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ON FINITELY LIPSCHITZ SPACE MAPPINGS

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Abstract. It is established that a ring Q-homeomorphism with respect to p-modulus in $\mathbb{R}^n$, $n \geqslant 2$, is finitely Lipschitz if $n-1 < p < n$ and if the mean integral value of $Q(x)$ over infinitesimal balls $B(x_0, \varepsilon)$ is finite everywhere.
Keywords: $Q-$homeomorphisms, p-modulus, p-capacity, finite Lipschitz.

1. Introduction

Recall that, given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\varrho : \mathbb{R}^n \to [0,\infty]$ is called *admissible* for Γ, abbr. $\varrho \in adm\,\Gamma$, if

$$\int\limits_{\gamma} \varrho\, ds \ \geq \ 1 \tag{1}$$

for all $\gamma \in \Gamma$.

The *p-modulus* of Γ is the quantity

$$M_p(\Gamma) \ = \ \inf_{\varrho \in adm\,\Gamma} \int\limits_{\mathbb{R}^n} \varrho^p(x)\ dm(x)\ . \tag{2}$$

Here the notation m refers to the Lebesgue measure in $\mathbb{R}^n$.

Let G and G' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $Q : G \to [0,\infty]$ be a measurable function. A homeomorphism $f : G \to G'$ is called a $Q-$*homeomorphism with respect*

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to the p-modulus if

$$M_p(f\Gamma) \leq \int\limits_G Q(x) \cdot \varrho^p(x)\ dm(x) \tag{3}$$

for every family Γ of paths in G and every admissible function ϱ for Γ.

This conception is a natural generalization of the geometric definition of a quasiconformal mapping: if $Q(x) \leq K < \infty$ a.e., then f is quasiconformal under $p = n$, see 13.1 and 34.6 in [26], and quasiisometric under $1 < p \neq n$, see [4].

This class of Q-homeomorphisms with respect to the n-modulus was first considered in the papers [16]-[18], see also the monograph [19]. The main goal of the theory of Q-homeomorphisms is to clear up various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. In particular, the problem of the local and boundary behavior of Q-homeomorphisms has been studied in $\mathbb{R}^n$ first in the case $Q \in BMO$ (bounded mean oscillation) in the papers [16]-[18] and then in the case of $Q \in FMO$ (finite mean oscillation) and other cases in the papers [11], [12], [21].

Note that the estimate of the type (3) was first established in the classical quasiconformal theory. Namely, it was obtained in [15], p. 221, for quasiconformal mappings in the complex plane that

$$M(f\Gamma)\ \leq\ \int\limits_{\mathbb{C}} K(z) \cdot \rho^2(z)\ dxdy \tag{4}$$

where

$$K(z)\ =\ \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \tag{5}$$

is a (local) maximal dilatation of the mapping f at a point z. Next, it was obtain in [1], Lemma 2.1, for quasiconformal mappings in space, $n \geq 2$, that

$$M(f\Gamma)\ \leq\ \int\limits_D K_I(x,f)\, \rho^n(x)\ dm(x) \tag{6}$$

where $K_I(x,f)$ stands for the inner dilatation of f at x, see (8) below.

Given a mapping $f : G \to \mathbb{R}^n$ with partial derivatives a.e., $f'(x)$ denotes the Jacobian matrix of f at $x \in D$ if it exists, $J(x) = J(x,f) = \det f'(x)$ the Jacobian of f at x, and $|f'(x)|$ the operator norm of $f'(x)$, i.e., $|f'(x)| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$. We also let $l(f'(x)) = \min\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$. The *outer dilatation* of f at x is defined by

$$K_O(x,f) = \begin{cases} \frac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x,f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \\ \infty, & \text{otherwise,} \end{cases} \tag{7}$$

the *inner dilatation* of f at x by

$$K_I(x,f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n}, & \text{if } J(x,f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \\ \infty, & \text{otherwise,} \end{cases} \tag{8}$$

The following notion generalizes and localizes the above notion of Q–homeomorphism. It is motivated by the ring definition of Gehring for quasiconformal mappings, see, e.g., [5], introduced first by V. Ryazanov, U. Srebro, and E. Yakubov in the plane and later on extended by V. Ryazanov and E. Sevostyanov to the space case, see, e.g., [22], [23] and Chapters 7 and 11 in [19].

Let $E,\ F\subset\mathbb{R}^n$ be arbitrary sets. Denote by $\Delta(E,F,G)$ a family of all curves $\gamma:[a,b]\to\overline{\mathbb{R}^n}$ joining E and F in G, i.e., $\gamma(a)\in E,\gamma(b)\in F$ and $\gamma(t)\in G$ for $t\in(a,b)$. Given a domain G in $\mathbb{R}^n$, $n\geq 2$, a (Lebesgue) measurable function $Q:G\to[0,\infty]$, $x_0\in G$, a homeomorphism $f:G\to\mathbb{R}^n$ is said to be a *ring Q–homeomorphism at the point x_0* if

$$M_p\left(f\left(\Delta\left(S_1,\,S_2,\,A\right)\right)\right)\ \leq\int\limits_{A} Q(x)\cdot\eta^p(|x-x_0|)\ dm(x) \tag{9}$$

for every ring $A=A(r_1,r_2,x_0)=\{x\in\mathbb{R}^n:r_1<|x-x_0|<r_2\}$ and the spheres $S_i=S(x_0,r_i)=\{x\in\mathbb{R}^n:|x-x_0|=r_i\}$, where $0<r_1<r_2<r_0:=\operatorname{dist}(x_0,\partial D)$, and every measurable function $\eta:(r_1,r_2)\to[0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\ dr\ \geq\ 1\,. \tag{10}$$

f is called a *ring $Q-$homeomorphisms with respect to the p-modulus in the domain G* if f is a ring Q–homeomorphism at every point $x_0\in G$.

Let $f:G\to\mathbb{R}^n,n\geq 2$ be a quasiconformal mapping. The *angular dilatation* of the mapping $f:G\to\mathbb{R}^n$ at the point $x\in G$ with respect to $x_0\in G,x_0\neq x$ is defined by

$$D_f(x,x_0)=\frac{J(x,f)}{l_f^n(x,x_0)}\,, \tag{11}$$

where

$$l_f(x,x_0)=\min_{|h|=1}\frac{|\partial_h f(x)|}{|\langle h,\frac{x-x_0}{|x-x_0|}\rangle|}.$$

Here $\partial_h f(x)$ denotes the derivative of f at x in the direction h and the minimum is taken over all unit vectors $h\in\mathbb{R}^n$, see [8].

We recall that the estimate of the type (9) was first established in the classical quasiconformal theory in complex plane, see [9]. Next, it was obtained in [8], for quasiconformal mappings in space, $n\geq 2$, that

$$M\left(f\left(\Delta\left(S_1,\,S_2,\,A\right)\right)\right)\ \leq\int\limits_{A} D_f(x,x_0)\cdot\eta^n(|x-x_0|)\ dm(x) \tag{12}$$

for every ring $A=A(x_0,r_1,r_2)=\{x\in\mathbb{R}^n:r_1<|x-x_0|<r_2\}$ and the spheres $S_i=S(x_0,r_i)=\{x\in\mathbb{R}^n:|x-x_0|=r_i\}$, where $0<r_1<r_2<r_0:=\operatorname{dist}(x_0,\partial G)$, and every measurable function $\eta:(r_1,r_2)\to[0,\infty]$ such that (10) holds.

Note that, in particular, homeomorphisms $f:G\to\mathbb{R}^n$ in the class $W^{1,n}_{loc}$ with $K_I(x,f)\in L^1_{loc}$ are ring Q–homeomorphisms as well as Q–homeomorphisms with $Q(x)=K_I(x,f)$, see, e.g., Theorem 6.10 and Corollary 4.9 in [16], or Theorem 4.1 in [19].

2. Preliminaries

Here a *condenser* is a pair $E = (A, C)$ where $A \subset \mathbb{R}^n$ is open and C is a non–empty compact set contained in A. E is a *ringlike condenser* if $B = A \setminus C$ is a ring, i.e., if B is a domain whose complement $\overline{\mathbb{R}^n} \setminus B$ has exactly two components where $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ is the one point compactification of $\mathbb{R}^n$. E is a *bounded condenser* if A is bounded. A condenser $E = (A, C)$ is said to be in a domain G if $A \subset G$.

The following proposition is immediate.

Proposition 1. *If $f : G \to \mathbb{R}^n$ is open and $E = (A, C)$ is a condenser in G, then $(fA,\, fC)$ is a condenser in fG.*

In the above situation we denote $fE = (fA,\, fC)$.

Let $E = (A,\, C)$ be a condenser. Then $W_0(E) = W_0(A,\, C)$ denotes the family of non–negative functions $u : A \to R^1$ such that (1) $u \in C_0(A)$, (2) $u(x) \geq 1$ for $x \in C$, and (3) u is ACL. We set

$$cap_p\, E \;=\; cap\; (A,\, C) \;=\; \inf_{u \,\in\, W_0(E)} \int\limits_A |\nabla u|^p \; dm \tag{13}$$

where

$$|\nabla u| \;=\; \left(\sum_{i=1}^{n} \left(\partial_i u\right)^2 \right)^{1/2}$$

and call the quantity (13) the *p-capacity* of the condenser E.

For the next statement, see, e.g., [3], [10] and [25].

Proposition 2. *Suppose $E = (A, C)$ is a condenser such that C is connected. Then*

$$\mathrm{cap}_p\, E = M_p(\Delta(\partial A, \partial C; A \setminus C))\,.$$

We give here also the following two useful statements, see Proposition 5 and 6 in [14].

Proposition 3. *Let $E = (A, C)$ be a condenser such that C is connected. Then*

$$\mathrm{cap}_p\, E \geqslant \frac{(\inf m_{n-1}\, \sigma)^p}{[m(A \setminus C)]^{p-1}}$$

where $m_{n-1}\, \sigma$ denotes the $(n-1)$-dimensional area of the C^∞-manifold σ that is the boundary $\sigma = \partial U$ of open set U containing C and contained along with its closure $\overline{U}$ in A and the infimum is taken over all such σ.

Proposition 4. *Let $E = (A, C)$ be a condenser such that C is connected. Then for $n - 1 < p \leq n$*

$$\left(cap_p\; E\right)^{n-1} \;\geq\; \gamma\, \frac{d(C)^p}{m(A)^{1-n+p}}$$

where γ is a positive constant that depends only on n and p, $d(A)$ is a diameter and $m(A)$ is the Lebesgue measure of A in $\mathbb{R}^n$.

3. Characterization of ring $Q-$homeomorphisms with respect to the p-modulus

The theorems of this section extend the corresponding results in [22], see also Section 7.3 in the monograph [19], from the case of $p = n$ to the case of $p \in (1, n]$. Below we use the standard conventions: $a/\infty = 0$ for $a \neq \infty$ and $a/0 = \infty$ if $a > 0$ and $0 \cdot \infty = 0$, see e.g. [24], p. 6.

Lemma 1. *Let G be a domain in $\mathbb{R}^n, n \geq 2$, $1 < p \leq n$, $Q : G \to [0, \infty]$ a measurable function and $q_{x_0}(r)$ the mean of $Q(x)$ over the sphere $|x - x_0| = r$. Set*

$$I = I(x_0, r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)} \tag{14}$$

and $S_j = \{x \in \mathbb{R}^n : |x - x_0| = r_j\}$, $j = 1, 2$, where $x_0 \in G$ and $0 < r_1 < r_2 < r_0 = dist\,(x_0, \partial G)$. Then whenever $f : G \to \mathbb{R}^n$ is a ring $Q-$homeomorphism with respect to the p-modulus at a point x_0

$$M_p\,(\Delta\,(fS_1, fS_2, fG)) \leq \frac{\omega_{n-1}}{I^{p-1}} \tag{15}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Proof. With no loss of generality, we may assume that $I \neq 0$ because otherwise (15) is trivial, and that $I \neq \infty$ because otherwise we can replace $Q(x)$ by $Q(x) + \delta$ with arbitrarily small $\delta > 0$ and then take the limit as $\delta \to 0$ in (15). The condition $I \neq \infty$ implies, in particular, that $q_{x_0}(r) \neq 0$ a.e. in (r_1, r_2). Set

$$\psi(t) = \begin{cases} 1/[t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)]\,, & t \in (r_1, r_2)\,, \\ 0\,, & t \notin (r_1, r_2)\,. \end{cases} \tag{16}$$

Then

$$\int\limits_{A} Q(x) \cdot \psi^p(|x - x_0|)\; dm(x) = \omega_{n-1} I \tag{17}$$

where $A = A(x_0, r_1, r_2)$.

Let Γ be a family of all paths joining the spheres S_1 and S_2 in A. Let also ψ^* be a Borel function such that $\psi^*(t) = \psi(t)$ for a.e. $t \in [0, \infty]$. Such a function ψ^* exists by the Lusin theorem, see, e.g., 2.3.5 in [2] and [24], p. 69. Then the function

$$\rho(x) = \psi^*(|x - x_0|)/I$$

is admissible for Γ and since f is a ring $Q-$homeomorphisms with respect to the p-modulus we get by (17) that

$$M_p(f\Gamma) \leq \int\limits_{A} Q(x) \cdot \rho^p(x)\; dm(x) = \frac{\omega_{n-1}}{I^{p-1}}\,. \tag{18}$$

and the proof is complete.

The following lemma shows that the estimate (15) cannot be improved for ring $Q-$homeomorphisms with respect to the p-modulus.

Lemma 2. *Let G be a domain in $\mathbb{R}^n$, $n \geq 2$, $1 < p \leq n$, $x_0 \in G$, $0 < r_1 < r_2 < r_0 = dist(x_0, \partial G)$, $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$, $Q :\to [0, \infty]$ be a measurable function. Set*

$$\eta_0(r) = \frac{1}{I r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)} \tag{19}$$

where $q_{x_0}(r)$ is the mean of $Q(x)$ over the sphere $|x - x_0| = r$ and I is given by (14). Then

$$\frac{\omega_{n-1}}{I^{p-1}} = \int\limits_A Q(x) \cdot \eta_0^p(|x - x_0|)\ dm(x) \leq \int\limits_A Q(x) \cdot \eta^p(|x - x_0|)\ dm(x) \tag{20}$$

whenever $\eta : (r_1, r_2) \to [0, \infty]$ is measurable and

$$\int\limits_{r_1}^{r_2} \eta(r)\ dr\ = 1\,. \tag{21}$$

Proof. If $I = \infty$, then the left hand side in (20) is equal to zero and the inequality is obvious. If $I = 0$, then $q_{x_0}(r) = \infty$ for a.e. $r \in (r_1, r_2)$ and the both sides in (20) are equal to ∞. Hence we may assume below that $0 < I < \infty$. Now, by (19) and (21) $q_{x_0}(r) \neq 0$ and $\eta(r) \neq \infty$ a.e. in (r_1, r_2). Set $\lambda(r) = r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)\eta(r)$ and $w(r) = 1/r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)$. Then by the standard conventions $\eta(r) = \lambda(r)w(r)$ a.e. in (r_1, r_2) and

$$C\ :\ =\ \int\limits_A Q(x) \cdot \eta^p(|x - x_0|)\, dm(x)\ =\ \omega_{n-1} \int\limits_{r_1}^{r_2} \lambda^p(r) \cdot w(r)\ dr\ . \tag{22}$$

By Jensen's inequality with weights, see, e.g., Theorem 2.6.2 in [20], applied to the convex function $\varphi(t) = t^p$ in the interval $\Omega = (r_1, r_2)$ with the probability measure

$$\nu\,(E)\ =\ \frac{1}{I} \int\limits_E w(r)\ dr \tag{23}$$

we obtain that

$$\left(⨍ \lambda^p(r) w(r)\ dr \right)^{1/p}\ \geq\ ⨍ \lambda(r) w(r)\ dr\ =\ \frac{1}{I} \tag{24}$$

where we also applied that $\eta(r) = \lambda(r)w(r)$ satisfies (21).

Thus

$$C\ \geq\ \frac{\omega_{n-1}}{I^{p-1}} \tag{25}$$

and the proof of (20) is complete.

Finally, combining Lemmas 1 and 2, we obtain the following statement.

Theorem 1. *Let G be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : G \to [0, \infty]$ a measurable function. A homeomorphism $f : G \to \mathbb{R}^n$ is ring $Q-$homeomorphism with respect to the p-modulus at a point $x_0 \in G$ if and only if for every $0 < r_1 < r_2 < d_0 = dist\,(x_0, \partial G)$*

$$M_p\,(\Delta\,(fS_1,\ fS_2\, fG))\ \leq\ \frac{\omega_{n-1}}{I^{p-1}} \tag{26}$$

where S_1 *and* S_2*,* $S_1 = \{x \in \mathbb{R}^n : \ |x-x_0| = r_1\}$ *and* $S_2 = \{x \in \mathbb{R}^n : \ |x-x_0| = r_2\}$*.* ω_{n-1} *is the area of the unit sphere in* $\mathbb{R}^n$*,* $I = I(x_0,r_1,r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}$*,* $q_{x_0}(r)$ *is the mean value of* Q *over the sphere* $|x-x_0| = r$*.*

Note that the infimum from the right hand side in (9) holds for the function

$$\eta_0(r) = \frac{1}{Ir^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}\,. \tag{27}$$

Theorem 1 will have many applications in the theory of ring $Q-$homeomorphisms with respect to the p-modulus, see, e.g., the next section.

4. On finitely by Lipschitz mappings

Given a mapping $\varphi : E \to \mathbb{R}^n$ and a point $x \in E \subseteq \mathbb{R}^n$, set

$$L(x,\varphi) \;=\; \limsup_{y\to x\; y\in E} \frac{|\varphi(y)-\varphi(x)|}{|y-x|} \tag{28}$$

and

$$l(x,\varphi) \;=\; \liminf_{y\to x\; y\in E} \frac{|\varphi(y)-\varphi(x)|}{|y-x|} \tag{29}$$

Given a set $A \subseteq \mathbb{R}^n$, $n \geq 1$, we say that a mapping $f : A \to \mathbb{R}^n$ is called *Lipschitz* if there is number $L > 0$ such that the inequality

$$|\,f(x)-f(y)| \;\leq L\,|\,x-y| \tag{30}$$

holds for all x and y in A. Given an open set $\Omega \subseteq \mathbb{R}^n$, we say that a mapping $f : \Omega \to \mathbb{R}^n$ is *finitely Lipschitz* if $L(x,f) < \infty$ for all $x \in \Omega$.

Lemma 3. *Let* G *and* G' *be bounded domains in* $\mathbb{R}^n$*,* $n \geq 2$*,* $Q : G \to [0,\infty]$ *be a measurable function and let* $f : G \to G'$ *be a ring* $Q-$*homeomorphism with respect to p-modulus at a point* $x_0 \in G$*,* $1 < p < n$*. Then*

$$m(fB(x_0,r_1)) \leqslant \frac{c_{n,p}}{I^{\frac{n(p-1)}{n-p}}(x_0,r_1,r_2)} \tag{31}$$

for every $0 < r_1 < r_2 < d_0 = dist\,(x_0,\partial G)$ *where* $I(x_0,r_1,r_2)$ *is defined by (14) and* $c_{n,p}$ *is a positive constant that depends only on* n *and* p*.*

Proof. Let us consider the condenser $(A_{t+\triangle t}, C_t)$, where $C_t = \overline{B(x_0,t)}$, $A_{t+\triangle t} = B(x_0, t+\triangle t)$. Note that $(fA_{t+\triangle t}, fC_t)$ is a ringlike condenser in $\mathbb{R}^n$ and according to Proposition 2, we have

$$\mathrm{cap}_p(fA_{t+\triangle t}, fC_t) = M_p(\triangle(\partial fA_{t+\triangle t}, \partial fC_t; fR_t)). \tag{32}$$

In view of Proposition 3, we obtain

$$\mathrm{cap}_p\,(fA_{t+\triangle t}, fC_t) \geqslant \frac{(\inf m_{n-1}\,\sigma)^p}{m\,(fA_{t+\Delta t}\setminus fC_t)^{p-1}}, \tag{33}$$

where $m_{n-1}\,\sigma$ denotes the $(n-1)$-dimensional area of a C^∞-manifold σ that is the boundary of an open set U containing fC_t with its closure $\overline{U}$ in $fA_{t+\triangle t}$ and the infimum is taken over all such σ.

On the other hand, by Lemma 1, we have

$$M_p(\triangle(\partial fA_{t+\triangle t},\partial fC_t;fR_t))\leqslant\frac{\omega_{n-1}}{\left(\int\limits_t^{t+\Delta t}\frac{ds}{s^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(s)}\right)^{p-1}}\,. \tag{34}$$

Combining (32)-(34), we obtain

$$\frac{(\inf m_{n-1}\,\sigma)^p}{m\,(fA_{t+\Delta t}\setminus fC_t)^{p-1}}\leqslant\frac{\omega_{n-1}}{\left(\int\limits_t^{t+\Delta t}\frac{ds}{s^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(s)}\right)^{p-1}}\,.$$

Applying the isoperimetric inequality to the numerator of the fraction on the left-hand side we came to the inequality

$$n\cdot\Omega_n^{\frac{1}{n}}\,(m(fC_t))^{\frac{n-1}{n}}\leq\omega_{n-1}^{\frac{1}{p}}\left(\frac{m\,(fA_{t+\Delta t}\setminus fC_t)}{\int\limits_t^{t+\Delta t}\frac{ds}{s^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(s)}}\right)^{\frac{p-1}{p}} \tag{35}$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Now, setting $\Phi(t):=m\,(fB_t)$, we see from (35) that

$$n\cdot\Omega_n^{\frac{1}{n}}\,\Phi^{\frac{n-1}{n}}(t)\leq\omega_{n-1}^{\frac{1}{p}}\left(\frac{\frac{\Phi(t+\Delta t)-\Phi(t)}{\Delta t}}{\frac{1}{\Delta t}\int\limits_t^{t+\Delta t}\frac{ds}{s^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(s)}}\right)^{\frac{p-1}{p}}. \tag{36}$$

Since the function $\Phi(t)$ is nondecreasing, has finite derivative $\Phi'(t)$ for a.e. t. Letting $\Delta t\to 0$ in (36) and taking into account that $\omega_{n-1}=n\Omega_n$, we obtain

$$\frac{n\Omega_n^{\frac{p-n}{n(p-1)}}}{t^{\frac{n-1}{p-1}}\,q_{x_0}^{\frac{1}{p-1}}(t)}\leqslant\frac{\Phi'(t)}{\Phi^{\frac{p(n-1)}{n(p-1)}}(t)}. \tag{37}$$

Integrating (37) under $1<p<n$ with respect to $t\in[r_1,r_2]$, since

$$\int\limits_{r_1}^{r_2}\frac{\Phi'(t)}{\Phi^{\frac{p(n-1)}{n(p-1)}}(t)}\,dt\leqslant\frac{n(p-1)}{p-n}\left(\Phi^{\frac{p-n}{n(p-1)}}(r_2)-\Phi^{\frac{p-n}{n(p-1)}}(r_1)\right),$$

see, e.g., Theorem IV. 7.4 in [24], we observe that

$$\Omega_n^{\frac{p-n}{n(p-1)}}\int\limits_{r_1}^{r_2}\frac{dt}{t^{\frac{n-1}{p-1}}\,q_{x_0}^{\frac{1}{p-1}}(t)}\leqslant\frac{p-1}{p-n}\left(\Phi^{\frac{p-n}{n(p-1)}}(r_2)-\Phi^{\frac{p-n}{n(p-1)}}(r_1)\right). \tag{38}$$

From (38) we conclude that

$$\Phi(r_1)\leqslant\left(\Phi^{\frac{p-n}{n(p-1)}}(r_2)+\Omega_n^{\frac{p-n}{n(p-1)}}\,\frac{n-p}{p-1}\int\limits_{r_1}^{r_2}\frac{dt}{t^{\frac{n-1}{p-1}}\,q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{\frac{n(p-1)}{p-n}}$$

and hence

$$\Phi(r_1) \leqslant \Omega_n \left(\frac{p-1}{n-p}\right)^{\frac{n(p-1)}{n-p}} \left(\int\limits_{r_1}^{r_2} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{-\frac{n(p-1)}{n-p}} .$$

Combining Lemmas 2 and 3, we have the following statement.

Lemma 4. *Let G and G' be bounded domains in $\mathbb{R}^n$, $n \geq 2$, $Q : G \to [0, \infty]$ be a measurable function and let $f : G \to G'$ be a ring $Q-$homeomorphism with respect to the p-modulus. Then for $1 < p < n$*

$$m(fB(x_0, r_1)) \leqslant c'_{n,p} \left[\int\limits_{A(x_0,r_1,r_2)} Q(x)\eta^p(|x - x_0|)\, dm(x)\right]^{\frac{n}{n-p}} . \tag{39}$$

for every ring $A = A(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0 = dist\,(x_0, \partial G)$ and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$, such that

$$\int\limits_{r_1}^{r_2} \eta(r)\ dr\ \geq\ 1\,. \tag{40}$$

where $c'_{n,p}$ is a positive constant that depends only on n and p.

Theorem 2. *Let G and G' be domains in $\mathbb{R}^n$, $n \geq 2$, and $Q : G \to [0, \infty]$ be a measurable function such that*

$$Q_0 = \overline{\lim_{r \to 0}}\, \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(x_0,\varepsilon)} Q(x)\, dm(x) < \infty. \tag{41}$$

Then for every ring $Q-$homeomorphism $f : G \to G'$ with respect to the p-modulus, $n - 1 < p < n$,

$$L(x_0, f) = \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq \lambda_{n,p}\, Q_0^{\frac{1}{n-p}} \tag{42}$$

where $\lambda_{n,p}$ is a positive constant that depends only on n and p.

The Theorem 2 under $Q(x) = const$ was obtain by Gehring in [4] see Theorem 3, p. 189.

Proof. Let us consider the spherical ring $A(x_0, \varepsilon, 2\varepsilon) = \{x : \varepsilon < |x - x_0| < 2\varepsilon\}$, $x \in G$, c $\varepsilon > 0$ such that $A(x_0, \varepsilon, 2\varepsilon) \subset G$. Since $\left(fB\,(x_0, 2\varepsilon)\,, \overline{fB\,(x_0, \varepsilon)}\right) = \left(fB\,(x_0, 2\varepsilon)\,, f\overline{B\,(x_0, \varepsilon)}\right)$ are ringlike condensers in G' and, according to Proposition 2, we obtain

$$cap_p\ (fB(x_0, 2\varepsilon), \overline{fB(x_0, \varepsilon)}) = M_p(\triangle(\partial fB(x_0, 2\varepsilon), \partial fB(x_0, \varepsilon); fA(x_0, \varepsilon, 2\varepsilon)))\,.$$

Note that, in view of the homeomorphism of f,

$$\triangle\,(\partial fB\,(x_0, 2\varepsilon)\,, \partial fB\,(x_0, \varepsilon)\,; fA(x_0, \varepsilon, 2\varepsilon)) =$$

$$f\,(\triangle\,(\partial B(x_0, 2\varepsilon), \partial B(x_0, \varepsilon); A(x_0, \varepsilon, 2\varepsilon)))\,.$$

By Proposition 4

$$\text{(43)} \qquad cap_p\ (fB(x_0,2\varepsilon), \overline{fB(x_0,\varepsilon)}) \geq \left(\gamma \frac{d^p(fB(x_0,\varepsilon))}{m^{1-n+p}(fB(x_0,2\varepsilon))}\right)^{\frac{1}{n-1}}$$

where γ is a positive constant that depends only on n and p, $d(A)$ is the diameter and $m(A)$ is the Lebesgue measure of A in $\mathbb{R}^n$.

By the definition of ring $Q-$homeomorphisms with respect to the p-modulus

$$\text{(44)} \qquad cap_p\ (fB(x_0,2\varepsilon), \overline{fB(x_0,\varepsilon)}) \leq \frac{1}{\varepsilon^p} \int\limits_{A(x_0,\varepsilon,2\varepsilon)} Q(x)\ dm(x)$$

because the function

$$\eta(t) \ = \left\{ \begin{array}{ll} \frac{1}{\varepsilon}, & \text{if } t\in(\varepsilon,2\varepsilon), \\ 0, & \text{if } t\in\mathbb{R}\setminus(\varepsilon,2\varepsilon) \end{array} \right.$$

satisfies (10) for $r_1=\varepsilon$ and $r_2=2\varepsilon$.

Next, the function

$$\widetilde{\eta}(t) \ = \left\{ \begin{array}{ll} \frac{1}{2\varepsilon}, & \text{if } t\in(2\varepsilon,4\varepsilon) \\ 0, & \text{if } t\in\mathbb{R}\setminus(2\varepsilon,4\varepsilon), \end{array} \right.$$

satisfies (10) for $r_1=2\varepsilon$ and $r_2=4\varepsilon$ and hence by Lemma 4 we have the following estimates:

$$\text{(45)} \qquad m(fB(x_0,2\varepsilon)) \leqslant c''_{n,p}\,\varepsilon^n \left[\frac{1}{m(B(x,4\varepsilon))} \int\limits_{B(x_0,4\varepsilon)} Q(x)\,dm(x)\right]^{\frac{n}{n-p}},$$

where $c''_{n,p}$ is a positive constant that depends only on n and p.

Combining (45), (44) and (43), we obtain

$$\frac{d(fB(x_0,\varepsilon))}{\varepsilon} \leq \lambda_{n,p}\,\cdot$$

$$\cdot\left(\frac{1}{m(B(x_0,4\varepsilon))} \int\limits_{B(x_0,4\varepsilon)} Q(y)\,dy\right)^{\frac{n(1-n+p)}{p(n-p)}} \left[\frac{1}{m(B(x,2\varepsilon))} \int\limits_{B(x_0,2\varepsilon)} Q(x)\,dm(x)\right]^{\frac{n-1}{p}}$$

and hence

$$L(x_0,f) = \limsup_{x\to x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} \leq \limsup_{\varepsilon\to 0} \frac{d(fB(x_0,\varepsilon))}{\varepsilon} \leq \lambda_{n,p}\,Q_0^{\frac{1}{n-p}},$$

where $\lambda_{n,p}$ is a positive constant that depends only on n and p.

Corollary 1. *Let G and G' be domains in $\mathbb{R}^n$, $n\geq 2$, $f:G\to G'$ be a ring $Q-$homeomorphism with respect to the p-modulus, $n-1<p<n$, such that*

$$\text{(46)} \qquad \overline{\lim_{r\to 0}}\ \frac{1}{\Omega_n\varepsilon^n} \int\limits_{B(x_0,\varepsilon)} Q(x)\,dm(x) < \infty \quad \forall\ x_0\in G.$$

Then f is finitely Lipschitz.

Note that the theory of ring $Q-$homeomorphisms with respect to p-modulus can be applied to mappings in the Orlich-Sobolev classes $W^{1,\varphi}_{loc}$ with a Calderon type condition on φ and, in particular, to the Sobolev classes $W^{1,p}_{loc}$ for $p > n-1$, cf. [13].

References

[1] C.J. Bishop, V.Ya. Gutlyanskii, O. Martio, M. Vuorinen. *On conformal dilatation in space* // Intern. Journ. Math. and Math. Scie., **22** (2003), 1397–1420. MR1980177

[2] B. Fuglede. *Extremal length and functional completion* // Acta Math., **98** (1957), 171–219. MR0097720

[3] F.W. Gehring. *Quasiconformal mappings in Complex Analysis and its Applications*. Vienna: International Atomic Energy Agency, **2**, 1976. MR0480997

[4] F.W. Gehring. *Lipschitz mappings and the p-capacity of ring in n-space* // Advances in the theory of Riemann surfaces (Proc. Conf. Stonybrook, N.Y., 1969), Ann. of Math. Studies, **66** (1971), 175–193. MR0310238

[5] F.W. Gehring. *Rings and quasiconformal mappings in space* // Trans. Amer. Math. Soc., **103** (1962), 353–393. MR0139735

[6] A. Golberg. *Differential properties of (α, Q)-homeomorphisms* // In: Further Progress in Analysis, World Scientific Publ, 2009, 218–228. MR2581624

[7] A. Golberg. *Integrally quasiconformal mappings in space* // In: Intern. Conf. "Analytic methods of mechanics and complex analysis"dedicated to N.A. Kilchevskii and V.A. Zmorovich on the occasion of their birthday centenary, Kiev, June 29 - July 5, 2009, 19.

[8] A. Golberg , V. Ya. Gutlyanskii. *On Lipschitz continuity of quasiconformal mappings in space* // Journal d'Analyse Mathematique, **109**:1, (2009), 233–251. MR2585395

[9] V. Gutlyanskii, T. Sugawa. *On Lipschitz Continuity of Quasiconformal Mappings* // Rep. Univ. Juvaskyla, **83** (2001), 91–108. MR1886616

[10] J. Hesse. *A $p-$extremal length and $p-$capacity equality* // Ark. Mat., **13** (1975), 131–144. MR0379871

[11] A. Ignat'ev, V. Ryazanov. *Finite mean oscillation in the mapping theory* // Ukrainian Math. Bull., **2**:3 (2005), 403–424. MR2325895

[12] A. Ignat'ev, V. Ryazanov. *To the theory of the boundary behavior of space mappings* // Ukrainian Math. Bull., **3**:2 (2006), 189–201. MR2325572

[13] D. Kovtonyuk, V. Ryazanov, R. Salimov, E. Sevost'yanov. *On mappings in the Orlicz-Sobolev classes*, ArXiv: 1012.5010 [math.CV], 2011, 69 p.

[14] V.I. Kruglikov. *Capacities of condensors and quasiconformal in the mean mappings in space* // Mat. Sb., **130**:2 (1986), 185–206. MR0854971

[15] O. Lehto, K. Virtanen. *Quasiconformal mappings in the Plane*, Berlin-Heidelberg-New York: Springer-Verlag, 1973. MR0344463

[16] O. Martio, V. Ryazanov, U. Srebro, E. Yakubov. *Mappings with finite length distortion* // J. Anal. Math., **93** (2004), 215–236. MR2110329

[17] O. Martio, V. Ryazanov, U. Srebro, E. Yakubov. *$Q-$homeomorphisms* // Contemporary Math., **364** (2004), 193–203. MR2099027

[18] O. Martio, V. Ryazanov, U. Srebro, E. Yakubov. *On $Q-$homeomorphisms* // Ann. Acad. Sci. Fenn., **30** (2005), 49–69. MR2140298

[19] O. Martio, V. Ryazanov, U. Srebro, E. Yakubov. *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematies, New York: Springer, 2009. MR2466579

[20] T. Ransford. *Potential Theory in the Complex Plane*, Cambridge: Cambridge University Press, 1995. MR1334766

[21] V. Ryazanov, E. Sevost'yanov. *Normal families of space mappings* // Sib. Electron. Math. Reports, **3** (2006), 216–231. MR2276021

[22] V. Ryazanov, E. Sevost'yanov. *Equicontinuous classes of ring Q-homeomorphisms* // Sib. Mat. Zh., **48**:6 (2007), 1361–1376. MR2397516
[23] V. Ryazanov, U. Srebro, E. Yakubov. *On ring solutions of Beltrami equation* // J. d'Analyse Math, **96** (2005), 117–150. MR2177183
[24] S. Saks. *Theory of the Integral*, New York: Dover Publ. Inc., 1964. MR0167578
[25] V.A. Shlyk. *On the equality between $p-$capacity and $p-$modulus* // Sib. Mat. Zh., **34**:6 (1993), 216–221; transl. in Siberian Math. J. **34**:6 (1993), 1196–1200. MR1268174
[26] J. Väisälä. *Lectures on n-Dimensional Quasiconformal Mappings*, Lecture Notes in Math. **229**, Berlin: Springer-Verlag, 1971. MR0454009

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