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SPORADIC COMPOSITION FACTORS OF FINITE GROUPS ISOSPECTRAL TO SIMPLE GROUPS

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Abstract. This paper considers a group which has a nonabelian sporadic composition factor and the same set of element orders as a finite simple group. It is proved that such group is isomorphic to $U_5(2)$ or a sporadic group.

Keywords: recognition by spectrum, sporadic groups, finite simple groups.

1. Introduction

The *spectrum* $\omega(G)$ of a finite group G is the set of its element orders. Two groups are said to be isospectral if they have the same spectra. Sometimes it is more convenient to consider the set of maximal elements of $\omega(G)$ with respect to divisibility. This set is denoted by $\mu(G)$ and clearly completely determines the spectrum.

Let $h(G)$ be the number of pairwise nonisomorphic groups isospectral to G. If $h(G) = 1$ then such group G is called *recognizable by spectrum*. A finite group G with $h(G) < \infty$ is said to be *almost recognizable by spectrum*. Finally a group G is *non-recognizable by spectrum* if $h(G) = \infty$. We say that the *problem of recognition by spectrum* is solved for G if the number $h(G)$ is known. A group G is called *quasirecognizable* by spectrum if every finite group isospectral to G has a unique nonabelian composition factor and this factor is isomorphic to G.

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There have been many articles on the recognition problem for different finite simple nonabelian groups L (e.g. see [1]). An important step during investigation of groups isospectral to a given group is studying of their composition structure. The present paper considers the following question

Which sporadic groups can be a composition factor of a group isospectral to a finite simple group?

We will assume that sporadic groups include 26 exceptional groups and the Tits group ${}^2F_4(2)'$. Obviously, a group isospectral to a quasirecognizable simple group, which is different from sporadic, cannot contain sporadic groups among composition factors. It was proved in [2, Theorem 2] that there are no sporadic groups among composition factors of a group isospectral to a simple symplectic or orthogonal group other than $S_4(3)$. The papers [1],[3], [4], [5] contain proofs and references to proofs of quasirecognizability for the other simple nonabelian groups except Alt_n, $L_n(q)$, $U_n(q)$, $G_2(q)$, $E_7(q)$ and sporadic groups. It was proved in [5, Theorem 2] that there are no sporadic groups among nonabelian composition factors of finite groups isospectral to $L_n(q)$, $U_n(q)$, where $n \geq 4$ and $(n, q) \neq (5, 2)$ in the case of unitary groups, and moreover an example of a group isospectral to $U_5(2)$ with the group M_{11} among composition factors was constructed in [6]. It was showed in [7] that all sporadic groups except J_2 are recognizable as well as the Tits group [8]. The survey [1] contains references to proofs of the recognizability of Alt_n where $n < 22$, $n \neq 6, 10$. Notice that $Alt_6 \simeq L_2(9)$ and it was proved in [9] that groups isospectral to Alt_{10} have no sporadic groups among composition factors. All remaining cases except alternating groups are considered in the present paper.

Theorem 1. *Let L be one of simple groups J_2, $L_2(q)$, $L_3(q)$, $U_3(q)$, $S_4(q)$, $G_2(q)$, $E_7(q)$ where q is a power of a prime p, and let G be a finite group isospectral to L. If G contains a composition factor S isomorphic to a sporadic group then either $L \simeq U_5(2)$ and $S \simeq M_{11}$ or $L \simeq S$.*

This theorem together with previous results (see references above) implies the following

Corollary 1. *If a finite group isospectral to a finite simple group L contains a sporadic group S among its composition factors and L is not isomorphic to Alt_n for $n \geq 22$ then either $L \simeq U_5(2)$ and $S \simeq M_{11}$, or $L \simeq S$.*

2. Preliminaries

Let G be a finite group and $\pi(G)$ be the set of prime divisors of its order. The *Gruenberg — Kegel graph* $GK(G)$, or the *prime graph*, of G is the graph with vertex set $\pi(G)$ in which two distinct vertices p and q are adjacent if and only if $pq \in \omega(G)$.

Recall that an *independent set of vertices*, or *coclique*, in a graph Γ is a set of vertices that are pairwise nonadjacent with each other in Γ. We write $t(\Gamma)$ to denote the independence number of Γ, i. e., the maximal number of vertices in its cocliques. Given a group G, put $t(G) = t(GK(G))$. By analogy, for each prime r, define the r-independence number $t(r, G)$ to be the maximal number of vertices in cocliques of $GK(G)$ containing the vertex r.

Lemma 1. *Let L be a finite nonabelian simple group such that $t(L) \geq 3$ and $t(2, L) \geq 2$, and G be a finite group with $\omega(G) = \omega(L)$. Then the following hold:*

1) *There exists a nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \operatorname{Aut} S$,*

where K is the maximal normal soluble subgroup of G.
2) *Every prime $r \in \pi(G)$ not adjacent with 2 in $GK(G)$ does not divide the product $|K| \cdot |\overline{G}/S|$. In particular, $t(2, S) \geq t(2, G)$.*

Proof. See [10, Theorem 2] and [11, Proposition 2].

We will use that 119 is the greatest element order of a sporadic group, this statement and other facts about element orders of sporadic groups can be checked using [12].

3. Classical groups

Lemma 2. *If $\omega(G) = \omega(L)$, where $L \in \{L_3(3), U_3(3), S_4(3)\}$, then G has no sporadic composition factors.*

Proof. Note that $\pi(L_3(3)) = \{2, 3, 13\}$, $\pi(U_3(3)) = \{2, 3, 7\}$, therefore, since 5 divides the order of any sporadic group, groups isospectral to one of those two groups have no sporadic groups among composition factors. Moreover, $\pi(S_4(3)) = \{2, 3, 5\}$ therefore this case is also impossible because orders of sporadic groups have a prime divisor greater than 5.

In [11, Proposition 4] it was proved that if a group G isospectral to a nonabelian simple group different from Alt_n, where $n \geq 5$, $L_3(3)$, $U_3(3)$, $S_4(3)$, then G has a unique nonabelian composition factor S, and hence $S \leq G/K \leq \mathrm{Aut}(S)$ for maximal normal soluble subgroup K in G. In particularly, Lemma 2 implies

Corollary 2. *If G is isospectral to a simple group from the statement of Theorem 1 and has a sporadic group among composition factors, then G has unique nonabelain composition factor.*

Lemma 3. *Let $\omega(G) = \omega(L_2(q))$, $q > 3$ and $S \leq G/K \leq \mathrm{Aut}(S)$ for a nonabelian simple group S. Then S is not sporadic.*

Proof. It is known that for $q > 3$, $q \neq 9$ groups $L_2(q)$ are recognizable by spectrum (see [13]). If $\omega(G) = \omega(L_2(9))$ then $\pi(G) = \{2, 3, 5, 7\}$. Therefore $S = J_2$ which is impossible since $8 \in \omega(J_2) \backslash \omega(L_2(9))$.

Lemma 4. *Let $\omega(G) = \omega(L_3(q))$ and $S \leq G/K \leq \mathrm{Aut}(S)$ for a nonabelian simple group S. Then S is not sporadic.*

Proof. The case $q = 3$ was considered above. It is known that the groups $L_3(q)$ for $q > 3$ are recognizable or almost recognizable by spectrum, moreover in the second case there are no sporadic groups among composition factors of groups isospectral to them (see [1]).

Lemma 5. *Let $\omega(G) = \omega(U_3(q))$, $q > 2$ and $S \leq G/K \leq \mathrm{Aut}(S)$ for a nonabelian simple group S. Then S is not sporadic.*

Proof. The main result in [14] implies that we have to consider only the following cases: either $q = 5$, or $q = 2^k - 1$ and $q^2 - q + 1$ is a prime. If $q = 5$ then because of the order of $U_3(5)$ we get that $S = J_2$. But $15 \in \omega(J_2) \backslash \omega(U_3(5))$ therefore this case is impossible. In the second case it follows from Tables 4, 6 and 8 in [15] that $U_3(q)$ satisfies to (1) of Lemma 1 and prime divisors of $(q^2 - q + 1)/(q + 1, 3)$ are not adjacent with 2 in the prime graph of $U_3(q)$. Therefore by (2) of Lemma 1, it is true that $(q^2 - q + 1)/(q + 1, 3) \in \omega(S)$. This number is greater than 127 for

$q \geq 20$, hence since $q = 2^k - 1$, we get that $q = 7$. This is impossible because $\pi(U_3(7)) = \{2, 3, 7, 43\}$ and there is no sporadic group whose set of prime order divisors is a subset of this set.

4. Exceptional groups of Lie type and J_2

It was proved in [16] that groups $G_2(3^n)$ are recognizable by spectrum. The recognizability of $G_2(4)$ was established in [17]. Since $G_2(q)$ is simple for $q > 2$ we assume further in this section that $q > 4$.

Lemma 6. *If q is a power of a prime p then*
1. $\mu(G_2(q) = \{8, 12, 2(q \pm 1), q^2 - 1, q^2 \pm q + 1\}$ *for* $p = 2$;
2. $\mu(G_2(q) = \{p^2, p(q \pm 1), 2(q \pm 1), q^2 - 1, q^2 \pm q + 1\}$ *for* $p = 3, 5$;
3. $\mu(G_2(q) = \{p(q \pm 1), 2(q \pm 1), q^2 - 1, q^2 \pm q + 1\}$ *for* $p > 5$.

Proof. It can be obtained from [18] and [19].

Lemma 7. *Let $\omega(G) = \omega(G_2(q))$, $q > 2$ and $S \leq G/K \leq \mathrm{Aut}(S)$ for a nonabelian simple group S. Then S is not sporadic.*

Proof. Assume to the contrary that S is sporadic. It is not hard to check from tables 5,7,9 in [15] that the group $G_2(q)$ for $q > 2$ satisfies Lemma 1. Moreover, Lemma 6 implies that the odd numbers $\frac{q^2+q+1}{(q-1,3)}$, $\frac{q^2-q+1}{(q+1,3)}$ are coprime to the other numbers from $\mu(G_2(q))$. In particular, their prime divisors are not adjacent with 2 in the prime graph. By (2) of Lemma 1 we have that $\frac{q^2+q+1}{(q-1,3)}, \frac{q^2-q+1}{(q+1,3)} \in \omega(S)$. Orders of elements of the sporadic groups do not exceed 119 therefore $\frac{q^2+q+1}{(q-1,3)} \leq 119$, $\frac{q^2-q+1}{(q+1,3)} \leq 119$, hence $q < 11$, $q \in \{3, 4, 5, 7, 8, 9\}$.

Let $q = 8$ or 9. In the first case $q^2+q+1 = 73$, in the second case $q^2-q+1 = 73$, in both cases we get that $73 \in \omega(G)$. This is a contradiction, since 73 does not divide orders of sporadic groups.

Let $q = 7$. Then $q^2 - q + 1 = 43 \in \omega(G)$ therefore $S \simeq J_4$. Since $37 \in \omega(J_4)\backslash\omega(G_2(7))$, this case is also impossible.

Let $q = 5$. Then $q^2+q+1 = 31 \in \omega(G)$. Clearly it is the maximal prime in $\omega(G)$, thus $S \simeq F_3$ or $S \simeq O'N$. In both cases $19 \in \omega(S)\backslash\omega(G_2(5))$, this is a contradiction.

Lemma 8. *Let $\omega(G) = \omega(E_7(q))$, $S \leq G/K \leq \mathrm{Aut}(S)$ for some nonabelian simple group S. Then S is not sporadic.*

Proof. The recognizability of $E_7(2)$ and $E_7(3)$ was proved in [20], so we assume that $q > 3$. Lemma 1.3 in [15] contains orders of maximal tori of the group $E_7(q)$ for each q which implies that prime divisors of $\frac{q^7-1}{(q-1)(q-1,7)}$ or $\frac{q^7+1}{(q+1)(q+1,7)}$, depending on q, are not adjacent with 2 in the prime graph of $E_7(q)$. From (2) of Lemma 1 it is concluded that S contains elements with such orders. As mentioned above orders of elements in sporadic groups do not exceed 119, but numbers $\frac{q^7-1}{(q-1)(q-1,7)}, \frac{q^7+1}{(q+1)(q+1,7)}$ are greater than 119 for $q > 2$. The lemma is proved.

Lemma 9. *Let $\omega(G) = \omega(J_2)$ and S be a nonabelian composition factor of G. If S is sporadic then $S \simeq J_2$.*

Proof. Note that $\pi(J_2) = \{2, 3, 5, 7\}$, the orders of other sporadic groups have prime divisors greater than 7, therefore since $\pi(S) \subseteq \pi(J_2)$, we have $S = J_2$. The lemma is proved.

References

[1] V. D. Mazurov, *Groups with prescribed spectrum* // Izv. Ural. Gos. Univ. Math. Mekh., **36**:7 (2005), 119–138. MR2190946

[2] A. V. Vasil′ev, M. A. Grechkoseeva, V. D. Mazurov, *On finite groups isospectral to simple symplectic and orthogonal groups* // Siberian Math. J., **50**:6 (2009), 965–981. MR2603865

[3] A. S. Kondrat′ev, *On recognizability by spectrum of finite simple orthogonal groups, II* // Vladikavkaz Math. J., **11**:4 (2009), 32–43. MR2768839

[4] A. S. Kondrat′ev, *Recognizability by spectrum of groups* $E_8(q)$ // Trudy Inst. Mat. i Mekh. UrO RAN, **16**:3 (2010), 146–149.

[5] A. V. Vasil′ev, M. A. Grechkoseeva, A. M. Staroletov, *On finite groups isospectral to simple linear and unitary groups* // Siberian Math. J., **52**:1 (2011), 30–40. Zbl pre05897315

[6] V. D. Mazurov, *Recognition of finite groups by a set of orders of their elements* // Algebra and Logic, **37**:6 (1998), 371–379. MR1680412

[7] V. D. Mazurov, W. J. Shi, *A note to the characterization of sporadic simple groups* // Ibid., **5**:3 (1998), 285–288. MR1679564

[8] S. Lipschutz, W. J. Shi, *Finite groups whose element orders do not exceed twenty* // Prog. Nat. Sci. **10**:1 (2000), 11–21. MR1765567

[9] A. M. Staroletov, *Groups isospectral to the degree 10 alternating group* // Siberian Math. J. **51**:3 (2010), 507–514. MR2683105

[10] A. V. Vasili′ev, I. B. Gorshkov, *On recognitioin of finite goups with connected prime graph* // Siberian Math. J., **50**:2 (2009), 233–238. MR2531755

[11] A. V. Vasil′ev, *On connection between the structure of a finite group and its prime graph* // Siberian Math. J., **46**:3 (2005), 396–404.

[12] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups: maximal subgroups and ordinary characters for simple groups*, Clarendon Press, Oxford, 1985. MR0827219

[13] R. Brandl, W. Shi, *The characterization of* $PSL(2, q)$ *by its element orders* // J. Algebra. **143**:2 (1991), 388–400. MR1132578

[14] A. V. Zavarnitsin, *Recognition of simple groups* $U_3(q)$ *by element orders* // Algebra and Logic, **45**:2 (2006), 106–116. MR2260330

[15] A. V. Vasil′ev, E. P. Vdovin, *An adjacency criterion for the prime graph of a finite simple group* // Algebra and Logic, **44**:6 (2005), 381–406. MR2213302

[16] A. V. Vasil′ev, *Recognizing Groups* $G_2(3^n)$ *by their element orders* // Algebra and Logic, **41**:2 (2002), 74–80. MR1922985

[17] V. D. Mazurov, *Recognition of finite simple groups* $S_4(q)$ *by their element orders* // Algebra and Logic, **41**:2 (2002), 93–110. MR1922988

[18] W. M. Kantor, A. Seress, *Prime power graphs for groups of Lie type* // J. Algebra, **247** (2002), 370–434. MR1877859

[19] D. Deriziotis, *Conjugacy classes and centralizers of semisimple elements in finite groups of Lie type*, Vorlesungen aus dem Fachbereich Mathematik der Universitat GH Essen, **11**, Universitat Essen, Fachbereich Mathematik, Essen, 1984.

[20] O. A. Alekseeva and A. S. Kondrat′ev, *Quasirecognition of one class of finite simple groups by the set of element orders* // Siberian Math. J., **44**:2 (2003), 195–207. MR1981363

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