac{2}{3}\mu_1\Big)m_1 + \Big(\nu_2 - \frac{2}{3}\mu_2\Big)m_2\Big]\nabla_x\mathrm{div}_x\frac{\partial\boldsymbol{w}_c}{\partial t}\\
&\quad + 2(\mu_1 m_1 + \mu_2 m_2)\mathrm{div}_x\mathbb{D}\Big(x, \frac{\partial\boldsymbol{w}_c}{\partial t}\Big) + \rho_f^0\boldsymbol{g}\\
&+ \frac{m_1m_2}{\rho_f^0}\Big\{\alpha_c^0(\rho_{1f}^0-\rho_{2f}^0)\nabla_x\mathrm{div}_x\boldsymbol{w}_r + \Big[\rho_{1f}^0\Big(\nu_2-\frac{2}{3}\mu_2\Big)-\rho_{2f}^0\Big(\nu_1-\frac{2}{3}\mu_1\Big)\Big]\nabla_x\mathrm{div}_x\frac{\partial\boldsymbol{w}_r}{\partial t}\\
&\quad + 2(\rho_{1f}^0\mu_2 - \rho_{2f}^0\mu_1)\mathrm{div}_x\mathbb{D}\Big(x, \frac{\partial\boldsymbol{w}_r}{\partial t}\Big)\Big\},
\end{aligned}$$

$$\begin{aligned}
\text{(5.3b)}\quad & 2\rho_{1f}^0\rho_{2f}^0\frac{m_1m_2}{\rho_f^0}\frac{\partial^2\boldsymbol{w}_r}{\partial t^2} + (\rho_{2f}^0m_2 - \rho_{1f}^0m_1)\frac{\partial^2\boldsymbol{w}_c}{\partial t^2} =\\
&\frac{m_1m_2}{\rho_f^0}\Big\{\alpha_r^0(\rho_{1f}^0-\rho_{2f}^0)\nabla_x\mathrm{div}_x\boldsymbol{w}_r + \Big[\rho_{2f}^0\Big(\nu_1-\frac{2}{3}\mu_1\Big)+\rho_{1f}^0\Big(\nu_2-\frac{2}{3}\mu_2\Big)\Big]\nabla_x\mathrm{div}_x\frac{\partial\boldsymbol{w}_r}{\partial t}\\
&\quad + 2(\rho_{2f}^0\mu_1 + \rho_{1f}^0\mu_2)\mathrm{div}_x\mathbb{D}\Big(x, \frac{\partial\boldsymbol{w}_r}{\partial t}\Big)\Big\} - 2K_F\frac{\partial\boldsymbol{w}_r}{\partial t}\\
&\quad + \alpha_r^0\nabla_x\mathrm{div}_x\boldsymbol{w}_c + \Big[\Big(\nu_2-\frac{2}{3}\mu_2\Big)m_2 - \Big(\nu_1-\frac{2}{3}\mu_1\Big)m_1\Big]\nabla_x\mathrm{div}_x\frac{\partial\boldsymbol{w}_c}{\partial t}\\
&\quad + 2(m_2\mu_2 - m_1\mu_1)\mathrm{div}_x\mathbb{D}\Big(x, \frac{\partial\boldsymbol{w}_c}{\partial t}\Big) + (\rho_{2f}^0m_2 - \rho_{1f}^0m_1)\boldsymbol{g},
\end{aligned}$$

where $\alpha_c^0 = c_{\rho2}\rho_{2f}^0\alpha_2^0 + c_{\rho1}\rho_{1f}^0\alpha_1^0$ and $\alpha_r^0 = c_{\rho2}\rho_{2f}^0\alpha_2^0 - c_{\rho1}\rho_{1f}^0\alpha_1^0$.

By $\chi(\boldsymbol{x})$ denote the characteristic function of the sub-domain Ω_f:

$$\chi(\boldsymbol{x}) := \begin{cases} 1, & \boldsymbol{x}\in\Omega_f,\\ 0, & \boldsymbol{x}\notin\Omega_f. \end{cases} \tag{5.4}$$

Introduce into considerations the vector-functions $\boldsymbol{u}_c$ and $\boldsymbol{u}_r$ by the formulas

$$\boldsymbol{u}_c := \chi \boldsymbol{w}_c + (1-\chi)\boldsymbol{w}, \quad (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{5.5a}$$

$$\boldsymbol{u}_r := \chi \boldsymbol{w}_r, \quad (\boldsymbol{x},t) \in \Omega \times (0,T) \tag{5.5b}$$

and the "uniform" density

$$\rho^0 := \chi \rho_f^0 + (1-\chi)\rho_s.$$

In terms of the just introduced notation the initial conditions (5.1j)–(5.1l) take the following forms

$$\boldsymbol{u}_c|_{t=0} = \boldsymbol{u}_c^0 := \chi\Big(\frac{m_1\rho_{1f}^0}{\rho_f^0}\boldsymbol{w}_1^0 + \frac{m_2\rho_{2f}^0}{\rho_f^0}\boldsymbol{w}_2^0\Big) + (1-\chi)\boldsymbol{w}^0, \tag{5.6a}$$

$$\boldsymbol{u}_r|_{t=0} = \boldsymbol{u}_r^0 := \chi(\boldsymbol{w}_2^0 - \boldsymbol{w}_1^0), \tag{5.6b}$$

$$\frac{\partial \boldsymbol{u}_c}{\partial t}\Big|_{t=0} = \boldsymbol{v}_c^0 := \chi\Big(\frac{m_1\rho_{1f}^0}{\rho_f^0}\boldsymbol{v}_1^0 + \frac{m_2\rho_{2f}^0}{\rho_f^0}\boldsymbol{v}_2^0\Big) + (1-\chi)\boldsymbol{v}^0, \tag{5.6c}$$

$$\frac{\partial \boldsymbol{u}_r}{\partial t}\Big|_{t=0} = \boldsymbol{v}_r^0 := \chi(\boldsymbol{v}_2^0 - \boldsymbol{v}_1^0). \tag{5.6d}$$

Fix an arbitrary $\tau \in (0,T]$, multiply (5.3a) and (5.1b) by an arbitrary smooth test vector-function $\boldsymbol{\Phi}(\boldsymbol{x},t)$ defined in the whole domain Ω and vanishing on $\partial\Omega$, integrate over $\Omega_f \times (0,\tau)$ and $\Omega_s \times (0,\tau)$, respectively, and integrate by parts with respect to $\boldsymbol{x}$ and t. Taking into account the interface condition (5.1d) we observe that the integrals over interface Γ_0 cancel, and taking into account initial data (5.6), we then end up with the integral equality

$$\begin{aligned}
&\int_0^\tau \int_\Omega \Big\{ \rho^0 \frac{\partial \boldsymbol{u}_c}{\partial t} \cdot \frac{\partial \boldsymbol{\Phi}}{\partial t} - \chi \alpha_c^0 \mathrm{div}_x \boldsymbol{u}_c \, \mathrm{div}_x \boldsymbol{\Phi} \\
&-\chi\Big[\Big(\nu_1 - \frac{2}{3}\mu_1\Big)m_1 + \Big(\nu_2 - \frac{2}{3}\mu_2\Big)m_2\Big]\mathrm{div}_x \frac{\partial \boldsymbol{u}_c}{\partial t}\,\mathrm{div}_x \boldsymbol{\Phi} \\
&-(1-\chi)\Big(\eta - \frac{2}{3}\lambda\Big)\mathrm{div}_x \boldsymbol{u}_c \mathrm{div}_x \boldsymbol{\Phi} \\
&-2\chi(\mu_1 m_1 + \mu_2 m_2)\mathbb{D}\Big(x, \frac{\partial \boldsymbol{u}_c}{\partial t}\Big) : \mathbb{D}(x,\boldsymbol{\Phi}) \\
&-2(1-\chi)\lambda \mathbb{D}(x,\boldsymbol{u}_c) : \mathbb{D}(x,\boldsymbol{\Phi}) \\
&-\chi \frac{m_1 m_2}{\rho_f^0}\Big[\alpha_c^0(\rho_{1f}^0 - \rho_{2f}^0)\mathrm{div}_x \boldsymbol{u}_r \mathrm{div}_x \boldsymbol{\Phi} \\
&+\Big(\rho_{1f}^0\Big(\nu_2 - \frac{2}{3}\mu_2\Big) - \rho_{2f}^0\Big(\nu_1 - \frac{2}{3}\mu_1\Big)\Big)\mathrm{div}_x \frac{\partial \boldsymbol{u}_r}{\partial t}\mathrm{div}_x \boldsymbol{\Phi} \\
&+2(\rho_{1f}^0\mu_2 - \rho_{2f}^0\mu_1)\mathbb{D}\Big(x, \frac{\partial \boldsymbol{u}_r}{\partial t}\Big) : \mathbb{D}(x,\boldsymbol{\Phi})\Big] + \rho^0 \boldsymbol{g}\cdot\boldsymbol{\Phi}\Big\} d\boldsymbol{x}dt = \\
&\int_\Omega \rho^0 \frac{\partial \boldsymbol{u}_c}{\partial t}(\boldsymbol{x},\tau)\cdot \boldsymbol{\Phi}(\boldsymbol{x},\tau)d\boldsymbol{x} - \int_\Omega \rho^0 \boldsymbol{v}_c^0 \cdot \boldsymbol{\Phi}(\boldsymbol{x},0)d\boldsymbol{x}.
\end{aligned} \tag{5.7a}$$

Analogously, fix arbitrary $\tau \in (0,T]$, multiply (5.3b) by an arbitrary test vector-function $\boldsymbol{\Psi}(\boldsymbol{x},t)$, vanishing on $\partial\Omega_f$, integrate with respect to $\boldsymbol{x}$ and t on $\Omega_f \times (0,\tau)$ by parts. Taking into account initial data (5.6), we arrive at the following integral equality:

$$\begin{aligned}
(5.7\text{b})\quad &\int_0^\tau\int_{\Omega_f}\Big\{2\rho_{1f}^0\rho_{2f}^0\frac{m_1m_2}{\rho_f^0}\frac{\partial \boldsymbol{u}_r}{\partial t}\cdot\frac{\partial \boldsymbol{\Psi}}{\partial t}+(\rho_{2f}^0m_2-\rho_{1f}^0m_1)\frac{\partial \boldsymbol{u}_c}{\partial t}\cdot\frac{\partial \boldsymbol{\Psi}}{\partial t}\\
&\quad-\frac{m_1m_2}{\rho_f^0}\Big[\alpha_r^0(\rho_{1f}^0-\rho_{2f}^0)\mathrm{div}_x\boldsymbol{u}_r\mathrm{div}_x\boldsymbol{\Psi}\\
&\quad+\Big(\rho_{2f}^0\Big(\nu_1-\frac{2}{3}\mu_1\Big)+\rho_{1f}^0\Big(\nu_2-\frac{2}{3}\mu_2\Big)\Big)\mathrm{div}_x\frac{\partial \boldsymbol{u}_r}{\partial t}\mathrm{div}_x\boldsymbol{\Psi}\\
&\quad+2(\rho_{2f}^0\mu_1+\rho_{1f}^0\mu_2)\mathbb{D}\Big(x,\frac{\partial \boldsymbol{u}_r}{\partial t}\Big):\mathbb{D}(x,\boldsymbol{\Psi})\Big]-2K_F\frac{\partial \boldsymbol{u}_r}{\partial t}\cdot\boldsymbol{\Psi}\\
&\quad-\alpha_r^0\mathrm{div}_x\boldsymbol{u}_c\mathrm{div}_x\boldsymbol{\Psi}-\Big(\Big(\nu_2-\frac{2}{3}\mu_2\Big)m_2-\Big(\nu_1-\frac{2}{3}\mu_1\Big)m_1\Big)\mathrm{div}_x\frac{\partial \boldsymbol{u}_c}{\partial t}\mathrm{div}_x\boldsymbol{\Psi}\\
&\quad-2(m_2\mu_2-m_1\mu_1)\mathbb{D}\Big(x,\frac{\partial \boldsymbol{u}_c}{\partial t}\Big):\mathbb{D}(x,\boldsymbol{\Psi})+(\rho_{2f}^0m_2-\rho_{1f}^0m_1)\boldsymbol{g}\cdot\boldsymbol{\Psi}\Big\}d\boldsymbol{x}dt=\\
&\int_{\Omega_f}\Big\{2\rho_{1f}^0\rho_{2f}^0\frac{m_1m_2}{\rho_f^0}\frac{\partial \boldsymbol{u}_r}{\partial t}(\boldsymbol{x},\tau)\cdot\boldsymbol{\Psi}(\boldsymbol{x},\tau)+(\rho_{2f}^0m_2-\rho_{1f}^0m_1)\frac{\partial \boldsymbol{u}_c}{\partial t}(\boldsymbol{x},\tau)\cdot\boldsymbol{\Psi}(\boldsymbol{x},\tau)\Big\}d\boldsymbol{x}\\
&\quad-\int_{\Omega_f}\Big\{2\rho_{1f}^0\rho_{2f}^0\frac{m_1m_2}{\rho_f^0}\boldsymbol{v}_r^0\cdot\boldsymbol{\Psi}(\boldsymbol{x},0)+(\rho_{2f}^0m_2-\rho_{1f}^0m_1)\boldsymbol{v}_c^0\cdot\boldsymbol{\Psi}(\boldsymbol{x},0)\Big\}d\boldsymbol{x}.
\end{aligned}$$

Now we are in a position to formulate a definition of weak generalized solution of Model IT_0:

Definition 1. *A pair of vector-functions* $\{\boldsymbol{u}_c(\boldsymbol{x},t),\ \boldsymbol{u}_r(\boldsymbol{x},t)\}$ *is called a weak generalized solution of Model* IT_0*, if these functions satisfy*

(1) *regularity conditions*

$$(5.8\text{a})\qquad \boldsymbol{u}_c,\ \boldsymbol{u}_r\ \in\ L^\infty(0,T;H_0^1(\Omega)),$$

$$(5.8\text{b})\qquad \frac{\partial \boldsymbol{u}_c}{\partial t},\ \frac{\partial \boldsymbol{u}_r}{\partial t}\ \in\ L^\infty(0,T;L^2(\Omega)),$$

$$(5.8\text{c})\qquad \frac{\partial \boldsymbol{u}_r}{\partial t}\ \in\ L^2(0,T;H_0^1(\Omega)),$$

$$(5.8\text{d})\qquad \chi\, div_x\frac{\partial \boldsymbol{u}_c}{\partial t},\ \chi\mathbb{D}\Big(x,\frac{\partial \boldsymbol{u}_c}{\partial t}\Big)\ \in\ L^2(0,T;L^2(\Omega)),$$

(2) *the initial conditions* (5.6a)–(5.6d) *in the trace sense on* Ω*,*
(3) *the integral equality* (5.7a) *for all* $\tau\in[0,T]$ *and all* $\boldsymbol{\Phi}\in C^1(\Omega\times[0,T])$ *such that* $\boldsymbol{\Phi}|_{\partial\Omega}=0$*,*
(4) *the integral equality* (5.7b) *for all* $\tau\in[0,T]$ *and all* $\boldsymbol{\Psi}\in C^1(\Omega_f\times[0,T])$ *such that* $\boldsymbol{\Psi}|_{\partial\Omega_f}=0$*,*
(5) *the equality* $(1-\chi)\boldsymbol{u}_r=0$ *a.e. in* Ω*, that is* $supp(\boldsymbol{u}_r)\subset\Omega_f$*.*

Remark 2. *Note that if* $\boldsymbol{u}_c$ *and* $\boldsymbol{u}_r$ *satisfy equations* (5.3b)*,* (5.3a) *and* (5.1b) *in the sense of distributions and satisfy regularity conditions* (5.8)*, then* $t\mapsto\dfrac{\partial \boldsymbol{u}_c}{\partial t}$ *and* $t\mapsto\dfrac{\partial \boldsymbol{u}_r}{\partial t}$ *are weakly continuous mappings of the interval* $[0,T]$ *into the Lebesgue space* $L^2(\Omega)$ [15, Ch. III, Sec. 1]*. Therefore, firstly, the first integral on the right-hand side in* (5.7a) *and the first integral on the right-hand side in* (5.7b) *are well defined* *for a generalized solution for any* $\tau \in [0,T]$*; secondly, the traces* $\boldsymbol{u}_c|_{t=0}$*,* $\boldsymbol{u}_r|_{t=0}$*,* $\dfrac{\partial \boldsymbol{u}_c}{\partial t}\Big|_{t=0}$*, and* $\dfrac{\partial \boldsymbol{u}_r}{\partial t}\Big|_{t=0}$ *are well defined in the initial conditions* (5.6a)–(5.6d).

Remark 3. *The regularity condition* (5.8a) *implies that* $\boldsymbol{u}_c$ *and* $\boldsymbol{u}_r$ *vanish on the surface* $\partial\Omega \times (0,T)$*. In turn, on the strength of representations* (5.2) *and* (5.5)*, this means that the boundary conditions* (5.1i) *in the formulation of Model* IT_0 *hold true in the strong trace sense.*

Remark 4. *The above derivation of the integral equalities* (5.7a) *and* (5.7b) *from the equations of Model* IT_0 *clearly shows that if there exists a classical solution of Model* IT_0*, then this solution is a generalized solution in the sense of Definition 1. The inverse assertion holds true as well: if* $\boldsymbol{g} \in C(\bar{\Omega} \times [0,T])$ *and there exists a generalized solution of Model* IT_0 *in the sense of Definition 1 such that* $\dfrac{\partial \boldsymbol{u}_c}{\partial t}, \dfrac{\partial \boldsymbol{u}_r}{\partial t} \in C([0,T];C^2(\Omega_f))$*,* $\dfrac{\partial^2 \boldsymbol{u}_c}{\partial t^2}, \dfrac{\partial^2 \boldsymbol{u}_r}{\partial t^2} \in C([0,T];C(\Omega_f))$*,* $\boldsymbol{u}_c \in C([0,T];C^2(\Omega_s))$*,* $\dfrac{\partial^2 \boldsymbol{u}_c}{\partial t^2} \in C([0,T];C(\Omega_s))$*, then introducing* $\boldsymbol{w}_1 := \boldsymbol{u}_c - \dfrac{m_2\rho^0_{2f}}{\rho^0_f}\boldsymbol{u}_r$ *and* $\boldsymbol{w}_2 := \boldsymbol{u}_c + \dfrac{m_1\rho^0_{1f}}{\rho^0_f}\boldsymbol{u}_r$ *for* $\boldsymbol{x} \in \Omega_f$ *and* $\boldsymbol{w} := \boldsymbol{u}_c$ *for* $\boldsymbol{x} \in \Omega_s$ *we see that the triple* $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w})$ *solves the equations* (5.1a) *and* (5.1b) *and the contact discontinuity conditions* (5.1c) *and* (5.1d)*, that is,* $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w})$ *is a classical solution of Model* IT_0*. Indeed, firstly, in* (5.7a) *take a test function* $\boldsymbol{\Phi}$ *with* $supp\, \Phi \subset \Omega_s \times [0,T]$ *and integrate by parts. Since* $\boldsymbol{u}_r = 0$ *on* Ω_s *then* (5.7a) *reduces to*

$$\int_0^\tau \int_{\Omega_s} \Big[-\rho_s \frac{\partial^2 \boldsymbol{w}}{\partial t^2} + \Big(\eta - \frac{2}{3}\lambda\Big)\nabla_x div_x \boldsymbol{w} + div_x(2\lambda \mathbb{D}(x,\boldsymbol{w})) + \rho_s \boldsymbol{g}\Big] \cdot \boldsymbol{\Phi} d\boldsymbol{x} dt = 0.$$

Since $\boldsymbol{\Phi}$ *is arbitrary enough, due to the DuBois–Raymond lemma this equality yields the equation* (5.1b) *everywhere in* $\Omega_s \times (0,T)$.

Secondly, in (5.7a) *take a test function* $\boldsymbol{\Phi} = \boldsymbol{\Psi}$ *with* $supp\, \boldsymbol{\Psi} \subset \Omega_f \times [0,T]$*, subtract the left hand and the right hand sides of* (5.7b) *from the left hand and the right hand sides of* (5.7a)*, respectively, and then divide the both sides of the resulting equality by two. After this, integrating by parts we end up with the integral equality*

$$\int_0^\tau \int_{\Omega_f} \Big\{-\rho^0_{1f} m_1 \frac{\partial^2 \boldsymbol{w}_1}{\partial t^2} + c_{\rho 1}\rho^0_{1f}\alpha^0_1 \nabla_x (m_1 div_x \boldsymbol{w}_1 + m_2 div_x \boldsymbol{w}_2)$$
$$+ div_x \Big[m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big)\Big(div_x \frac{\partial \boldsymbol{w}_1}{\partial t}\Big)\mathbb{I} + 2m_1\mu_1 \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_1}{\partial t}\Big)\Big]$$
$$+ m_1\rho^0_{1f}\boldsymbol{g} - \boldsymbol{F}_{12}\Big\} \cdot \boldsymbol{\Psi} d\boldsymbol{x} dt = 0.$$

Since $\boldsymbol{\Psi}$ *is arbitrary enough this equality yields the equation* (5.1a) *(with* $i = 1$*) everywhere in* $\Omega_f \times (0,T)$*. Analogously we also derive* (5.1a) *with* $i = 2$ *everywhere in* $\Omega_f \times (0,T)$.

Thirdly, integrate by parts in (5.7a) *so that derivatives of* Φ *disappear. On the strength of equations* (5.1a) *and* (5.1b) *the sum of all integrals over* $\Omega_s \times (0,T)$ *and*

$\Omega_f \times (0,T)$ vanishes. Thus we get

$$\int_0^\tau \int_{\partial\Omega_f \cap \Gamma_0} \Big\{ \alpha_c^0 (m_1\, div_x \boldsymbol{w}_1 + m_2\, div_x \boldsymbol{w}_2)\boldsymbol{n} + m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big)\Big(div_x \frac{\partial \boldsymbol{w}_1}{\partial t}\Big)\boldsymbol{n} + m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big)\Big(div_x \frac{\partial \boldsymbol{w}_2}{\partial t}\Big)\boldsymbol{n} + 2m_1\mu_1\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_1}{\partial t}\Big)\boldsymbol{n} + 2m_2\mu_2\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_2}{\partial t}\Big)\boldsymbol{n}\Big\} \cdot \boldsymbol{\Phi}\, d\sigma dt - \int_0^\tau \int_{\partial\Omega_s \cap \Gamma_0} \Big(\eta - \frac{2}{3}\lambda\Big)(div_x \boldsymbol{w})\boldsymbol{n} + 2\lambda\mathbb{D}(x,\boldsymbol{w})\boldsymbol{n}\Big\} \cdot \boldsymbol{\Phi}\, d\sigma dt = 0.$$

Here recall that $\boldsymbol{n}$ is a unit normal to Γ_0 pointing into Ω_f. Since $\boldsymbol{\Phi}$ is arbitrary enough this equality yields the contact discontinuity condition (5.1d) *everywhere on $\Gamma_0 \times (0,T)$. This equality means also that the condition* (5.1d) *in the notion of generalized solution is understood in the trace sense.*

Finally we notice that the contact discontinuity condition (5.1c) *holds true since $\boldsymbol{u}_r$ vanishes on $\Gamma_0 \times (0,T)$ and by this we finish this observation of consistency of the notion of generalized solutions in the sense of Definition 1 with the notion of classical solutions.*

6. Galerkin's approximations

We construct Galerkin's system associated with Model IT_0 as follows.

First, multiply (5.3a) and (5.1b) by an arbitrary smooth test vector-function $\boldsymbol{\varphi}(\boldsymbol{x})$ defined on the whole domain Ω and vanishing on $\partial\Omega$, integrate over Ω_f and Ω_s, respectively, and integrate by parts in $\boldsymbol{x}$. Taking into account the interface condition (5.1d), we see that the integrals over interface Γ_0 cancel and thus (formally) arrive at the integral equality

$$\text{(6.1a)} \quad \int_\Omega \Big\{ \rho^0 \frac{\partial^2 \boldsymbol{u}_c}{\partial t^2} \cdot \boldsymbol{\varphi} + \chi\alpha_c^0 \mathrm{div}_x \boldsymbol{u}_c \mathrm{div}_x \boldsymbol{\varphi} + \chi\Big[\Big(\nu_1 - \frac{2}{3}\mu_1\Big)m_1 + \Big(\nu_2 - \frac{2}{3}\mu_2\Big)m_2\Big]\mathrm{div}_x \frac{\partial \boldsymbol{u}_c}{\partial t}\mathrm{div}_x \boldsymbol{\varphi} + (1-\chi)\Big(\eta - \frac{2}{3}\lambda\Big)\mathrm{div}_x \boldsymbol{u}_c \mathrm{div}_x \boldsymbol{\varphi} + 2\chi(\mu_1 m_1 + \mu_2 m_2)\mathbb{D}\Big(x, \frac{\partial \boldsymbol{u}_c}{\partial t}\Big) : \mathbb{D}(x,\boldsymbol{\varphi}) + 2(1-\chi)\lambda\mathbb{D}(x,\boldsymbol{u}_c) : \mathbb{D}(x,\boldsymbol{\varphi}) + \chi\frac{m_1 m_2}{\rho_f^0}\Big[\alpha_c^0(\rho_{1f}^0 - \rho_{2f}^0)\mathrm{div}_x \boldsymbol{u}_r \mathrm{div}_x \boldsymbol{\varphi} + \Big(\rho_{1f}^0\Big(\nu_2 - \frac{2}{3}\mu_2\Big) - \rho_{2f}^0\Big(\nu_1 - \frac{2}{3}\mu_1\Big)\Big)\mathrm{div}_x \frac{\partial \boldsymbol{u}_r}{\partial t}\mathrm{div}_x \boldsymbol{\varphi} + 2(\rho_{1f}^0\mu_2 - \rho_{2f}^0\mu_1)\mathbb{D}\Big(x, \frac{\partial \boldsymbol{u}_r}{\partial t}\Big) : \mathbb{D}(x,\boldsymbol{\varphi})\Big]\Big\} d\boldsymbol{x} = \int_\Omega \rho^0 \boldsymbol{g} \cdot \boldsymbol{\varphi} d\boldsymbol{x}.$$

Analogously, multiplying (5.3b) by an arbitrary smooth test vector-function $\boldsymbol{\psi}(\boldsymbol{x})$ vanishing on $\partial\Omega_f$, and integrating with respect to $\boldsymbol{x}$ on Ω_f by parts, we (formally, again) get the integral equality

$$\text{(6.1b)}\quad \int_{\Omega_f}\Big\{2\rho^0_{1f}\rho^0_{2f}\frac{m_1m_2}{\rho^0_f}\frac{\partial^2\boldsymbol{u}_r}{\partial t^2}\cdot\boldsymbol{\psi}+(\rho^0_{2f}m_2-\rho^0_{1f}m_1)\frac{\partial^2\boldsymbol{u}_c}{\partial t^2}\cdot\boldsymbol{\psi}$$
$$+\frac{m_1m_2}{\rho^0_f}\Big[\alpha^0_r(\rho^0_{1f}-\rho^0_{2f})\mathrm{div}_x\boldsymbol{u}_r\mathrm{div}_x\boldsymbol{\psi}$$
$$+\Big(\rho^0_{2f}\Big(\nu_1-\frac{2}{3}\mu_1\Big)+\rho^0_{1f}\Big(\nu_2-\frac{2}{3}\mu_2\Big)\Big)\mathrm{div}_x\frac{\partial\boldsymbol{u}_r}{\partial t}\mathrm{div}_x\boldsymbol{\psi}$$
$$+2(\rho^0_{2f}\mu_1+\rho^0_{1f}\mu_2)\mathbb{D}\Big(x,\frac{\partial\boldsymbol{u}_r}{\partial t}\Big):\mathbb{D}(x,\boldsymbol{\psi})\Big]+2K_F\frac{\partial\boldsymbol{u}_r}{\partial t}\cdot\boldsymbol{\psi}$$
$$+\alpha^0_r\mathrm{div}_x\boldsymbol{u}_c\mathrm{div}_x\boldsymbol{\psi}+\Big(\Big(\nu_2-\frac{2}{3}\mu_2\Big)m_2-\Big(\nu_1-\frac{2}{3}\mu_1\Big)m_1\Big)\mathrm{div}_x\frac{\partial\boldsymbol{u}_c}{\partial t}\mathrm{div}_x\boldsymbol{\psi}$$
$$+2(m_2\mu_2-m_1\mu_1)\mathbb{D}\Big(x,\frac{\partial\boldsymbol{u}_c}{\partial t}\Big):\mathbb{D}(x,\boldsymbol{\psi})\Big\}d\boldsymbol{x}=\int_{\Omega_f}(\rho^0_{2f}m_2-\rho^0_{1f}m_1)\boldsymbol{g}\cdot\boldsymbol{\psi}d\boldsymbol{x}.$$

Let $\{\boldsymbol{\psi}_l,\boldsymbol{\varphi}_l\}_{l=1}^{\infty}$ be a total orthogonal system in $H^1_0(\Omega)$ and at the same time be a total orthonormal system in $L^2(\Omega)$ such that $\mathrm{supp}(\boldsymbol{\psi}_l)\subset\Omega_f$. Such a vector-function system (called Galerkin's basis) can be constructed, for example, as the set of solutions of the eigenfunction problems

$$\text{(6.2a)}\qquad \triangle_x\boldsymbol{\psi}=\lambda\boldsymbol{\psi},\quad \boldsymbol{\psi}|_{\partial\Omega_f}=0,\quad\text{and}$$
$$\text{(6.2b)}\qquad \triangle_x\boldsymbol{\varphi}=\lambda\boldsymbol{\varphi},\quad \boldsymbol{\varphi}|_{\partial\Omega}=0.$$

Here note that any solution of the former eigenfunction problem upon zero-extension beyond Ω_f becomes a solution of the latter problem [16, Sec. II.4].

We seek for Galerkin's approximations of vectors $\boldsymbol{u}_c(\boldsymbol{x},t)$ and $\boldsymbol{u}_r(\boldsymbol{x},t)$ in the forms

$$\text{(6.3)}\quad \boldsymbol{u}^n_c(\boldsymbol{x},t)=\sum_{l=1}^n a^n_l(t)\boldsymbol{\psi}_l(\boldsymbol{x})+\sum_{l=1}^n b^n_l(t)\boldsymbol{\varphi}_l(\boldsymbol{x}),\quad \boldsymbol{u}^n_r(\boldsymbol{x},t)=\sum_{l=1}^n c^n_l(t)\boldsymbol{\psi}_l(\boldsymbol{x}).$$

where functions $a^n_l(t)$, $b^n_l(t)$ and $c^n_l(t)$ $(l=1,\dots,n)$ are unknown. In order to find them, in equations (6.1a) and (6.1b) we substitute representations (6.3) on the places of $\boldsymbol{u}_c(\boldsymbol{x},t)$ and $\boldsymbol{u}_r(\boldsymbol{x},t)$, and Galerkin's basic functions $\boldsymbol{\psi}_l$ and $\boldsymbol{\varphi}_l$ $(l=1,\dots,n)$ on the places of $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$. Thus we arrive at the following system of the second-order ordinary differential equations, called Galerkin's system:

(6.4a)
$$\rho^0_f\frac{d^2\boldsymbol{a}^n}{dt^2}+\alpha^0_c\big[\mathbb{B}^n_2\boldsymbol{a}^n+\mathbb{B}^n_1\boldsymbol{b}^n\big]+\Big(\Big(\nu_1-\frac{2}{3}\mu_1\Big)m_1+\Big(\nu_2-\frac{2}{3}\mu_2\Big)m_2\Big)\Big[\mathbb{B}^n_2\frac{d\boldsymbol{a}^n}{dt}+\mathbb{B}^n_1\frac{d\boldsymbol{b}^n}{dt}\Big]$$
$$+2(\mu_1m_1+\mu_2m_2)\Big[\mathbb{C}^n_2\frac{d\boldsymbol{a}^n}{dt}+\mathbb{C}^n_1\frac{d\boldsymbol{b}^n}{dt}\Big]+\frac{m_1m_2}{\rho^0_f}\Big[\alpha^0_c(\rho^0_{1f}-\rho^0_{2f})\mathbb{B}^n_2\boldsymbol{c}^n$$
$$+\Big(\rho^0_{1f}\Big(\nu_2-\frac{2}{3}\mu_2\Big)-\rho^0_{2f}\Big(\nu_1-\frac{2}{3}\mu_1\Big)\Big)\mathbb{B}^n_2\frac{d\boldsymbol{c}^n}{dt}+2(\rho^0_{1f}\mu_2-\rho^0_{2f}\mu_1)\mathbb{C}^n_2\frac{d\boldsymbol{c}^n}{dt}\Big]=\rho^0_f\boldsymbol{g}^n_1,$$

$$
\begin{aligned}
\text{(6.4b)}\quad & \big[\rho_f^0\mathbb{A}_1^n + \rho_s\mathbb{A}_2^n\big]\frac{d^2\boldsymbol{b}^n}{dt^2} + \alpha_c^0\big[(\mathbb{B}_1^n)^t\boldsymbol{a}^n + \mathbb{B}_3^n\boldsymbol{b}^n\big] \\
& + \Big(\Big(\nu_1 - \frac{2}{3}\mu_1\Big)m_1 + \Big(\nu_2 - \frac{2}{3}\mu_2\Big)m_2\Big)\Big[(\mathbb{B}_1^n)^t\frac{d\boldsymbol{a}^n}{dt} + \mathbb{B}_3^n\frac{d\boldsymbol{b}^n}{dt}\Big] + \Big(\eta - \frac{2}{3}\lambda\Big)\mathbb{B}_4^n\boldsymbol{b}^n \\
& + 2(\mu_1 m_1 + \mu_2 m_2)\Big[(\mathbb{C}_1^n)^t\frac{d\boldsymbol{a}^n}{dt} + \mathbb{C}_3^n\frac{d\boldsymbol{b}^n}{dt}\Big] + 2\lambda\mathbb{C}_4^n\boldsymbol{b}^n \\
& + \frac{m_1 m_2}{\rho_f^0}\Big[\alpha_c^0(\rho_{1f}^0 - \rho_{2f}^0)(\mathbb{B}_1^n)^t\boldsymbol{c}^n + \Big(\rho_{1f}^0\Big(\nu_2 - \frac{2}{3}\mu_2\Big) - \rho_{2f}^0\Big(\nu_1 - \frac{2}{3}\mu_1\Big)\Big)(\mathbb{C}_1^n)^t\frac{d\boldsymbol{c}^n}{dt} \\
& + 2(\rho_{1f}^0\mu_2 - \rho_{2f}^0\mu_1)(\mathbb{C}_1^n)^t\frac{d\boldsymbol{c}^n}{dt}\Big] = \rho_f^0\boldsymbol{g}_2^n + \rho_s\boldsymbol{g}_3^n,
\end{aligned}
$$

$$
\begin{aligned}
\text{(6.4c)}\quad & 2\rho_{1f}^0\rho_{2f}^0\frac{m_1 m_2}{\rho_f^0}\frac{d^2\boldsymbol{c}^n}{dt^2} + (\rho_{2f}^0 m_2 - \rho_{1f}^0 m_1)\frac{d^2\boldsymbol{a}^n}{dt^2} \\
& + \frac{m_1 m_2}{\rho_f^0}\Big[\alpha_r^0(\rho_{1f}^0 - \rho_{2f}^0)\mathbb{B}_2^n\boldsymbol{c}^n + \Big(\rho_{2f}^0\Big(\nu_1 - \frac{2}{3}\mu_1\Big) + \rho_{1f}^0\Big(\nu_2 - \frac{2}{3}\mu_2\Big)\Big)\mathbb{B}_2^n\frac{d\boldsymbol{c}^n}{dt} \\
& + 2(\rho_{2f}^0\mu_1 + \rho_{1f}^0\mu_2)\mathbb{C}_2^n\frac{d\boldsymbol{c}^n}{dt}\Big] + 2K_F\frac{d\boldsymbol{c}^n}{dt} + \alpha_r^0\big[\mathbb{B}_2^n\boldsymbol{a}^n + \mathbb{B}_1^n\boldsymbol{b}^n\big] \\
& + \Big(\Big(\nu_2 - \frac{2}{3}\mu_2\Big)m_2 - \Big(\nu_1 - \frac{2}{3}\mu_1\Big)m_1\Big)\Big[\mathbb{B}_2^n\frac{d\boldsymbol{a}^n}{dt} + \mathbb{B}_1^n\frac{d\boldsymbol{b}^n}{dt}\Big] \\
& + 2(m_2\mu_2 - m_1\mu_1)\Big[\mathbb{C}_2^n\frac{d\boldsymbol{a}^n}{dt} + \mathbb{C}_1^n\frac{d\boldsymbol{b}^n}{dt}\Big] = (\rho_{2f}^0 m_2 - \rho_{1f}^0 m_1)\boldsymbol{g}_1^n.
\end{aligned}
$$

Here we have set

$$
\begin{aligned}
&\boldsymbol{a}^n(t) = (a_1^n(t), \dots, a_n^n(t)), \\
&\boldsymbol{b}^n(t) = (b_1^n(t), \dots, b_n^n(t)), \\
&\boldsymbol{c}^n(t) = (c_1^n(t), \dots, c_n^n(t)), \\
&\big(\mathbb{A}_1^n\big)_{lk} = \int_{\Omega_f} \boldsymbol{\varphi}_l(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_k(\boldsymbol{x})d\boldsymbol{x}, \\
&\big(\mathbb{A}_2^n\big)_{lk} = \int_{\Omega_s} \boldsymbol{\varphi}_l(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_k(\boldsymbol{x})d\boldsymbol{x}, \\
&\big(\mathbb{B}_1^n\big)_{lk} = \int_{\Omega_f} \mathrm{div}_x\boldsymbol{\varphi}_l(\boldsymbol{x})\mathrm{div}_x\boldsymbol{\psi}_k(\boldsymbol{x})d\boldsymbol{x}, \\
&\big(\mathbb{B}_2^n\big)_{lk} = \int_{\Omega_f} \mathrm{div}_x\boldsymbol{\psi}_l(\boldsymbol{x})\mathrm{div}_x\boldsymbol{\psi}_k(\boldsymbol{x})d\boldsymbol{x}, \\
&\big(\mathbb{B}_3^n\big)_{lk} = \int_{\Omega_f} \mathrm{div}_x\boldsymbol{\varphi}_l(\boldsymbol{x})\mathrm{div}_x\boldsymbol{\varphi}_k(\boldsymbol{x})d\boldsymbol{x}, \\
&\big(\mathbb{B}_4^n\big)_{lk} = \int_{\Omega_s} \mathrm{div}_x\boldsymbol{\varphi}_l(\boldsymbol{x})\mathrm{div}_x\boldsymbol{\varphi}_k(\boldsymbol{x})d\boldsymbol{x}, \\
&\big(\mathbb{C}_1^n\big)_{lk} = \int_{\Omega_f} \mathbb{D}(x, \boldsymbol{\varphi}_l(\boldsymbol{x})) : \mathbb{D}(x, \boldsymbol{\psi}_k(\boldsymbol{x}))d\boldsymbol{x}, \\
&\big(\mathbb{C}_2^n\big)_{lk} = \int_{\Omega_f} \mathbb{D}(x, \boldsymbol{\psi}_l(\boldsymbol{x})) : \mathbb{D}(x, \boldsymbol{\psi}_k(\boldsymbol{x}))d\boldsymbol{x},
\end{aligned}
$$

$$\left(\mathbb{C}_3^n\right)_{lk} = \int_{\Omega_f} \mathbb{D}(x, \boldsymbol{\varphi}_l(\boldsymbol{x})) : \mathbb{D}(x, \boldsymbol{\varphi}_k(\boldsymbol{x})) d\boldsymbol{x},$$

$$\left(\mathbb{C}_4^n\right)_{lk} = \int_{\Omega_s} \mathbb{D}(x, \boldsymbol{\varphi}_l(\boldsymbol{x})) : \mathbb{D}(x, \boldsymbol{\varphi}_k(\boldsymbol{x})) d\boldsymbol{x},$$

$$\left(\boldsymbol{g}_1^n\right)_k = \int_{\Omega_f} \boldsymbol{g} \cdot \boldsymbol{\psi}_k(\boldsymbol{x}) d\boldsymbol{x},$$

$$\left(\boldsymbol{g}_2^n\right)_k = \int_{\Omega_f} \boldsymbol{g} \cdot \boldsymbol{\varphi}_k(\boldsymbol{x}) d\boldsymbol{x}$$

$$\left(\boldsymbol{g}_3^n\right)_{lk} = \int_{\Omega_s} \boldsymbol{g} \cdot \boldsymbol{\varphi}_k(\boldsymbol{x}) d\boldsymbol{x}.$$

System (6.4a)–(6.4c) is supplemented with the initial data

$$\text{(6.5a)}\quad a_l^n|_{t=0} = a_l^0 := \int_\Omega \boldsymbol{u}_c^0(\boldsymbol{x}) \cdot \boldsymbol{\psi}_l(\boldsymbol{x}) d\boldsymbol{x}, \quad \frac{da_l^n}{dt}|_{t=0} = a_l' := \int_\Omega \boldsymbol{v}_c^0(\boldsymbol{x}) \cdot \boldsymbol{\psi}_l(\boldsymbol{x}) d\boldsymbol{x},$$

$$\text{(6.5b)}\quad b_l^n|_{t=0} = b_l^0 := \int_\Omega \boldsymbol{u}_c^0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_l(\boldsymbol{x}) d\boldsymbol{x}, \quad \frac{db_l^n}{dt}|_{t=0} = b_l' := \int_\Omega \boldsymbol{v}_c^0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_l(\boldsymbol{x}) d\boldsymbol{x},$$

$$\text{(6.5c)}\quad c_l^n|_{t=0} = c_l^0 := \int_\Omega \boldsymbol{u}_r^0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_l(\boldsymbol{x}) d\boldsymbol{x}, \quad \frac{dc_l^n}{dt}|_{t=0} = c_l' := \int_\Omega \boldsymbol{v}_r^0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_l(\boldsymbol{x}) d\boldsymbol{x}.$$

Lemma 1. *The matrix $\rho_f^0 \mathbb{A}_1^n + \rho_s \mathbb{A}_2^n$ is positive definite and therefore invertible.*

Proof. Since $\rho_f^0 > 0$, $\rho_s > 0$ and the set $\{\boldsymbol{\varphi}_l\}$ is orthonormal in $L^2(\Omega)$, for any $\boldsymbol{\xi} \in \mathbb{R}^n$ we get

$$(\rho_f^0 \mathbb{A}_1^n + \rho_s \mathbb{A}_2^n)\boldsymbol{\xi} \cdot \boldsymbol{\xi} = \rho_f^0 \int_{\Omega_f} \left|\sum_{i=1}^n \xi_i \boldsymbol{\varphi}_i\right|^2 d\boldsymbol{x} + \rho_s \int_{\Omega_s} \left|\sum_{i=1}^n \xi_i \boldsymbol{\varphi}_i\right|^2 d\boldsymbol{x} \geq$$

$$\geq \min\{\rho_f^0, \rho_s\} \int_\Omega \left|\sum_{i=1}^n \xi_i \boldsymbol{\varphi}_i\right|^2 d\boldsymbol{x} = \min\{\rho_f^0, \rho_s\} |\boldsymbol{\xi}|^2,$$

which completes the proof. □

On the strength of the classical theory of systems of second-order ordinary linear differential equations, the following assertion holds true due to Lemma 1:

Proposition 1. *Galerkin's system* (6.4)*, supplemented with initial data* (6.5)*, has a unique smooth solution $(\boldsymbol{a}^n(t), \boldsymbol{b}^n(t), \boldsymbol{c}^n(t))$ for any $n \in \mathbb{N}$.*

Remark 5. *Note that n-th Galerkin's approximations $\boldsymbol{u}_c^n(\boldsymbol{x}, t)$ and $\boldsymbol{u}_r^n(\boldsymbol{x}, t)$ satisfy*

(1) *the equality $(1 - \chi)\boldsymbol{u}_r^n = 0$ a.e. in Ω,*
(2) *the initial data* (5.6) *with*

$$\text{(6.6a)}\qquad \boldsymbol{u}_c^{0n} := \sum_{l=1}^n a_l^0 \boldsymbol{\psi}_l(\boldsymbol{x}) + \sum_{l=1}^n b_l^0 \boldsymbol{\varphi}_l(\boldsymbol{x}), \qquad \boldsymbol{u}_r^{0n} := \sum_{l=1}^n c_l^0 \boldsymbol{\psi}_l(\boldsymbol{x}),$$

$$\text{(6.6b)}\qquad \boldsymbol{v}_c^{0n} := \sum_{l=1}^n a_l' \boldsymbol{\varphi}_l + \sum_{l=1}^n b_l' \boldsymbol{\psi}_l, \qquad \boldsymbol{v}_r^{0n} := \sum_{l=1}^n c_l' \boldsymbol{\psi}_l,$$

i.e., the approximate initial vectors are the partial Fourier sums of $\boldsymbol{u}_c^0$, $\boldsymbol{u}_r^0$, $\boldsymbol{v}_c^0$, $\boldsymbol{v}_r^0$.

(3) *the integral equality* (5.7a) *for test function* $\boldsymbol{\Phi} = \zeta(t)\boldsymbol{\varphi}(x)$ *with an arbitrary smooth* $\zeta(t)$ *and with* $\boldsymbol{\varphi}(x)$ *being an arbitrary function from the linear span of the set* $\{\boldsymbol{\psi}_l, \boldsymbol{\varphi}_l\}_{l=1}^m$, $\forall\ m \le n$,

(4) *the integral equality* (5.7b) *for test function* $\boldsymbol{\Psi} = \zeta(t)\boldsymbol{\psi}(x)$ *with an arbitrary smooth* $\zeta(t)$ *and with* $\boldsymbol{\psi}(x)$ *being an arbitrary function from the linear span of the set* $\{\boldsymbol{\psi}_l\}_{l=1}^m$, $\forall\ m \le n$.

7. Uniform estimates of Galerkin's approximations

We start with building up *the energy identity*, which manifests the law of conservation of mechanical energy. Firstly, we subtract the left hand and the right hand sides of (5.7b) from the left hand and the right hand sides of (5.7a), respectively, and then divide the both sides of resulting equality by two. Secondly, we add the left hand and the right hand sides of (5.7b) to the left hand and the right hand sides of (5.7a), respectively, and then divide the both sides of resulting equality by two. Thus we obtain two integral equalities for $\boldsymbol{u}_c^n$ and $\boldsymbol{u}_r^n$; each of them contains two test functions, $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$.

We take

$$\boldsymbol{\Phi} = \frac{\partial}{\partial t}\Big(\boldsymbol{u}_c^n - \frac{m_2\rho_{2f}^0}{\rho_f^0}\boldsymbol{u}_r^n\Big), \quad \boldsymbol{\Psi} = \frac{\partial \boldsymbol{u}_r^n}{\partial t}$$

in the first of these equalities and

$$\boldsymbol{\Phi} = \frac{\partial}{\partial t}\Big(\boldsymbol{u}_c^n + \frac{m_1\rho_{1f}^0}{\rho_f^0}\boldsymbol{u}_r^n\Big), \quad \boldsymbol{\Psi} = \frac{\partial \boldsymbol{u}_r^n}{\partial t}$$

in the second of these equalities. This choice of test functions is legal due to Remark 5. Denote

$$\boldsymbol{w}_1^n := \boldsymbol{u}_c^n - \frac{m_2\rho_{2f}^0}{\rho_f^0}\boldsymbol{u}_r^n, \quad \boldsymbol{w}_2^n := \boldsymbol{u}_c^n + \frac{m_1\rho_{1f}^0}{\rho_f^0}\boldsymbol{u}_r^n, \tag{7.1}$$

$$\boldsymbol{w}_1^{0n} := \boldsymbol{u}_c^{0n} - \frac{m_2\rho_{2f}^0}{\rho_f^0}\boldsymbol{u}_r^{0n}, \quad \boldsymbol{w}_2^{0n} := \boldsymbol{u}_c^{0n} + \frac{m_1\rho_{1f}^0}{\rho_f^0}\boldsymbol{u}_r^{0n}, \tag{7.2}$$

$$\boldsymbol{v}_1^{0n} := \boldsymbol{v}_c^{0n} - \frac{m_2\rho_{2f}^0}{\rho_f^0}\boldsymbol{v}_r^{0n}, \quad \boldsymbol{v}_2^{0n} := \boldsymbol{v}_c^{0n} + \frac{m_1\rho_{1f}^0}{\rho_f^0}\boldsymbol{v}_r^{0n} \tag{7.3}$$

for the sake of brevity. Summing the obtained equations we arrive at the energy identity:

$$\begin{aligned}
&\frac{1}{2}\rho_{1f}^0 m_1 \int_\Omega \chi\Big|\frac{\partial \boldsymbol{w}_1^n}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} \\
&\quad + \frac{1}{2}\rho_{2f}^0 m_2 \int_\Omega \chi\Big|\frac{\partial \boldsymbol{w}_2^n}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} + \frac{1}{2}\rho_s \int_\Omega (1-\chi)\Big|\frac{\partial \boldsymbol{u}_c^n}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} \\
&+ \frac{1}{2}c_{\rho 1}\rho_{1f}^0\alpha_1^0 m_1 \int_\Omega \chi\,|\mathrm{div}_x \boldsymbol{w}_1^n(\tau)|^2\,d\boldsymbol{x} + \frac{1}{2}c_{\rho 2}\rho_{2f}^0\alpha_2^0 m_2 \int_\Omega \chi\,|\mathrm{div}_x \boldsymbol{w}_2^n(\tau)|^2\,d\boldsymbol{x} \\
&+ \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big)\int_\Omega (1-\chi)\,|\mathrm{div}_x \boldsymbol{u}_c^n(\tau)|^2\,d\boldsymbol{x} + \frac{1}{2}\lambda\int_\Omega (1-\chi)\,|\mathbb{D}(x,\boldsymbol{u}_c^n(\tau))|^2\,d\boldsymbol{x} \\
&+m_1\Big(\nu_1-\frac{2}{3}\mu_1\Big)\int_0^\tau\int_\Omega \chi\Big|\mathrm{div}_x\frac{\partial \boldsymbol{w}_1^n}{\partial t}\Big|^2 d\boldsymbol{x}dt + m_2\Big(\nu_2-\frac{2}{3}\mu_2\Big)\int_0^\tau\int_\Omega \chi\Big|\mathrm{div}_x\frac{\partial \boldsymbol{w}_2^n}{\partial t}\Big|^2 d\boldsymbol{x}dt
\end{aligned} \tag{7.4}$$

$$+ m_1\mu_1 \int_0^\tau \int_\Omega \chi \Big|\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_1^n}{\partial t}\Big)\Big|^2 d\boldsymbol{x}dt$$
$$+ m_2\mu_2 \int_0^\tau \int_\Omega \chi \Big|\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_2^n}{\partial t}\Big)\Big|^2 d\boldsymbol{x}dt + K_F \int_0^\tau \int_\Omega \chi \Big|\frac{\partial \boldsymbol{u}_r^n}{\partial t}\Big|^2 d\boldsymbol{x}dt$$
$$= \frac{1}{2}\rho_{1f}^0 m_1 \int_\Omega \chi |\boldsymbol{v}_1^{0n}|^2 d\boldsymbol{x} + \frac{1}{2}\rho_{2f}^0 m_2 \int_\Omega \chi |\boldsymbol{v}_2^{0n}|^2 d\boldsymbol{x} + \frac{1}{2}\int_\Omega (1-\chi)|\boldsymbol{v}_c^{0n}|^2 d\boldsymbol{x}$$
$$+ \frac{1}{2} c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_1 \int_\Omega \chi \left|\mathrm{div}_x \boldsymbol{w}_1^{0n}\right|^2 d\boldsymbol{x}$$
$$+ \frac{1}{2} c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_2 \int_\Omega \chi \left|\mathrm{div}_x \boldsymbol{w}_2^{0n}\right|^2 d\boldsymbol{x}$$
$$+ \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big) \int_\Omega (1-\chi) \left|\mathrm{div}_x \boldsymbol{u}_c^{0n}\right|^2 d\boldsymbol{x} + \frac{1}{2}\lambda \int_\Omega (1-\chi) \left|\mathbb{D}(x, \boldsymbol{u}_c^{0n})\right|^2 d\boldsymbol{x}$$
$$- \int_0^\tau \int_\Omega \chi c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_2 \mathrm{div}_x \boldsymbol{w}_2^n \mathrm{div}_x \frac{\partial \boldsymbol{w}_1^n}{\partial t} d\boldsymbol{x}dt$$
$$- \int_0^\tau \int_\Omega \chi c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_1 \mathrm{div}_x \boldsymbol{w}_1^n \mathrm{div}_x \frac{\partial \boldsymbol{w}_2^n}{\partial t} d\boldsymbol{x}dt$$
$$+ \int_0^\tau \int_\Omega \chi m_1 \rho_{1f}^0 \boldsymbol{g} \cdot \frac{\partial \boldsymbol{w}_1^n}{\partial t} d\boldsymbol{x}dt + \int_0^\tau \int_\Omega \chi m_2 \rho_{2f}^0 \boldsymbol{g} \cdot \frac{\partial \boldsymbol{w}_2^n}{\partial t} d\boldsymbol{x}dt$$
$$+ \int_0^\tau \int_\Omega (1-\chi)\rho_s \boldsymbol{g} \cdot \frac{\partial \boldsymbol{u}_c^n}{\partial t} d\boldsymbol{x}dt.$$

Remark 6. *Formally, the above described procedure of derivation of the integral equality* (7.4) *is equivalent to the following one: firstly, to multiply the equation* $(5.1\mathrm{a})_1$ *by* $\partial \boldsymbol{w}_1/\partial t$*, the equation* $(5.1\mathrm{a})_2$ *by* $\partial \boldsymbol{w}_2/\partial t$*, and the equation* (5.1b) *by* $\partial \boldsymbol{w}/\partial t$*; secondly, to integrate thus obtained equalities over* Q_T *and sum them; thirdly, to integrate by parts in the certain terms, using the initial and boundary data in Model* IT_0*; finally, to introduce the notations* (5.2a), (5.2b), (5.5a), *and* (5.5b).

Applying Young's inequality to the terms in the right hand side of the energy identity (7.4) we get

$$(7.5)\quad \left|\int_0^\tau \int_\Omega \chi m_1 \rho_{1f}^0 \boldsymbol{g} \cdot \frac{\partial \boldsymbol{w}_1^n}{\partial t} d\boldsymbol{x}dt\right| \le \frac{m_1\rho_{1f}^0}{2\varepsilon_1} \int_0^\tau \int_\Omega \chi |\boldsymbol{g}|^2 d\boldsymbol{x}dt + \frac{m_1\rho_{1f}^0 \varepsilon_1}{2} \int_0^\tau \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_1^n}{\partial t}\Big|^2 d\boldsymbol{x}dt,$$

$$(7.6)\quad \left|\int_0^\tau \int_\Omega \chi m_2 \rho_{2f}^0 \boldsymbol{g} \cdot \frac{\partial \boldsymbol{w}_2^n}{\partial t} d\boldsymbol{x}dt\right| \le \frac{m_2\rho_{2f}^0}{2\varepsilon_2} \int_0^\tau \int_\Omega \chi |\boldsymbol{g}|^2 d\boldsymbol{x}dt + \frac{m_2\rho_{2f}^0 \varepsilon_1}{2} \int_0^\tau \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_2^n}{\partial t}\Big|^2 d\boldsymbol{x}dt,$$

$$(7.7)\quad \left|\int_0^\tau \int_\Omega (1-\chi)\rho_s \boldsymbol{g} \cdot \frac{\partial \boldsymbol{u}_c^n}{\partial t} d\boldsymbol{x}dt\right| \le \frac{\rho_s}{2\varepsilon_3} \int_0^\tau \int_\Omega (1-\chi) |\boldsymbol{g}|^2 d\boldsymbol{x}dt + \frac{\rho_s \varepsilon_3}{2} \int_0^\tau \int_\Omega (1-\chi) \Big|\frac{\partial \boldsymbol{u}_c^n}{\partial t}\Big|^2 d\boldsymbol{x}dt,$$

$$\text{(7.8)} \quad \Big|\int_0^\tau \int_\Omega \chi c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_2 \mathrm{div}_x \boldsymbol{w}_2^n \mathrm{div}_x \frac{\partial \boldsymbol{w}_1^n}{\partial t} d\boldsymbol{x}dt\Big| \leq$$
$$\frac{c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_2}{2\varepsilon_4} \int_0^\tau \int_\Omega \chi |\mathrm{div}_x \boldsymbol{w}_2^n|^2 d\boldsymbol{x}dt + \frac{c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_2 \varepsilon_4}{2} \int_0^\tau \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_1^n}{\partial t}\Big|^2 d\boldsymbol{x}dt,$$

$$\text{(7.9)} \quad \Big|\int_0^\tau \int_\Omega \chi c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_1 \mathrm{div}_x \boldsymbol{w}_1^n \mathrm{div}_x \frac{\partial \boldsymbol{w}_2^n}{\partial t} d\boldsymbol{x}dt\Big| \leq$$
$$\frac{c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_1}{2\varepsilon_5} \int_0^\tau \int_\Omega \chi |\mathrm{div}_x \boldsymbol{w}_1^n|^2 d\boldsymbol{x}dt + \frac{c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_1 \varepsilon_5}{2} \int_0^\tau \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_2^n}{\partial t}\Big|^2 d\boldsymbol{x}dt.$$

Choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \dfrac{1}{2}$, $\varepsilon_4 = \dfrac{m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big)}{c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_2}$, $\varepsilon_5 = \dfrac{m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big)}{c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_1}$, and estimate the right hand side of (7.4) by virtue of (7.5)–(7.9). Thus we arrive at the inequality

$$\text{(7.10)} \quad \frac{1}{4}\rho_{1f}^0 m_1 \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_1^n}{\partial t}(\tau)\Big|^2 d\boldsymbol{x}$$
$$+ \frac{1}{4}\rho_{2f}^0 m_2 \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_2^n}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} + \frac{1}{4}\rho_s \int_\Omega (1-\chi)\Big|\frac{\partial \boldsymbol{u}_c^n}{\partial t}(\tau)\Big|^2 d\boldsymbol{x}$$
$$+ \frac{1}{2}c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_1 \int_\Omega \chi \left|\mathrm{div}_x \boldsymbol{w}_1^n(\tau)\right|^2 d\boldsymbol{x} + \frac{1}{2}c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_2 \int_\Omega \chi \left|\mathrm{div}_x \boldsymbol{w}_2^n(\tau)\right|^2 d\boldsymbol{x}$$
$$+ \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big) \int_\Omega (1-\chi)\left|\mathrm{div}_x \boldsymbol{u}_c^n(\tau)\right|^2 d\boldsymbol{x} + \frac{1}{2}\lambda \int_\Omega (1-\chi)\left|\mathbb{D}(x, \boldsymbol{u}_c^n(\tau))\right|^2 d\boldsymbol{x}$$
$$+ \frac{1}{2}m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big) \int_0^\tau \int_\Omega \chi \Big|\mathrm{div}_x \frac{\partial \boldsymbol{w}_1^n}{\partial t}\Big|^2 d\boldsymbol{x}dt$$
$$+ \frac{1}{2}m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big) \int_0^\tau \int_\Omega \chi \Big|\mathrm{div}_x \frac{\partial \boldsymbol{w}_2^n}{\partial t}\Big|^2 d\boldsymbol{x}dt$$
$$+ m_1\mu_1 \int_0^\tau \int_\Omega \chi \Big|\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_1^n}{\partial t}\Big)\Big|^2 d\boldsymbol{x}dt$$
$$+ m_2\mu_2 \int_0^\tau \int_\Omega \chi \Big|\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_2^n}{\partial t}\Big)\Big|^2 d\boldsymbol{x}dt + K_F \int_0^\tau \int_\Omega \chi \Big|\frac{\partial \boldsymbol{u}_r^n}{\partial t}\Big|^2 d\boldsymbol{x}dt$$
$$\leq \frac{1}{2}\rho_{1f}^0 m_1 \int_\Omega \chi |\boldsymbol{v}_1^{0n}|^2 d\boldsymbol{x} + \frac{1}{2}\rho_{2f}^0 m_2 \int_\Omega \chi |\boldsymbol{v}_2^{0n}|^2 d\boldsymbol{x} + \frac{1}{2}\int_\Omega (1-\chi)|\boldsymbol{v}_c^{0n}|^2 d\boldsymbol{x}$$
$$+ \frac{1}{2}c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_1 \int_\Omega \chi \left|\mathrm{div}_x \boldsymbol{w}_1^{0n}\right|^2 d\boldsymbol{x} + \frac{1}{2}c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_2 \int_\Omega \chi \left|\mathrm{div}_x \boldsymbol{w}_2^{0n}\right|^2 d\boldsymbol{x}$$
$$+ \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big) \int_\Omega (1-\chi)\left|\mathrm{div}_x \boldsymbol{u}_c^{0n}\right|^2 d\boldsymbol{x} + \frac{1}{2}\lambda \int_\Omega (1-\chi)\left|\mathbb{D}(x, \boldsymbol{u}_c^{0n})\right|^2 d\boldsymbol{x}$$
$$+ \frac{(c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_2)^2}{2m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big)} \int_0^\tau \int_\Omega \chi |\mathrm{div}_x \boldsymbol{w}_2^n|^2 d\boldsymbol{x}dt + \frac{(c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_1)^2}{2m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big)} \int_0^\tau \int_\Omega \chi |\mathrm{div}_x \boldsymbol{w}_1^n|^2 d\boldsymbol{x}dt$$
$$+ m_1\rho_{1f}^0 \int_0^\tau \int_\Omega \chi |\boldsymbol{g}|^2 d\boldsymbol{x}dt + m_2\rho_{2f}^0 \int_0^\tau \int_\Omega \chi |\boldsymbol{g}|^2 d\boldsymbol{x}dt$$
$$+ \rho_s \int_0^\tau \int_\Omega (1-\chi)|\boldsymbol{g}|^2 d\boldsymbol{x}dt.$$

Now we denote

$$\varepsilon_6 = \max\left\{ \frac{(c_{\rho 1}\rho_{1f}^0\alpha_1^0 m_2)^2}{m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big)c_{\rho 2}\rho_{2f}^0\alpha_2^0 m_2};\ \frac{(c_{\rho 2}\rho_{2f}^0\alpha_2^0 m_1)^2}{m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big)c_{\rho 1}\rho_{1f}^0\alpha_1^0 m_1} \right\}, \tag{7.11}$$

estimate the right hand side of (7.10) using the inequality

$$\begin{aligned} &\frac{(c_{\rho 1}\rho_{1f}^0\alpha_1^0 m_2)^2}{2m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big)}\int_0^\tau\int_\Omega \chi|\mathrm{div}_x \boldsymbol{w}_2^n|^2 d\boldsymbol{x}dt \\ &\qquad + \frac{(c_{\rho 2}\rho_{2f}^0\alpha_2^0 m_1)^2}{2m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big)}\int_0^\tau\int_\Omega \chi|\mathrm{div}_x \boldsymbol{w}_1^n|^2 d\boldsymbol{x}dt \\ &\qquad \le \varepsilon_6\Big(\frac{1}{2}c_{\rho 1}\rho_{1f}^0\alpha_1^0 m_1\int_0^\tau\int_\Omega \chi\,|\mathrm{div}_x \boldsymbol{w}_1^n|^2\, d\boldsymbol{x}dt \\ &\qquad\qquad + \frac{1}{2}c_{\rho 2}\rho_{2f}^0\alpha_2^0 m_2\int_0^\tau\int_\Omega \chi\,|\mathrm{div}_x \boldsymbol{w}_2^n|^2\, d\boldsymbol{x}dt\Big), \end{aligned} \tag{7.12}$$

discard all terms on the left hand side except for

$$\frac{1}{2}c_{\rho 1}\rho_{1f}^0\alpha_1^0 m_1\int_\Omega \chi\,|\mathrm{div}_x \boldsymbol{w}_1^n(\tau)|^2\, d\boldsymbol{x} + \frac{1}{2}c_{\rho 2}\rho_{2f}^0\alpha_2^0 m_2\int_\Omega \chi\,|\mathrm{div}_x \boldsymbol{w}_2^n(\tau)|^2\, d\boldsymbol{x},$$

and apply Grownwall's lemma to thus obtained inequality. Thus we arrive at the estimate

$$\begin{aligned} &\frac{1}{2}c_{\rho 1}\rho_{1f}^0\alpha_1^0 m_1\int_\Omega \chi\,|\mathrm{div}_x \boldsymbol{w}_1^n(t)|^2\, d\boldsymbol{x} \\ &\qquad + \frac{1}{2}c_{\rho 2}\rho_{2f}^0\alpha_2^0 m_2\int_\Omega \chi\,|\mathrm{div}_x \boldsymbol{w}_2^n(t)|^2\, d\boldsymbol{x} \\ &\le e^{\varepsilon_6 t}\Bigg(\frac{1}{2}\rho_{1f}^0 m_1\int_\Omega \chi|\boldsymbol{v}_1^{0n}|^2 d\boldsymbol{x} + \frac{1}{2}\rho_{2f}^0 m_2\int_\Omega \chi|\boldsymbol{v}_2^{0n}|^2 d\boldsymbol{x} \\ &\qquad + \frac{1}{2}\int_\Omega (1-\chi)|\boldsymbol{v}_c^{0n}|^2 d\boldsymbol{x} \\ &\qquad + \frac{1}{2}c_{\rho 1}\rho_{1f}^0\alpha_1^0 m_1\int_\Omega \chi\,\big|\mathrm{div}_x \boldsymbol{w}_1^{0n}\big|^2\, d\boldsymbol{x} \\ &\qquad + \frac{1}{2}c_{\rho 2}\rho_{2f}^0\alpha_2^0 m_2\int_\Omega \chi\,\big|\mathrm{div}_x \boldsymbol{w}_2^{0n}\big|^2\, d\boldsymbol{x} \\ &\qquad + \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big)\int_\Omega (1-\chi)\,\big|\mathrm{div}_x \boldsymbol{u}_c^{0n}\big|^2\, d\boldsymbol{x} \\ &\qquad + \frac{1}{2}\lambda\int_\Omega (1-\chi)\,\big|\mathbb{D}(x,\boldsymbol{u}_c^{0n})\big|^2\, d\boldsymbol{x} \\ &\qquad + \int_0^t\int_\Omega \rho^0|\boldsymbol{g}(\boldsymbol{x},s)|^2 e^{-\varepsilon_6 s} ds\Bigg) \quad \forall\, t\in[0,T]. \end{aligned} \tag{7.13}$$

Collecting altogether (7.10), (7.12), and (7.13) we derive *the energy inequality*

$$
\begin{aligned}
(7.14)\quad & \frac{1}{4}\rho^0_{1f}m_1\int_\Omega\chi\Big|\frac{\partial \boldsymbol{w}^n_1}{\partial t}(\tau)\Big|^2d\boldsymbol{x}+\frac{1}{4}\rho^0_{2f}m_2\int_\Omega\chi\Big|\frac{\partial \boldsymbol{w}^n_2}{\partial t}(\tau)\Big|^2d\boldsymbol{x}\\
&+\frac{1}{4}\rho_s\int_\Omega(1-\chi)\Big|\frac{\partial \boldsymbol{u}^n_c}{\partial t}(\tau)\Big|^2d\boldsymbol{x}\\
&+\frac{1}{2}c_{\rho1}\rho^0_{1f}\alpha^0_1m_1\int_\Omega\chi\,|\mathrm{div}_x\boldsymbol{w}^n_1(\tau)|^2\,d\boldsymbol{x}+\frac{1}{2}c_{\rho2}\rho^0_{2f}\alpha^0_2m_2\int_\Omega\chi\,|\mathrm{div}_x\boldsymbol{w}^n_2(\tau)|^2\,d\boldsymbol{x}\\
&+\frac{1}{2}\Big(\eta-\frac{2}{3}\lambda\Big)\int_\Omega(1-\chi)\,|\mathrm{div}_x\boldsymbol{u}^n_c(\tau)|^2\,d\boldsymbol{x}+\frac{1}{2}\lambda\int_\Omega(1-\chi)\,|\mathbb{D}(x,\boldsymbol{u}^n_c(\tau))|^2\,d\boldsymbol{x}\\
&+\frac{1}{2}m_1\Big(\nu_1-\frac{2}{3}\mu_1\Big)\int_0^\tau\int_\Omega\chi\Big|\mathrm{div}_x\frac{\partial \boldsymbol{w}^n_1}{\partial t}\Big|^2d\boldsymbol{x}dt\\
&+\frac{1}{2}m_2\Big(\nu_2-\frac{2}{3}\mu_2\Big)\int_0^\tau\int_\Omega\chi\Big|\mathrm{div}_x\frac{\partial \boldsymbol{w}^n_2}{\partial t}\Big|^2d\boldsymbol{x}dt\\
&+m_1\mu_1\int_0^\tau\int_\Omega\chi\Big|\mathbb{D}\Big(x,\frac{\partial \boldsymbol{w}^n_1}{\partial t}\Big)\Big|^2d\boldsymbol{x}dt\\
&+m_2\mu_2\int_0^\tau\int_\Omega\chi\Big|\mathbb{D}\Big(x,\frac{\partial \boldsymbol{w}^n_2}{\partial t}\Big)\Big|^2d\boldsymbol{x}dt+K_F\int_0^\tau\int_\Omega\chi\Big|\frac{\partial \boldsymbol{u}^n_r}{\partial t}\Big|^2d\boldsymbol{x}dt\\
&\le\frac{1}{2}\rho^0_{1f}m_1\int_\Omega\chi|\boldsymbol{v}^{0n}_1|^2d\boldsymbol{x}+\frac{1}{2}\rho^0_{2f}m_2\int_\Omega\chi|\boldsymbol{v}^{0n}_2|^2d\boldsymbol{x}+\frac{1}{2}\int_\Omega(1-\chi)|\boldsymbol{v}^{0n}_c|^2d\boldsymbol{x}\\
&+\frac{1}{2}c_{\rho1}\rho^0_{1f}\alpha^0_1m_1\int_\Omega\chi\,\big|\mathrm{div}_x\boldsymbol{w}^{0n}_1\big|^2\,d\boldsymbol{x}+\frac{1}{2}c_{\rho2}\rho^0_{2f}\alpha^0_2m_2\int_\Omega\chi\,\big|\mathrm{div}_x\boldsymbol{w}^{0n}_2\big|^2\,d\boldsymbol{x}\\
&+\frac{1}{2}\Big(\eta-\frac{2}{3}\lambda\Big)\int_\Omega(1-\chi)\,\big|\mathrm{div}_x\boldsymbol{u}^{0n}_c\big|^2\,d\boldsymbol{x}+\frac{1}{2}\lambda\int_\Omega(1-\chi)\,\big|\mathbb{D}(x,\boldsymbol{u}^{0n}_c)\big|^2\,d\boldsymbol{x}\\
&+m_1\rho^0_{1f}\int_0^\tau\int_\Omega\chi|\boldsymbol{g}|^2d\boldsymbol{x}dt+m_2\rho^0_{2f}\int_0^\tau\int_\Omega\chi|\boldsymbol{g}|^2d\boldsymbol{x}dt+\rho_s\int_0^\tau\int_\Omega(1-\chi)|\boldsymbol{g}|^2d\boldsymbol{x}dt\\
&+(e^{\varepsilon_6\tau}-1)\Big(\frac{1}{2}\rho^0_{1f}m_1\int_\Omega\chi|\boldsymbol{v}^{0n}_1|^2d\boldsymbol{x}+\frac{1}{2}\rho^0_{2f}m_2\int_\Omega\chi|\boldsymbol{v}^{0n}_2|^2d\boldsymbol{x}+\frac{1}{2}\int_\Omega(1-\chi)|\boldsymbol{v}^{0n}_c|^2d\boldsymbol{x}\\
&+\frac{1}{2}c_{\rho1}\rho^0_{1f}\alpha^0_1m_1\int_\Omega\chi\,\big|\mathrm{div}_x\boldsymbol{w}^{0n}_1\big|^2\,d\boldsymbol{x}+\frac{1}{2}c_{\rho2}\rho^0_{2f}\alpha^0_2m_2\int_\Omega\chi\,\big|\mathrm{div}_x\boldsymbol{w}^{0n}_2\big|^2\,d\boldsymbol{x}\\
&+\frac{1}{2}\Big(\eta-\frac{2}{3}\lambda\Big)\int_\Omega(1-\chi)\,\big|\mathrm{div}_x\boldsymbol{u}^{0n}_c\big|^2\,d\boldsymbol{x}+\frac{1}{2}\lambda\int_\Omega(1-\chi)\,\big|\mathbb{D}(x,\boldsymbol{u}^{0n}_c)\big|^2\,d\boldsymbol{x}\Big)\\
&+\varepsilon_6\int_0^\tau e^{\varepsilon_6t}\Big(\int_0^t\int_\Omega\rho^0|\boldsymbol{g}(\boldsymbol{x},s)|^2e^{-\varepsilon_6s}ds\Big)\,dt\quad\forall\,\tau\in[0,T].
\end{aligned}
$$

Bessel's inequalities

$$
\big\|\boldsymbol{u}^{0n}_c\big\|^2_{H^1_0(\Omega)}\le\big\|\boldsymbol{u}^0_c\big\|^2_{H^1_0(\Omega)},\quad\big\|\boldsymbol{u}^{0n}_r\big\|^2_{H^1_0(\Omega_f)}\le\big\|\boldsymbol{u}^0_r\big\|^2_{H^1_0(\Omega_f)},
$$

$$
\big\|\boldsymbol{v}^{0n}_c\big\|^2_{L^2(\Omega)}\le\big\|\boldsymbol{v}^0_c\big\|^2_{L^2(\Omega)},\quad\big\|\boldsymbol{v}^{0n}_r\big\|^2_{L^2(\Omega_f)}\le\big\|\boldsymbol{v}^0_r\big\|^2_{L^2(\Omega_f)},
$$

and formulas (7.2) and (7.3) imply that the right hand side of the energy inequality (7.14) is bounded from above by some constant C_0, which depends on T, $\big\|\boldsymbol{u}^0_c\big\|_{H^1_0(\Omega)}$, $\big\|\boldsymbol{u}^0_r\big\|_{H^1_0(\Omega_f)}$, $\big\|\boldsymbol{v}^0_c\big\|_{L^2(\Omega)}$, $\big\|\boldsymbol{v}^0_r\big\|_{L^2(\Omega)}$, $\|\boldsymbol{g}\|_{L^2(\Omega\times(0,T))}$, and the constant coefficients in equations (5.1a), and (5.1b), but does not depend on $n\in\mathbb{N}$.

Due to this remark, on the strength of the estimates (7.13) and (7.14) and the formulas (7.2) and (7.3), we establish the following.

Proposition 2. *Let* $\boldsymbol{g} \in L^2(\Omega \times (0,T))$, $\boldsymbol{u}_c^0, \boldsymbol{u}_r^0 \in H_0^1(\Omega)$, *and* $\boldsymbol{v}_c^0, \boldsymbol{v}_r^0 \in L^2(\Omega)$. *Then the following assertions hold true.*

(1) *The sets* $\left\{\dfrac{\partial \boldsymbol{u}_c^n}{\partial t}\right\}$, $\left\{\dfrac{\partial \boldsymbol{u}_r^n}{\partial t}\right\}$ *are uniformly bounded in* $L^\infty(0,T;L^2(\Omega))$,

(2) *The set* $\left\{(1-\chi)\mathbb{D}\big(x, \boldsymbol{u}_c^n(\tau)\big)\right\}$ *is uniformly bounded in* $L^\infty(0,T;L^2(\Omega))$,

(3) *The sets* $\left\{\chi\mathbb{D}\Big(x, \dfrac{\partial \boldsymbol{u}_c^n}{\partial t}\Big)\right\}$, $\left\{\chi \, div_x \dfrac{\partial \boldsymbol{u}_c^n}{\partial t}\right\}$, $\left\{\mathbb{D}\Big(x, \dfrac{\partial \boldsymbol{u}_r^n}{\partial t}\Big)\right\}$ *are uniformly bounded in* $L^2(\Omega \times (0,T))$.

Corollary 1. *The sets* $\{\boldsymbol{u}_c^n\}$, $\{\boldsymbol{u}_r^n\}$ *are uniformly bounded in* $L^\infty(0,T;H_0^1(\Omega))$.

Proof. The inequality

(7.15)
$$\frac{1}{2}\operatorname*{ess\,sup}_{t\in[0,\tau]} \|\chi\mathbb{D}(x,\boldsymbol{\varphi}(t))\|^2_{L_2(\Omega)} \le \tau\Big\|\chi\mathbb{D}\big(x, \frac{\partial \boldsymbol{\varphi}}{\partial t}\big)\Big\|^2_{L^2(\Omega\times(0,\tau))} + \|\mathbb{D}(x,\boldsymbol{\varphi}(0))\|^2_{L^2(\Omega)}$$

is valid for any $\tau \in [0,T]$ and for any vector-function $\boldsymbol{\varphi}(\boldsymbol{x},t)$, for which the right hand side is defined [17, Corollary 4.4]. On the strength of items 2 and 3 of Proposition 2, taking $\boldsymbol{\varphi} = \boldsymbol{u}_c^n$ and $\boldsymbol{\varphi} = \boldsymbol{u}_r^n$ in (7.15) we establish that the sets $\left\{\mathbb{D}\big(x, \boldsymbol{u}_c^n\big)\right\}$ and $\left\{\mathbb{D}\big(x, \boldsymbol{u}_r^n\big)\right\}$ are uniformly bounded in $L^\infty(0,T;L^2(\Omega))$. Using these bounds and Korn's inequality

$$\|\boldsymbol{\varphi}\|_{H_0^1(\Omega)} \le C_k(\Omega)\|\mathbb{D}(x,\boldsymbol{\varphi})\|_{L^2(\Omega)},$$

which is valid for any function $\boldsymbol{\varphi} \in L^2(\Omega)$ such that $\mathbb{D}(x,\boldsymbol{\varphi}) \in L^2(\Omega)$ and $\boldsymbol{\varphi}$ vanishes in the trace sense on some open subset of $\partial\Omega$ [18, Ch. III, Sec. 3.2], we complete the proof. □

Corollary 2. *The set* $\left\{\dfrac{\partial \boldsymbol{u}_r^n}{\partial t}\right\}$ *is uniformly bounded in* $L^2(0,T;H_0^1(\Omega))$.

Proof. Due to item 3 of proposition 2 and the Fubini theorem [19, §I.4.45] the set $\left\{\mathbb{D}\Big(x, \dfrac{\partial \boldsymbol{u}_r^n}{\partial t}(\tau)\Big)\right\}$ is uniformly bounded in $L^2(\Omega)$ for a.e $\tau \in (0,T)$. Using Korn's inequality we get

$$\Big\|\frac{\partial \boldsymbol{u}_r^n}{\partial t}(\tau)\Big\|_{H_0^1(\Omega)} \le C_k(\Omega)\Big\|\mathbb{D}\Big(x, \frac{\partial \boldsymbol{u}_r^n}{\partial t}(\tau)\Big)\Big\|_{L^2(\Omega)}, \quad \text{for a.e } \tau \in (0,T).$$

Integrating this inequality with respect to τ on $[0,T]$ we complete the proof. □

8. Well-posedness of Model IT_0

Theorem 1. *Whenever* $\boldsymbol{g} \in L^2(\Omega \times (0,T))$, $\boldsymbol{u}_c^0, \boldsymbol{u}_r^0 \in H_0^1(\Omega)$, *and* $\boldsymbol{v}_c^0, \boldsymbol{v}_r^0 \in L^2(\Omega)$, *there exists a unique generalized solution* $\{\boldsymbol{u}_c, \boldsymbol{u}_r\}$ *of Model* IT_0 *in the sense of Definition 1.*

Furthermore, the solution satisfies the energy identity

$$
\begin{aligned}
(8.1)\quad & \frac{1}{2}\rho_{1f}^0 m_1 \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_1}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} \\
& + \frac{1}{2}\rho_{2f}^0 m_2 \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_2}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} + \frac{1}{2}\rho_s \int_\Omega (1-\chi) \Big|\frac{\partial \boldsymbol{u}_c}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} \\
& + \frac{1}{2} c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_1 \int_\Omega \chi \left|div_x \boldsymbol{w}_1(\tau)\right|^2 d\boldsymbol{x} + \frac{1}{2} c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_2 \int_\Omega \chi \left|div_x \boldsymbol{w}_2(\tau)\right|^2 d\boldsymbol{x} \\
& + \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big) \int_\Omega (1-\chi) \left|div_x \boldsymbol{u}_c(\tau)\right|^2 d\boldsymbol{x} + \frac{1}{2}\lambda \int_\Omega (1-\chi) \left|\mathbb{D}(x, \boldsymbol{u}_c(\tau))\right|^2 d\boldsymbol{x} \\
& + m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big) \int_0^\tau \int_\Omega \chi \Big|div_x \frac{\partial \boldsymbol{w}_1}{\partial t}\Big|^2 d\boldsymbol{x}dt + m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big) \int_0^\tau \int_\Omega \chi \Big|div_x \frac{\partial \boldsymbol{w}_2}{\partial t}\Big|^2 d\boldsymbol{x}dt \\
& + m_1\mu_1 \int_0^\tau \int_\Omega \chi \Big|\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_1}{\partial t}\Big)\Big|^2 d\boldsymbol{x}dt \\
& + m_2\mu_2 \int_0^\tau \int_\Omega \chi \Big|\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_2}{\partial t}\Big)\Big|^2 d\boldsymbol{x}dt + K_F \int_0^\tau \int_\Omega \chi \Big|\frac{\partial \boldsymbol{u}_r}{\partial t}\Big|^2 d\boldsymbol{x}dt \\
& = \frac{1}{2}\rho_{1f}^0 m_1 \int_\Omega \chi |\boldsymbol{v}_1^0|^2 d\boldsymbol{x} + \frac{1}{2}\rho_{2f}^0 m_2 \int_\Omega \chi |\boldsymbol{v}_2^0|^2 d\boldsymbol{x} + \frac{1}{2}\int_\Omega (1-\chi)|\boldsymbol{v}_c^0|^2 d\boldsymbol{x} \\
& + \frac{1}{2} c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_1 \int_\Omega \chi \left|div_x \boldsymbol{w}_1^0\right|^2 d\boldsymbol{x} + \frac{1}{2} c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_2 \int_\Omega \chi \left|div_x \boldsymbol{w}_2^0\right|^2 d\boldsymbol{x} \\
& + \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big) \int_\Omega (1-\chi) \left|div_x \boldsymbol{u}_c^0\right|^2 d\boldsymbol{x} + \frac{1}{2}\lambda \int_\Omega (1-\chi) \left|\mathbb{D}(x, \boldsymbol{u}_c^0)\right|^2 d\boldsymbol{x} \\
& - \int_0^\tau \int_\Omega \chi c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_2\, div_x \boldsymbol{w}_2\, div_x \frac{\partial \boldsymbol{w}_1}{\partial t} d\boldsymbol{x}dt \\
& - \int_0^\tau \int_\Omega \chi c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_1\, div_x \boldsymbol{w}_1\, div_x \frac{\partial \boldsymbol{w}_2}{\partial t} d\boldsymbol{x}dt \\
& + \int_0^\tau \int_\Omega \chi m_1 \rho_{1f}^0 \boldsymbol{g} \cdot \frac{\partial \boldsymbol{w}_1}{\partial t} d\boldsymbol{x}dt + \int_0^\tau \int_\Omega \chi m_2 \rho_{2f}^0 \boldsymbol{g} \cdot \frac{\partial \boldsymbol{w}_2}{\partial t} d\boldsymbol{x}dt \\
& + \int_0^\tau \int_\Omega (1-\chi)\rho_s \boldsymbol{g} \cdot \frac{\partial \boldsymbol{u}_c}{\partial t} d\boldsymbol{x}dt
\end{aligned}
$$

and the energy inequality

(8.2)

$$
\begin{aligned}
& \frac{1}{4}\rho_{1f}^0 m_1 \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_1}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} + \frac{1}{4}\rho_{2f}^0 m_2 \int_\Omega \chi \Big|\frac{\partial \boldsymbol{w}_2}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} + \frac{1}{4}\rho_s \int_\Omega (1-\chi) \Big|\frac{\partial \boldsymbol{u}_c}{\partial t}(\tau)\Big|^2 d\boldsymbol{x} \\
& + \frac{1}{2} c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_1 \int_\Omega \chi \left|div_x \boldsymbol{w}_1(\tau)\right|^2 d\boldsymbol{x} + \frac{1}{2} c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_2 \int_\Omega \chi \left|div_x \boldsymbol{w}_2(\tau)\right|^2 d\boldsymbol{x} \\
& + \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big) \int_\Omega (1-\chi) \left|div_x \boldsymbol{u}_c(\tau)\right|^2 d\boldsymbol{x} + \frac{1}{2}\lambda \int_\Omega (1-\chi) \left|\mathbb{D}(x, \boldsymbol{u}_c(\tau))\right|^2 d\boldsymbol{x} \\
& + \frac{1}{2} m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big) \int_0^\tau \int_\Omega \chi \Big|div_x \frac{\partial \boldsymbol{w}_1}{\partial t}\Big|^2 d\boldsymbol{x}dt + \frac{1}{2} m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big) \int_0^\tau \int_\Omega \chi \Big|div_x \frac{\partial \boldsymbol{w}_2}{\partial t}\Big|^2 d\boldsymbol{x}dt \\
& + m_1\mu_1 \int_0^\tau \int_\Omega \chi \Big|\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_1}{\partial t}\Big)\Big|^2 d\boldsymbol{x}dt
\end{aligned}
$$

$$+ m_2\mu_2 \int_0^\tau \int_\Omega \chi \Big| \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}_2}{\partial t}\Big)\Big|^2 d\boldsymbol{x}dt + K_F \int_0^\tau \int_\Omega \chi \Big| \frac{\partial \boldsymbol{u}_r}{\partial t}\Big|^2 d\boldsymbol{x}dt$$

$$\le \frac{1}{2}\rho^0_{1f} m_1 \int_\Omega \chi |\boldsymbol{v}^0_1|^2 d\boldsymbol{x} + \frac{1}{2}\rho^0_{2f} m_2 \int_\Omega \chi |\boldsymbol{v}^0_2|^2 d\boldsymbol{x} + \frac{1}{2}\int_\Omega (1-\chi)|\boldsymbol{v}^0_c|^2 d\boldsymbol{x}$$

$$+ \frac{1}{2} c_{\rho 1}\rho^0_{1f}\alpha^0_1 m_1 \int_\Omega \chi \left| div_x \boldsymbol{w}^0_1\right|^2 d\boldsymbol{x} + \frac{1}{2} c_{\rho 2}\rho^0_{2f}\alpha^0_2 m_2 \int_\Omega \chi \left| div_x \boldsymbol{w}^0_2\right|^2 d\boldsymbol{x}$$

$$+ \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big) \int_\Omega (1-\chi) \left| div_x \boldsymbol{u}^0_c\right|^2 d\boldsymbol{x} + \frac{1}{2}\lambda \int_\Omega (1-\chi) \left|\mathbb{D}(x, \boldsymbol{u}^0_c)\right|^2 d\boldsymbol{x}$$

$$+ m_1\rho^0_{1f} \int_0^\tau \int_\Omega \chi |\boldsymbol{g}|^2 d\boldsymbol{x}dt + m_2\rho^0_{2f} \int_0^\tau \int_\Omega \chi |\boldsymbol{g}|^2 d\boldsymbol{x}dt + \rho_s \int_0^\tau \int_\Omega (1-\chi)|\boldsymbol{g}|^2 d\boldsymbol{x}dt$$

$$+ (e^{\varepsilon_6\tau} - 1)\Bigg(\frac{1}{2}\rho^0_{1f} m_1 \int_\Omega \chi |\boldsymbol{v}^0_1|^2 d\boldsymbol{x} + \frac{1}{2}\rho^0_{2f} m_2 \int_\Omega \chi |\boldsymbol{v}^0_2|^2 d\boldsymbol{x} + \frac{1}{2}\int_\Omega (1-\chi)|\boldsymbol{v}^0_c|^2 d\boldsymbol{x}$$

$$+ \frac{1}{2} c_{\rho 1}\rho^0_{1f}\alpha^0_1 m_1 \int_\Omega \chi \left| div_x \boldsymbol{w}^0_1\right|^2 d\boldsymbol{x} + \frac{1}{2} c_{\rho 2}\rho^0_{2f}\alpha^0_2 m_2 \int_\Omega \chi \left| div_x \boldsymbol{w}^0_2\right|^2 d\boldsymbol{x}$$

$$+ \frac{1}{2}\Big(\eta - \frac{2}{3}\lambda\Big) \int_\Omega (1-\chi) \left| div_x \boldsymbol{u}^0_c\right|^2 d\boldsymbol{x} + \frac{1}{2}\lambda \int_\Omega (1-\chi) \left|\mathbb{D}(x, \boldsymbol{u}^0_c)\right|^2 d\boldsymbol{x}\Bigg)$$

$$+ \varepsilon_6 \int_0^\tau e^{\varepsilon_6 t}\left(\int_0^t \int_\Omega \rho^0 |\boldsymbol{g}(\boldsymbol{x}, s)|^2 e^{-\varepsilon_6 s} ds\right) dt \quad \forall\, \tau \in [0, T].$$

In (8.1) *and* (8.2) *the following notation is in use:*

$$\boldsymbol{w}_1 := \boldsymbol{u}_c - \frac{m_2\rho^0_{2f}}{\rho^0_f}\boldsymbol{u}_r, \qquad \boldsymbol{w}_2 := \boldsymbol{u}_c + \frac{m_1\rho^0_{1f}}{\rho^0_f}\boldsymbol{u}_r, \tag{8.3}$$

$$\boldsymbol{w}^0_1 := \boldsymbol{u}^0_c - \frac{m_2\rho^0_{2f}}{\rho^0_f}\boldsymbol{u}^0_r, \qquad \boldsymbol{w}^0_2 := \boldsymbol{u}^0_c + \frac{m_1\rho^0_{1f}}{\rho^0_f}\boldsymbol{u}^0_r, \tag{8.4}$$

$$\boldsymbol{v}^0_1 := \boldsymbol{v}^0_c - \frac{m_2\rho^0_{2f}}{\rho^0_f}\boldsymbol{v}^0_r, \qquad \boldsymbol{v}^0_2 := \boldsymbol{v}^0_c + \frac{m_1\rho^0_{1f}}{\rho^0_f}\boldsymbol{v}^0_r, \tag{8.5}$$

$$\varepsilon_6 := \max\left\{ \frac{(c_{\rho 1}\rho^0_{1f}\alpha^0_1 m_2)^2}{m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big)c_{\rho 2}\rho^0_{2f}\alpha^0_2 m_2};\ \frac{(c_{\rho 2}\rho^0_{2f}\alpha^0_2 m_1)^2}{m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big)c_{\rho 1}\rho^0_{1f}\alpha^0_1 m_1} \right\}. \tag{8.6}$$

Proof. First we prove existence. On the strength of Proposition 2, and Corollaries 1 and 2, there exist subsequences $\{\boldsymbol{u}^{n_k}_c\}$, $\{\boldsymbol{u}^{n_k}_r\}$ and functions $\boldsymbol{u}_c$, $\boldsymbol{u}_r$ such that

$$\boldsymbol{u}^{n_k}_c \to \boldsymbol{u}_c, \quad \boldsymbol{u}^{n_k}_r \to \boldsymbol{u}_r \qquad *\text{-weakly in } L^\infty(0,T; H^1_0(\Omega)), \tag{8.7a}$$

$$\frac{\partial \boldsymbol{u}^{n_k}_c}{\partial t} \to \frac{\partial \boldsymbol{u}_c}{\partial t}, \quad \frac{\partial \boldsymbol{u}^{n_k}_r}{\partial t} \to \frac{\partial \boldsymbol{u}_r}{\partial t} \qquad *\text{-weakly in } L^\infty(0,T; L^2(\Omega)), \tag{8.7b}$$

$$\chi\mathbb{D}\Big(x, \frac{\partial \boldsymbol{u}^{n_k}_c}{\partial t}\Big) \to \chi\mathbb{D}\Big(x, \frac{\partial \boldsymbol{u}_c}{\partial t}\Big) \qquad \text{weakly in } L^2(Q_T), \tag{8.7c}$$

$$\chi \mathrm{div}_x \frac{\partial \boldsymbol{u}^{n_k}_c}{\partial t} \to \chi \mathrm{div}_x \frac{\partial \boldsymbol{u}_c}{\partial t} \qquad \text{weakly in } L^2(Q_T) \tag{8.7d}$$

$$\frac{\partial \boldsymbol{u}^{n_k}_r}{\partial t} \to \frac{\partial \boldsymbol{u}_r}{\partial t} \qquad \text{weakly in } L^2(0,T; H^1_0(\Omega)), \tag{8.7e}$$

as $k \nearrow \infty$. On the strength of these limiting relations and Remark 5 vector-functions $\boldsymbol{u}_c$ and $\boldsymbol{u}_r$ satisfy items 1, 2, and 5 of Definition 1, and integral equalities (5.7a) and

(5.7b) with the test functions $\mathbf{\Phi}_{\boldsymbol{m}} = \zeta(t)\boldsymbol{\varphi}(x)$ and $\mathbf{\Psi}_{\boldsymbol{m}} = \zeta(t)\boldsymbol{\psi}(x)$, where $\zeta(t)$ is arbitrary and smooth, and $\boldsymbol{\varphi}(x)$ and $\boldsymbol{\psi}(x)$ are arbitrary functions belonging to the linear spans of the sets $\{\boldsymbol{\psi}_l, \boldsymbol{\varphi}_l\}_{l=1}^m$ and $\{\boldsymbol{\psi}_l\}_{l=1}^m$, correspondingly. Working out the limiting transition as $m \nearrow \infty$ we conclude that the pair of vector-functions $\{\boldsymbol{u}_c,\ \boldsymbol{u}_r\}$ satisfies integral equalities (5.7a) and (5.7b) with arbitrary test functions $\mathbf{\Phi}$ and $\mathbf{\Psi}$, i.e., admits items 3 and 4 of Definition 1, and therefore is a weak generalized solution of Model IT_0.

Now let us prove that the energy identity (8.1) hold true. To this end we combine the techniques of Sec. 7 with the standard method in the theory of evolutionary partial differential equations. This method is based on the special choice of test functions in integral equalities (5.7a) and (5.7b).

Firstly, like in Sec. 7, we subtract the left hand and the right hand sides of (5.7b) from the left hand and the right hand sides of (5.7a), respectively, and then divide the both sides of resulting equality by two. Secondly, we add the left hand and the right hand sides of (5.7b) to the left hand and the right hand sides of (5.7a), respectively, and then divide the both sides of resulting equality by two. Thus we obtain two integral equalities for $\boldsymbol{u}_c$ and $\boldsymbol{u}_r$; each of them contains two test functions, $\mathbf{\Phi}$ and $\mathbf{\Psi}$. Thirdly, following the track of considerations of [20, Ch. 2, Sec. 5.2], we introduce a continuous piece-wise linear function on $[0,\tau]$ such that $\phi_m(t) = 1$ if $(2/m) < t < \tau - (2/m)$ and $\phi_m(t) = 0$ if $t > \tau - (1/m)$ and $t < (1/m)$, and a regularizing sequence $\omega_k \in C_0^\infty(\mathbb{R})$ such that

$$\omega_k(t) = \omega_k(-t), \quad \omega_k(t) \geq 0, \quad \int_{-\infty}^{\infty} \omega_k(t)dt = 1, \quad \operatorname{supp}\omega_k \subset \Big[-\frac{1}{k}, \frac{1}{k}\Big].$$

For $k > 2m$, we set

$$\mathbf{\Phi}^1_{mk} = \Big(\Big(\phi_m \frac{\partial \boldsymbol{w}_1}{\partial t}\Big) * \omega_k * \omega_k\Big)\phi_m, \tag{8.8}$$

$$\mathbf{\Phi}^2_{mk} = \Big(\Big(\phi_m \frac{\partial \boldsymbol{w}_2}{\partial t}\Big) * \omega_k * \omega_k\Big)\phi_m, \tag{8.9}$$

$$\mathbf{\Psi}_{mk} = \Big(\Big(\phi_m \frac{\partial \boldsymbol{u}_r}{\partial t}\Big) * \omega_k * \omega_k\Big)\phi_m. \tag{8.10}$$

In (8.8)–(8.10) the asterisk $*$ means the integral convolution in $\mathbb{R}$ and $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are defined in (8.3). We substitute $\mathbf{\Phi}^1_{mk}$ and $\mathbf{\Psi}_{mk}$ for $\mathbf{\Phi}$ and $\mathbf{\Psi}$ into the first of the obtained integral equalities and $\mathbf{\Phi}^2_{mk}$ and $\mathbf{\Psi}_{mk}$ for $\mathbf{\Phi}$ and $\mathbf{\Psi}$ into the second one. Clearly, this choice of test functions is valid due to regularity properties (5.8). Finally, we sum the two resulting integral equalities and pass to the limit as $k \nearrow \infty$ and $m \nearrow \infty$ precisely using the techniques [20, Ch. 2, Sec. 5.2]. Thus we end up with the required energy identity (8.1).

Applying to energy identity (8.1) exactly the same arguments as used in Sec. 7 for derivation of the estimate (7.14), we arrive at the energy inequality (8.2).

Since model IT_0 is linear, the uniqueness assertion amounts to the proposition that, if $\boldsymbol{g}$, $\boldsymbol{u}_c^0$, $\boldsymbol{u}_r^0$, $\boldsymbol{v}_c^0$, $\boldsymbol{v}_r^0$ are equal to zero, then there is only trivial solution. The latter proposition is obvious due to (8.2). □

Appendix. List of the most frequently used notation

Notation

$$\varepsilon_6 := \max\left\{ \frac{(c_{\rho 1}\rho^0_{1f}\alpha^0_1 m_2)^2}{m_1\Big(\nu_1 - \frac{2}{3}\mu_1\Big)c_{\rho 2}\rho^0_{2f}\alpha^0_2 m_2};\ \frac{(c_{\rho 2}\rho^0_{2f}\alpha^0_2 m_1)^2}{m_2\Big(\nu_2 - \frac{2}{3}\mu_2\Big)c_{\rho 1}\rho^0_{1f}\alpha^0_1 m_1} \right\}$$

was introduced in (7.11) and (8.6);

$$\alpha^0_i = \frac{m_i}{c_{\rho i}\rho^0_{if}}\,\frac{c_{\rho 1}\rho^0_{1f}c_{\rho 2}\rho^0_{2f}}{m_1 c_{\rho 2}\rho^0_{2f} + m_2 c_{\rho 1}\rho^0_{1f}}, \quad i = 1, 2,$$ was introduced in (4.1b);

$$\alpha^0_c = c_{\rho 2}\rho^0_{2f}\alpha^0_2 + c_{\rho 1}\rho^0_{1f}\alpha^0_1$$ was introduced in (5.3a);

$$\alpha^0_r = c_{\rho 2}\rho^0_{2f}\alpha^0_2 - c_{\rho 1}\rho^0_{1f}\alpha^0_1$$ was introduced in (5.3b);

$$\rho^0_f = m_1\rho^0_{1f} + m_2\rho^0_{2f}$$ was introduced in (5.2a) and (5.2b);

$$\rho^0 = \chi\rho^0_f + (1-\chi)\rho_s$$ was introduced after formula (5.5b);

$$\chi(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \Omega_f, \\ 0, & \boldsymbol{x} \notin \Omega_f, \end{cases}$$ was introduced in (5.4);

$$\boldsymbol{u}_c = \chi\boldsymbol{w}_c + (1-\chi)\boldsymbol{w}, \quad (\boldsymbol{x}, t) \in Q_T = \Omega \times (0, T),$$ was introduced in (5.5a);

$$\boldsymbol{u}_r = \chi\boldsymbol{w}_r, \quad (\boldsymbol{x}, t) \in Q_T = \Omega \times (0, T),$$ was introduced in (5.5b);

$$\boldsymbol{v}^{0n}_1 := \boldsymbol{v}^{0n}_c - \frac{m_2\rho^0_{2f}}{\rho^0_f}\boldsymbol{v}^{0n}_r$$ was introduced in (7.3);

$$\boldsymbol{v}^{0n}_2 := \boldsymbol{v}^{0n}_c + \frac{m_1\rho^0_{1f}}{\rho^0_f}\boldsymbol{v}^{0n}_r$$ was introduced in (7.3);

$$\boldsymbol{w}^{n}_1 := \boldsymbol{u}^{n}_c - \frac{m_2\rho^0_{2f}}{\rho^0_f}\boldsymbol{u}^{n}_r$$ was introduced in (7.1);

$$\boldsymbol{w}^{n}_2 := \boldsymbol{u}^{n}_c + \frac{m_1\rho^0_{1f}}{\rho^0_f}\boldsymbol{u}^{n}_r$$ was introduced in (7.1);

$$\boldsymbol{w}^{0n}_1 := \boldsymbol{u}^{0n}_c - \frac{m_2\rho^0_{2f}}{\rho^0_f}\boldsymbol{u}^{0n}_r$$ was introduced in (7.2);

$$\boldsymbol{w}^{0n}_2 := \boldsymbol{u}^{0n}_c + \frac{m_1\rho^0_{1f}}{\rho^0_f}\boldsymbol{u}^{0n}_r$$ was introduced in (7.2);

$$\boldsymbol{w}_c = \frac{m_1\rho^0_{1f}}{\rho^0_f}\boldsymbol{w}_1 + \frac{m_2\rho^0_{2f}}{\rho^0_f}\boldsymbol{w}_2$$ was introduced in (5.2a);

$$\boldsymbol{w}_r = \boldsymbol{w}_2 - \boldsymbol{w}_1$$ was introduced in (5.2b).

References

[1] M. Massoud, *Engineering Thermofluids. Thermodynamics, Fluid Mechanics, and Heat Transfer*, Springer, Berlin, 2005.

[2] R.I. Nigmatulin, *Dynamics of Multiphase Systems, Vol. 1*, Washington, Hemisphere, 1990.

[3] P. Germain, *Course of Mechanics of Continuous Media, Vol. 1*, Moscow, Vysshaia Shkola, 1983. (Russian translation of *Cours de Mécanique des Milieux Continus. Tome 1: Théorie générale*, Masson et Cie, Paris, 1973 (in French).)

[4] S.A. Sazhenkov, *Effective thermoviscoelasticity of a saturated porous ground*, Vestnik, Quart. J. of Novosibirsk State Univ., Series: Math. mech. and informatics, **8**:2 (2008), 105–129.

[5] A. Meirmanov, *Double porosity models for liquid filtration in incompressible poroelastic media*, Mathematical Models and Methods in Applied Sciences, **20** (2010), 635–659. MR 2647035

[6] A.M. Meirmanov, *A description of seismic acoustic wave propagation in porous media via homogenization*, SIAM J. Math. Anal., **40**:3 (2008), 1272–1289. MR 2452887

[7] R.P. Gilbert, A. Mikelić, *Homogenizing the acoustic properties of the seabed: Part I*, Nonlinear Anal., **40** (2000), 185–212. MR 1768409

[8] T. Clopeau, J.L. Ferrin, R.P. Gilbert, A. Mikelić, *Homogenizing the acoustic properties of the seabed: Part II*, Math. Comp. Modelling, **33** (2001), 821–841. MR 1826537

[9] R.A. Rakhmatulin, *Foundations of gas dynamics of self-penetrative motions of continuous media*, PMM, 1956, vol. 20, no. 2. (In Russian.)

[10] L.D. Landau, E.M. Lifshitz, *Theory of Elasticity: Course of Theoretical Physics, Vol. 7*, Oxford, Butterworth – Heinemann, 1995.

[11] L.V. Ovsiannikov, *Introduction to Mechanics of Continuous Media, Part 2*, Novosibirsk, Novosibirsk State University, 1977 (in Russian).

[12] *Bear J.* Dynamics of Fluids in Porous Media. Dover Publications, New York, 1988. Zbl 1191.76002

[13] L.D. Landau, E.M. Lifshitz, *Fluid Mechanics: Course of Theoretical Physics, Vol. 6*, Oxford, Butterworth – Heinemann, 1995.

[14] L.I. Sedov, *Mechanics of Continuous Media, Vol. 1*, Singapore, World Scientific, 1997. MR 1470716

[15] S.N. Antontsev, A.V. Kazhikhov, V.N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, Amsterdam, North-Holland, 1990. MR 1035212

[16] O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics, Applied Mathematical Sciences* **49**, New York, Springer, 1985. MR 0793735

[17] A.M. Meirmanov, S.A. Sazhenkov, *Generalized solutions to linearized equations of thermoelastic solid and viscous thermofluid*, Electronic Journal of Differential Equations, **2007**:41 (2007), 1–29. MR 2299595

[18] Yu.G. Reshetnyak, *Stability Theorems in Geometry and Analysis, Mathematics and Its Appl.*, **304**, Dordrecht, Kluwer, 1994. MR 1326375

[19] J. Warga. *Optimal Control of Differential and Functional Equations* (New York and London: Academic Press, 1972). Zbl 0253.49001

[20] J.L. Lions. *Some Methods of Solution for Nonlinear Problems*, Moscow, Mir, 1972. (Russian translation of *Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaire*, Paris, Dunod Gauthier–Villars, 1969 (in French).)

Sergei A. Sazhenkov
Lavrentyev Institute for Hydrodynamics,
Siberian Division of the Russian Academy of Sciences,
pr. Acad. Lavrentyeva, 15,
630090, Novosibirsk, Russia
E-mail address: sazhenkovs@yandex.ru

Elena V. Sazhenkova
Novosibirsk State University of Economics and Management,
Institute for Applied Informatics,
Kamenskaya st., 56,
630099, Novosibirsk, Russia
E-mail address: sazhenkovs@yandex.ru