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INDUCED PERFECT COLORINGS

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Abstract. We introduce the operator that maps an eigenspace of a halved graph of a distance-2 biregular graph to an eigenspace of the second halved graph. Using the introduced notion, we show that Steiner triple and Steiner quadruple systems give new infinite series of perfect 2-colorings of the Johnson graphs $J(n, 4)$ and $J(n, 5)$.

Keywords: distance-biregular graphs, graph spectra, perfect colorings

1. Preliminaries

Let $G = (V, E)$ be a graph. By G_2 we denote the graph having the same vertex set as G and edges connecting vertices at distance 2 in G. If G is a connected bipartite graph, then the connected components of G_2 are called the *halved graphs* of G [2] and denoted by G' and G''. Note that the vertex sets of the halved graphs coincide with parts V' and V'' of graph G.

For two vertices x and y of a graph G at distance two let $t(x, y)$ denote the number of vertices that are at distance one from x and y in G. A bipartite biregular graph G with parts V' and V'' of degrees c' and c'' is called a (c', c'', t', t'') *distance-2 biregular graph* if $t(x, y)$ is equal to t' or t'' depending on belonging of x and y to V' or V''.

The notion of a distance-2 biregular graph generalizes the notion of a distance-biregular graph, introduced in [4]. It is easy to see that the graphs of infinite rectangular and hexagonal grids are not distance-biregular but are distance-2

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biregular. As far as finite graphs are concerned, any bipartite biregular graph of girth more than four is a distance-2 biregular graph.

Let M be the adjacency matrix of a graph G. A function $\varphi : V \to R$ is called an λ-*eigenfunction of* G if the vector $\overline{\varphi}$ consisting of the values of this function is an eigenvector of M corresponding to the eigenvalue λ, i.e. : $M\overline{\varphi} = \lambda\overline{\varphi}$.

Let us introduce the main notion of this paper. Let G be a distance-2 biregular graph, φ be a real-valued function defined on the vertices of G'. Consider the function defined on the vertices of G'' by the following equality:

$$\varphi_{G''}(x) = \sum_{y:(x,y)\in E} \varphi(y).$$

We say that $\varphi_{G''}(x)$ is the *function induced in* G'' by φ.

Tight connection between completely regular codes, perfect colorings (also known as equitable partitions [8]) in association schemes and eigenfunctions of these schemes has been exploited intensely starting from Delsarte's work [5] for establishing various properties of codes from this class.

In [7] Godsil used induced eigenfunctions for studying properties of the Johnson association scheme.

In this paper we show that inducing preserves the property of being an eigenfunction in case of an arbitrary distance-2 biregular graph. We introduce the notion of induced perfect coloring and show that in some cases inducing preserves the property of being perfect 2-coloring. This approach applied to a pair of Johnson graphs yields two new infinite series of perfect 2-colorings of the Johnson graphs which arise from Steiner triple and Steiner quadruple systems.

The notion of perfect coloring generalizes the concept of completely regular codes given by Delsarte [5]. This class of codes with good algebraic-combinatorial properties includes well-known codes such as perfect, Preparata and some BCH codes in the Hamming scheme and other important combinatorial objects including Steiner triple systems and Steiner quadruple systems [10] in the Johnson scheme.

For more results on perfect colorings and completely regular codes in the Johnson scheme see [1], [10].

2. Induced eigenfunctions

In this section we show that induced eigenfunction of a halved graph of distance-2 biregular graph is an eigenfunction of the second halved graph.

Theorem 1. *Let G be a (c', c'', t', t'') distance-2 biregular graph with the halved graphs G' and G'', φ be an λ'-eigenfunction of G'. Then*

1. The induced function $\varphi_{G''}$ is the constant zero function iff $\lambda' = -\frac{c'}{t'}$;

2. If λ' is not equal to $-\frac{c'}{t'}$, then $\varphi_{G''}$ is an $\frac{c'-c''+t'\lambda}{t''}$-eigenfunction of G'';

3. It is true that $(\varphi_{G''})_{G'} = (\lambda' t' + c')\varphi$.

Доказательство. First of all, we derive some useful equalities.

Let S be the $|V'|\times|V''|$ incidence matrix that shows relationship between vertices of the parts V' and V'' in G. Let M' and M'' be the adjacency matrices of the graphs G' and G'', respectively, and I' and I'' be the identity matrices of the same size as M' and M'' respectively. The definition of distance-2 biregular graph implies the following two equalities:

$$S^T S = t'M' + c'I', \tag{1}$$

$$SS^T = t''M'' + c''I''. \tag{2}$$

Since the matrix $S^T S$ is a polynomial of M', then the vector $\overline{\varphi}$ is its eigenvector:

$$S^T S\overline{\varphi} = (\lambda' t' + c')\overline{\varphi}. \tag{3}$$

The definition of $\overline{\varphi}_{G''}$ can be rewritten in the following form:

$$\overline{\varphi}_{G''} = S\overline{\varphi}. \tag{4}$$

1. The necessity part of the first statement follows directly from (3) and (4).

Suppose that the equality $\lambda' = -\frac{c'}{t'}$ and the inequality $S\overline{\varphi} \neq 0$ both hold. By (3) the last equality implies

$$\overline{\varphi}^T S^T S\overline{\varphi} = 0$$

whereas the inequality gives

$$(S\overline{\varphi})^T S\overline{\varphi} \neq 0,$$

a contradiction.

2. From (2) and (4) we get:

$$M''\overline{\varphi_{G''}} = \frac{1}{t''}SS^T S\overline{\varphi} - \frac{c''}{t''}S\overline{\varphi}.$$

Substituting the expression for $S^T S$ from (1) in the previous equality, we obtain

$$M''\overline{\varphi_{G''}} = \frac{t'}{t''}SM'\overline{\varphi} + \frac{c' - c''}{t''}S\overline{\varphi}.$$

Now by (4) we get

$$M''\overline{\varphi_{G''}} = (\frac{t'\lambda' + c' - c''}{t''})\overline{\varphi_{G''}},$$

i.e. $\overline{\varphi_{G''}}$ is an eigenvector of G''.

3. The third statement of the theorem follows directly from the equality (3).

□

3. Induced perfect colorings

In the previous section we proved that inducing preserves the property of being eigenfunction. Note that a perfect 2-coloring of a graph can be treated as an eigenfunction of the graph taking just two different values. In this section we show that inducing preserves the property of being perfect 2-coloring if the initial coloring is (α, β)-balanced.

A mapping T from the vertex set of an arbitrary graph G to the set of m colors $\{1, \dots, m\}$ is called a *perfect m-coloring* with a matrix $A = \{a_{ij}\}_{i,j=1,\dots,m}$ (also known as a graph divisor [3]) if for any i and j in $\{1, \dots, m\}$ an arbitrary vertex of color i has exactly a_{ij} neighbors of color j. For convenience' sake in case of two colors the first color is called white and the second color is called black.

Consider a perfect 2-coloring T of a graph with the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Let us adjust the values of the function T in the following way: if x is a white colored vertex, then $\widetilde{T}(x)$ equals $-a_{21}$, otherwise $\widetilde{T}(x)$ equals a_{12}. By the definition of perfect coloring the altered function $\widetilde{T}$ is an $(a_{21} - a_{11})$-eigenfunction of the graph if it is regular [3]. Now suppose we are given an eigenfunction of a graph taking two different values. Define the coloring with vertices of the same color being vertices where the eigenfunction takes the same values. From the definition of eigenfunction it is easy to see that the coloring is a perfect 2-coloring of the graph. Thus we obtain:

Proposition 1. *A perfect 2-coloring of a regular graph with the matrix* $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *is an* $(a_{11}-a_{21})$*-eigenfunction of the graph taking two different values per se. The conversion is also true.*

Let G now be a distance-2 biregular graph. In accordance with the notion of an induced eigenfunction, we introduce the notion of an induced coloring. Given a coloring T of a halved graph G' of G, we define the induced coloring $T_{G''}$ of the halved graph G'': two vertices of G'' are colored in the same color if the color decompositions of their neighborhoods in graph G coincide.

The following natural question arises: "In which cases induced coloring is perfect?". We now introduce one more definition in order to give a complete answer to this question when the initial coloring is perfect 2-coloring.

Given a distance-2 biregular graph G, a perfect 2-coloring of its halved graph G' is called (α,β)*-balanced* in G if any vertex of V'' is adjacent with α or β vertices of white color in graph G. To be definite we assume that the vertices of white color in the induced coloring $T_{G''}$ are those that are adjacent with α vertices from G'.

Theorem 2. *Let G be a (c',c'',t',t'') distance-2 biregular graph with the halved graphs G' and G'', T be a (α,β)-balanced perfect 2-coloring of G' with the matrix* $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. *Then $T_{G''}$ is a perfect 2-coloring of G'' with the matrix*

$$B=\begin{pmatrix} \frac{c''(c'-1)}{t''}-\frac{a_{21}t'c''-\alpha t'(a_{12}+a_{21})}{(\beta-\alpha)t''} & \frac{a_{21}t'c''-\alpha t'(a_{12}+a_{21})}{(\beta-\alpha)t''} \\ \frac{\beta t'(a_{12}+a_{21})-a_{21}t'c''}{(\beta-\alpha)t''} & \frac{c''(c'-1)}{t''}-\frac{\beta t'(a_{12}+a_{21})-a_{21}t'c''}{(\beta-\alpha)t''} \end{pmatrix}.$$

Доказательство. By Theorem 1 and Proposition 1, the eigenfunction $\widetilde{T}_{G''}$ corresponding to $T_{G''}$ is an $\frac{c'-c''+t'(a_{11}-a_{21})}{t''}$-eigenfunction. Note that $\widetilde{T}_{G''}$ takes two different values due to T being balanced. In other words, $T_{G''}$ is a perfect 2-coloring with the matrix $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, such that

$$b_{11}-b_{21}=\frac{c'-c''+t'(a_{11}-a_{21})}{t''}. \tag{5}$$

Note that the definition of (c',c'',t',t'') distance-2 biregular graph implies that G' and G'' are regular graphs of degrees $\frac{c'(c''-1)}{t'}$ and $\frac{c''(c'-1)}{t''}$ respectively. Therefore, the following equalities hold:

$$b_{11}+b_{12}=\frac{c''(c'-1)}{t''},\ \ a_{11}+a_{12}=\frac{c'(c''-1)}{t'}. \tag{6}$$

Combining (5) and (6), we get

$$b_{12}+b_{21}=\frac{t'}{t''}(a_{12}+a_{21}). \tag{7}$$

We now count the number N of edges (x, y) in G, where x is a white vertex of G' and y is a vertex of G'' of any color. On one hand, for a fixed vertex x there exists exactly c' possibilities for y. Therefore, N is equal to the number of white vertices in G' multiplied by the degree of part V' in G:

$$N = (\frac{a_{21}}{a_{12} + a_{21}}|V'|)c'.$$

On the other hand, for a fixed vertex y there exists α or β possibilities for x depending on the color of the vertex y. Hence, we get:

$$N = \frac{\alpha b_{21}|V''|}{b_{12} + b_{21}} + \frac{\beta b_{12}|V''|}{b_{12} + b_{21}}.$$

Therefore, we get:

$$(\frac{a_{21}}{a_{12} + a_{21}}|V'|)c' = \frac{\alpha b_{21}|V''|}{b_{12} + b_{21}} + \frac{\beta b_{12}|V''|}{b_{12} + b_{21}}.$$

Since G is a bipartite biregular graph, we have the following equality:

$$|V'|c' = |V''|c''.$$

From (7) and two previous equalities we obtain

$$\alpha + (\beta - \alpha)\frac{b_{12}t''}{t'(a_{12} + a_{21})} = \frac{a_{21}}{a_{21} + a_{12}}c'',$$

which gives the required value for b_{12}. The values of the remaining elements b_{11}, b_{22}, b_{21} of the matrix B are obtained using equalities (6) and (7).

□

4. Induced perfect 2-colorings of the Johnson graphs

In this section we illustrate an application of the described approach for perfect 2-colorings of the Johnson graph.

We need few more definitions. Given a collection C of vertices of a graph G, define a pair of vertices to be colored in the same color if they are at the same distance from C. The obtained coloring is called the *distance coloring* with respect to C. The *Johnson graph* $J(n, w)$ has vertices given by the w-subsets of $\{1, \dots, n\}$, with two vertices connected iff their intersection is of size $w - 1$. A collection of w-subsets of $\{1, \dots, n\}$ is $t-(n, w, \lambda)$*-design* if every w-element subset of $\{1, \dots, n\}$ occurs exactly in λ blocks.

4.1. **(w-1)-(n,w,1)-designs.** Consider a bipartite graph G with the parts $V(J(n, w))$ and $V(J(n, w + 1))$ and adjacency showing the inclusion of a w-subset into a $w + 1$-subset. The graph G is known to be distance-biregular [2].

Suppose a $(w-1)-(n, w, 1)$-design D is given. The block set of this design admits the obvious embedding into the vertex set of $J(n, w)$. Martin [10] noted that the distance coloring of $J(n, w)$ with respect to D is a perfect coloring of $J(n, w)$ with the matrix

$$\begin{pmatrix} 0 & w(n-w) \\ w & w(n-w-1) \end{pmatrix}.$$

Note that this coloring is (0,1)-balanced in G, because no pair of vertices of D is adjacent in $J(n, w)$. Applying Theorem 2 to the coloring, we obtain

Proposition 2. *Let D be a $(w-1)-(n,w,1)$-design. Then there exists a perfect 2-coloring of $J(n,w+1)$ with the matrix*

$$\begin{pmatrix} (w+1)(n-2w-1) & (w+1)w \\ w(n-2w) & w^2-2w-1+n \end{pmatrix}.$$

Taking into account existence of Steiner triple and quadruple systems [6], [9] we get:

Proposition 3. *For every n, $n \equiv 1,3 (mod\ 6)$ there exists a perfect 2-coloring of $J(n,4)$ with the matrix*

$$\begin{pmatrix} 4n-28 & 12 \\ 3(n-6) & n+2 \end{pmatrix}.$$

Proposition 4. *For every n, $n \equiv 2,4 (mod\ 6)$ there exists a perfect 2-coloring of $J(n,5)$ with the matrix*

$$\begin{pmatrix} 5(n-5) & 20 \\ 4(n-8) & n+7 \end{pmatrix}.$$

4.2. **Perfect codes in the Johnson graphs.** The long-standing hypothesis proposed by Delsarte in [5] states that no perfect codes in the Johnson graphs exist. We show that the existence of 1-perfect code in the Johnson graph $J(n,w)$ would imply the existence of particular perfect 2-coloring of $J(n,w+1)$.

Suppose that 1-perfect code in $J(n,w)$ exists. It is not hard to see that the distance coloring T of $J(n,w)$ with respect to a 1-perfect code is a perfect coloring with the matrix $\begin{pmatrix} 0 & w(n-w) \\ 1 & w(n-w)-1 \end{pmatrix}$. Note that the coloring T is $(0,1)$-balanced in G, so, by Theorem 2, induced coloring $T_{J(n,w+1)}$ is perfect. Thus, we obtain

Proposition 5. *If there exists a 1-perfect code in $J(n,w)$ then there exists a perfect 2-coloring of $J(n,w+1)$ with the matrix:*

$$\begin{pmatrix} (w+1)(n-w-2) & w+1 \\ w(n-w-1) & n-w-1 \end{pmatrix}.$$

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