S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 8, стр. 333–340 (2011) УДК 512.54

MSC 20D06

FIXED POINTS OF LARGE PRIME-ORDER ELEMENTS IN THE EQUICHARACTERISTIC ACTION OF LINEAR AND UNITARY GROUPS

ANDREI V. ZAVARNITSINE

Abstract. We find sufficient arithmetic conditions for element of large prime order of a finite simple linear or unitary group to have a nontrivial fixed point in equicharacteristic modules. Some applications are made to the study of the prime graph of simple groups.

Keywords: finite simple groups, fixed points, equicharacteristic action, prime graph.

1. Introduction

The existence of nontrivial fixed points of elements of simple groups in their representations has been a popular subject of investigation. In this paper, we are interested in the action of semisimple elements of "large" prime order of the simple groups $\mathrm{L}_n^\varepsilon(q)$, where $\varepsilon \in \{+,-\}$, and $q = p^m$ for a prime p, in their *equicharacteristic* modules (i. e. modules over a field of the defining characteristic p). Here we adopt the common notation $\mathrm{L}_n^+(q)$ for $\mathrm{PSL}_n(q)$ and $\mathrm{L}_n^-(q)$ for $\mathrm{PSU}_n(q)$. An element order of a group is "large" (or r-maximal) if, when multiplied by a prime r, it is no longer an element order. The question of existence of fixed points of large-order elements in r-modular representations arises naturally in some recognition problems. In the

Zavarnitsine, A.V., Fixed points of large prime-order elements in the equicharacteristic action of linear and unitary groups.

Supported by the Russian Foundation for Basic Research (projects 10–01–90007, 11–01–00456); by the Council of the President grants (project NSc–3669.2010.1); by the Program "Development of the Scientific Potential of Higher School" (project 2.1.1.10726); by the Grant Council of the Russian Federation (MK-2136.2010.1); by the Russian Federal Program "Scientific and pedagogic people of the innovative Russia" (contract 14.740.11.0346).

Received July, 25, 2011, published November, 3, 2011.

last section, we give an application of the main results to the recognition problem related to the prime graph of a group.

2. Preliminaries

In order to give precise statements of the results, we first introduce some notation and definitions.

Given $m, n \in \mathbb{Z}$, we set $(m, n) = \gcd(m, n)$. For a natural n and a prime r, we denote by n_r the r-part of n (i. e. the maximal power of r dividing n). We also write $n_{r'}$ for n/n_r.

Given a finite group G and an element $g \in G$, we denote by $|g|$ the order of g. The *spectrum* $\omega(G)$ of a finite group G is the set of its element orders. In other words,

$$\omega(G) = \{ n \in \mathbb{N} \mid \exists\, g \in G : |g| = n \}.$$

Let r be a prime. An element $g \in G$ is said to have *r-maximal order*, if $r|g| \notin \omega(G)$

In what follows, we will use the symbol ε to denote either ± 1 or the sign '+' or '−'.

Let $t > 1$ and n be natural numbers. If there exists a prime that divides $t^n - \varepsilon^n$ and does not divide $t^i - \varepsilon^i$ for $1 \leqslant i < n$, then we denote this prime by $t_{[\varepsilon n]}$ and call it a *primitive prime divisor* of $t^n - \varepsilon^n$. In general, a primitive divisor need neither exist nor be unique.

Observe that the value of $t_{[\varepsilon n]}$ depends on t, ε, n. It is not correct to ask what the primitive prime divisor of a given number is without specifying a representation of this number in the form $t^n - \varepsilon^n$. For example, one can write $1023 = 2^{10} - 1 = 2^{10} - (-1)^{10} = 32^2 - 1$ and obtain $2_{[+10]} = 11$, $2_{[-10]} = 31$, $32_{[+2]} \in \{3, 11\}$.

The following lemma is a generalization of the well-known Zsigmondy's theorem, see [7, Lemma 5]:

Lemma 1. *Let $t, n > 1$ be natural numbers and $\varepsilon = \pm 1$. There exists a primitive divisor $t_{[\varepsilon n]}$ of $t^n - \varepsilon^n$, except in the following cases:*

(*i*) $\varepsilon = 1$, $n = 6$, $t = 2$;
(*ii*) $\varepsilon = 1$, $n = 2$, *and* $t = 2^l - 1$ *for some* $l \geqslant 2$;
(*iii*) $\varepsilon = -1$, $n = 3$, $t = 2$;
(*iv*) $\varepsilon = -1$, $n = 2$, *and* $t = 2^l + 1$ *for some* $l \geqslant 0$.

Henceforth, we use the symbol $q_{[\varepsilon n]}$ implicitly assuming that the parameters ε, n, q are such that a primitive prime divisor of $q^n - \varepsilon^n$ exists.

The following description of semisimple elements of $\mathrm{L}_n^\varepsilon(q)$ of p-maximal prime order is a consequence of [9, Proposition 3.1]:

Lemma 2. *Let $G = \mathrm{L}_n^\varepsilon(q)$ be a simple group, where $q = p^m$ for a prime p. Let $r \in \pi(G)$, $r \neq p$. Then r is a p-maximal element order in $\omega(G)$ if and only if one of the following holds:*

(1) *r has the form $q_{[\varepsilon n]}$ or $q_{[\varepsilon(n-1)]}$;*
(2) *$r = 3$, $n = 3$, and $(q - \varepsilon)_3 = 3$.*

In connection with Lemma 2, we note that there is an exceptional situation when $n = 2$ and $q + \varepsilon \equiv 0 \pmod 4$. In this case, the prime $r = 2$ has the form $q_{[\varepsilon(n-1)]}$, however the elements of $\mathrm{L}_n^\varepsilon(q)$ of p-maximal order r (like those of order $q_{[\varepsilon n]}$) arise from the maximal torus $\mathbb{Z}_{q+\varepsilon}$ of $\mathrm{SL}_n^\varepsilon(q)$.

Lemma 3. *If a Frobenius group FC with kernel F and cyclic complement C of order n acts faithfully on a vector space V over a field of nonzero characteristic p coprime with $|F|$ then $pn \in V \leftthreetimes C$*

Proof. See [5, Lemma 1]. □

We now recall some facts from the representation theory of algebraic groups. For details, see e. g. [3].

Let $\mathbb{G} = \mathrm{SL}_n(F)$, where F is an algebraically closed field of characteristic p. Then $\mathbb{G}$ is a simply connected simple algebraic group of type A_l, where $l = n - 1$. Denote by ω_0 the zero weight and by $\omega_1, \ldots, \omega_l$ the fundamental weights of $\mathbb{G}$ (with respect to a fixed maximal torus of $\mathbb{G}$ and a system of positive roots). Let $\Omega = \mathbb{Z}\omega_1 \oplus \ldots \oplus \mathbb{Z}\omega_l$ be the *weight lattice* and let the root system of $\mathbb{G}$ have simple roots $\alpha_1, \ldots, \alpha_l$. Also, let $\Omega_0 = \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_l$ be the set of *radical weights.*

Define the map $\delta : \Omega \to \mathbb{Z}_{l+1}$ by the rule

$$\delta : a_1\omega_1 + \ldots + a_l\omega_l \quad \mapsto \quad a_1 + 2a_2 + \ldots + la_l \pmod{l+1}.$$

Lemma 4. *The map $\delta : \Omega \to \mathbb{Z}_{l+1}$ is a group epimorphism with* $\ker\delta = \Omega_0$.

Proof. Clearly, δ is additive and $\delta(\omega_1) = 1$. Hence, δ is a group epimorphism. Take an arbitrary weight $\lambda = a_1\omega_1 + \ldots + a_l\omega_l$. If $\lambda = \alpha_i$ is a simple root then $(a_1, \ldots, a_l)$ is the ith row of the Cartan matrix of type A_l and it is readily verified that $\delta(\lambda) = 0$. Since Ω_0 is spanned by the roots, we have $\Omega_0 \subseteq \ker\delta$. It is known that $\Omega/\Omega_0 \cong \mathbb{Z}_{l+1}$, see [1, sec. 7(VIII), §4, chap. VI]. Hence, surjectivity of δ implies that $\ker\delta = \Omega_0$. □

A weight $a_1\omega_1 + \ldots + a_l\omega_l \in \Omega$ is called *k-restricted* $0 \leqslant a_i < k$, $i = 1, \ldots, l$, where k is usually a power of p.

For an irreducible (rational, finite dimensional) $\mathbb{G}$-module L, denote by $\Omega(L)$ the set of weights of L The irreducible $\mathbb{G}$-module of highest weight λ is customarily denoted by $L(\lambda)$. Obviously, $\Omega(L(\omega_0)) = \{\omega_0\}$. The structure of the modules $L(\omega_i)$, $i = 1, \ldots, l$ is well known and described in the following lemma (see, e.g. [3, II.2.15]):

Lemma 5. *Let $\mathbb{G} = \mathrm{SL}_n(F)$ and let $V = F^n$ be the natural $\mathbb{G}$-module with the canonical basis $e_1, \ldots, e_n$. Choose the diagonal subgroup $\mathbb{H}$ for a fixed maximal torus of $\mathbb{G}$. Then e_i is an eigenvector for $\mathbb{H}$ with the corresponding weight ε_i. Choose a system of positive roots $\{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}$. Then, for $1 \leqslant k < n$, we have $\omega_k = \varepsilon_1 + \ldots + \varepsilon_k$ and the module $L(\omega_k)$ is isomorphic to the k-th exterior power $\wedge^k V$ and has the set of weights*

$$\Omega(L(\omega_k)) = \{\varepsilon_{i_1} + \varepsilon_{i_2} + \ldots + \varepsilon_{i_k} \mid 1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n\}.$$

The following assertion is a refinement of the Suprunenko–Zalesskii lemma for groups of type A_l, see [12, Lemma 13].

Lemma 6. *Let $\mathbb{G} = \mathrm{SL}_n(F)$ and let L be an irreducible $\mathbb{G}$-module whose highest weight λ is p-restricted. Then $\Omega(L(\omega_i)) \subseteq \Omega(L)$, where $i = \delta(\lambda)$.*

Let $S = \mathrm{SL}_n^\varepsilon(q)$, $G = \mathrm{L}_n^\varepsilon(q)$, $q = p^m$. There is a Frobenius endomorphism σ of $\mathbb{G}$ such that $\mathrm{C}_\mathbb{G}(\sigma) = S$. Every absolutely irreducible FG-module W can be lifted to S and, by the Steinberg theorem [8, Theorem 43], is the restriction to S of a $\mathbb{G}$-module $L(\lambda)$, where λ is q-restricted. Abusing the language we will sometimes identify W with $L(\lambda)$, e. g. when we say that λ is the highest weight of W, etc.

The question of which irreducible $\mathbb{G}$-modules $L(\lambda)$ upon restriction to S give rise to FG-modules is answered by the following assertion:

Lemma 7. *Let $S = \mathrm{SL}_n^\varepsilon(q)$, $q = p^m$, $d = (n, q-\varepsilon)$. Then the F-representation of S corresponding to an irreducible $\mathbb{G}$-module $L(\lambda)$ contains $\mathrm{Z}(S)$ in its kernel if and only if $\delta(\lambda) \equiv 0 \pmod d$.*

Proof. We assume that S is contained in $\mathbb{G}$ as explained above. Set $W = L(\lambda)$ and $l = n-1$. The center $\mathrm{Z}(\mathbb{G})$ is a cyclic group of order $n_{p'}$ which is shown in [4, sec. 6.2] to contain a generator z such that $(\omega_1(z), \dots, \omega_l(z)) = (\zeta, \zeta^2, \dots, \zeta^l)$, where $\zeta \in F^\times$ is of order $n_{p'}$. Thus, for any weight $\mu = a_1\omega_1 + \ldots + a_l\omega_l$, we have

$$\mu(z) = z^{a_1+2a_2+\ldots+la_l} = z^{\delta(\mu)}.$$

By [3, Proposition II.2.4], $\Omega(W)$ lies in a single coset of $\Omega : \Omega_0$. Hence, δ is constant on $\Omega(W)$ and $\mu(z) = \lambda(z)$ for all $\mu \in \Omega(W)$.

Since the center $\mathrm{Z}(S)$ is generated by the element $z_0 = z^{n_{p'}/d}$ of order d, the inclusion of $\mathrm{Z}(S)$ into the kernel holds if and only if $\mu(z_0) = 1$ for all $\mu \in \Omega(W)$. By the above, this is equivalent to $\lambda(z_0) = 1$. However, $\lambda(z_0) = (z_0)^{\delta(\lambda)}$ and the claim follows. □

3. Main results

We can now state and prove the principal result of this paper.

Theorem 8. *Let $G = \mathrm{L}_n^\varepsilon(q)$, $q = p^m$, be a simple group, let W be an absolutely irreducible module for G in characteristic p. Suppose that the highest weight λ of W is p-restricted. Denote $k = \delta(\lambda)$ and $H = W \leftthreetimes G$.*

(1) *If $(n,k) > 1$ then $p\,q_{[\varepsilon n]} \in \omega(H)$.*

(2) *If $(n-1,k) > 1$ or $(n-1,k-1) > 1$ then $p\,q_{[\varepsilon(n-1)]} \in \omega(H)$.*

Proof. In case (ii) we may assume that $n \geqslant 3$. Let $S = \mathrm{SL}_n^\varepsilon(q)$ and set n_0 equal to n or $n-1$ according as case (i) or (ii) holds. Denote $r = q_{[\varepsilon n_0]}$ and let $g \in S$ be an element of order r whose characteristic values constitute the set

$$\Theta = \{\theta, \theta^{\varepsilon q}, \theta^{(\varepsilon q)^2}, \dots, \theta^{(\varepsilon q)^{n_0-1}}\}$$

in case (i) and the set $\{1\} \cup \Theta$ in case (ii), where $\theta \in F^\times$ and $|\theta| = r$. Since $\langle g \rangle \cap \mathrm{Z}(S) = 1$, it suffices to show that g fixes in W a nontrivial vector.

We will first show that there are k distinct characteristic values of g whose product equals 1. Set $b = (n_0, k_0)$, where k_0 is k or $k-1$ according as $(n_0,k) > 1$ or $(n_0, k-1) > 1$. Since $b > 1$, we have

$$r \;\Big|\; \frac{(\varepsilon q)^{n_0} - 1}{(\varepsilon q)^{n_0/b} - 1} = 1 + t + t^2 + \ldots + t^{b-1}, \quad t = (\varepsilon q)^{n_0/b}. \tag{1}$$

The set Θ is the union of $f = n_0/b$ pairwise disjoint subsets of size b

$$\{\theta, \theta^t, \dots, \theta^{t^{b-1}}\},\ \{\theta^{\varepsilon q}, \theta^{t(\varepsilon q)}, \dots, \theta^{t^{b-1}(\varepsilon q)}\},$$
$$\dots,\ \{\theta^{(\varepsilon q)^{f-1}}, \theta^{t(\varepsilon q)^{f-1}}, \dots, \theta^{t^{b-1}(\varepsilon q)^{f-1}}\}$$

in each of which the product of all elements equals 1 due to (1). Since

$$k_0 \leqslant k = \delta(\lambda) \leqslant n-1 \leqslant n_0,$$

we have $k_0/b \leqslant f$, and the union of arbitrary k_0/b of the above subsets contains k_0 distinct characteristic values of g with product 1. If $k_0 = k$, we have the desired set. If $k_0 = k-1$, then we are in case (ii) and thus can adjoin to this set the kth characteristic value 1.

Now assume that $S = \mathrm{C}_{\mathbb{G}}(\sigma)$, where σ is a suitable Frobenius endomorphism of $\mathbb{G}$. Choose the diagonal subgroup $\mathbb{H}$ for a fixed maximal torus of $\mathbb{G}$. The canonical basis elements $e_1, \dots, e_n$ correspond to the “coordinate” weights $\varepsilon_1, \dots, \varepsilon_n$ so that $\varepsilon_i(h)$ is the ith characteristic value of h for every $h \in \mathbb{H}$. There exists an element $a \in \mathbb{G}$ such that $h = {}^a g \in \mathbb{H}$. We set the weight μ equal to the sum of those k coordinate weights ε_i corresponding to the above-chosen k characteristic values of g whose product is 1. Then we have $\mu(h) = 1$ and also, due to Lemmas 6 and 5, we have $\mu \in \Omega(L(\omega_k)) \subseteq \Omega(W)$. Thus, there is a weight vector $w_0 \in W$ of weight μ such that $w_0 h = \mu(h) w_0 = w_0$. Denote $w = w_0 a$. Then

$$wg = w_0 a g = w_0 h a = w_0 a = w.$$

Hence, g has a nontrivial fixed point in W which implies that $pr \in \omega(H)$ as claimed. □

Although Theorem 8 shows that in general the existence of nontrivial fixed points of p-maximal elements depends on the module W, in some cases this dependence disappears.

Corollary 9. *Let $G = \mathrm{L}_n^{\varepsilon}(q)$, $q = p^m$, be a simple group which acts absolutely irreducibly on a vector space W over a field of characteristic p. Denote $H = W \rtimes G$.*

(1) *If $q = p$ and $(n, q-\varepsilon) > 1$ then $p\, q_{[\varepsilon n]} \in \omega(H)$.*
(2) *If n is odd then $p\, q_{[\varepsilon(n-1)]} \in \omega(H)$.*
(3) *If $n = 3$ and $(q-\varepsilon)_3 = 3$ then $3p \in \omega(H)$.*
(4) *If $n = 2$ and q is odd then $2p \in \omega(H)$.*

Proof. We consider W as an FG-module and denote by λ the highest weight of W.

(i) Let $k = \delta(\lambda)$. Since $q = p$, it follows that λ is p-restricted. Denote $d = (n, q-\varepsilon)$. By Lemma 7, we have $k \equiv 0 \pmod d$. Hence, d divides (n, k), which yields $(n, k) > 1$. By Theorem 8(i), we have $p\, q_{[\varepsilon n]} \in \omega(H)$.

(ii) As above, we assume that W lifts to the restriction to $S = \mathrm{SL}_n^{\varepsilon}(q)$ of the irreducible module $L(\lambda)$ for the ambient algebraic group $\mathbb{G} = \mathrm{SL}_n(F)$. We may write

$$\lambda = \lambda_0 + p\lambda_1 + \ldots + p^{m-1}\lambda_{m-1},$$

where λ_i are p-restricted. By the Steinberg Tensor Product Theorem [8, Theorem 41], we have

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{\rho} \otimes \ldots \otimes L(\lambda_{m-1})^{\rho^{m-1}},$$

where ρ denotes the twisting by the Frobenius map corresponding to the automorphism $x \mapsto x^p$ of F. Let $k_i = \delta(\lambda_i)$, $i = 0, \dots, m-1$. Since either k_i or $k_i - 1$ is even, we have $(n-1, k_i) > 1$ or $(n-1, k_i - 1) > 1$. From the proof of Theorem 8(ii), it follows that there is an element $g \in S$ of order $q_{[\varepsilon(n-1)]}$ that fixes a nontrivial vector $w_i \in L(\lambda_i)$ for each $i = 0, \dots, m-1$. Define

$$w = w_0 \otimes w_1^{\rho} \otimes \ldots \otimes w_{m-1}^{\rho^{m-1}}.$$

Clearly, $w \in L(\lambda)$ is fixed by g and the claim follows.

(iii)–(iv) For $n = 2$ and $n = 3$, G includes a Frobenius subgroup of the shape $(q+\eta)/2 : 2$, where $q \equiv \eta \pmod 4$, and $(q^2 + \varepsilon q + 1)/3 : 3$, respectively. Hence, the claim follows by Lemma 3. □

We included cases (*iii*)–(*iv*) of Corollary 9 for the sake of completeness. They cover the exceptional cases of p-maximal elements; namely, case (*ii*) of Lemma 2 and the situation mentioned after Lemma 2.

4. Application

The information about fixed points of prime-order elements can be useful in the study of recognition problems related to the prime graph.

Recall that the *prime graph* $\Gamma(G)$ of a finite group G is the graph whose vertex set is the set $\pi(G)$ of prime divisors of the order $|G|$ in which two distinct vertices $p, q \in \pi(G)$ are joined by an edge if and only if G contains an element of order pq.

Of particular interest are the finite groups that are uniquely determined by their prime graph. The group G is said to be *recognizable by graph* if, for every finite group H, the equality of vertex-labeled graphs $\Gamma(H) = \Gamma(G)$ implies the isomorphism $H \cong G$. Examples of recognizable by graph groups are the simple groups J_1, M_{22}, M_{23}, M_{24}, Co_2, $\mathrm{G}_2(7)$, J_4, ${}^2\mathrm{G}_2(q)$, $q > 3$, see [2, 10].

The recognition problem for G includes checking whether $\Gamma(G) = \Gamma(H)$ for some proper *covering group* H of G (where H is said to *cover* G if G is a homomorphic image of H). This situation can be reduced to the study of irreducible modular representations of G as shows the next result.

Lemma 10. *If H is a proper covering of G such that $\Gamma(G) = \Gamma(H)$ then there is a vector space W over a field of positive characteristic on which G acts absolutely irreducibly and $\Gamma(G) = \Gamma(W \rtimes G)$.*

Proof. We proceed along the lines of the proof of [12, Lemma 9]. Let $M = N.G$ be a proper covering of minimal order such that $\Gamma(G) = \Gamma(M)$. We have $|M| \leqslant |H|$ and $N \neq 1$. Let $r \in \pi(N)$ and let S be a Sylow r-subgroup of N. Then $M = N_M(S)N$ and thus $N_M(S)/(N_M(S) \cap N) \cong G$. By the minimality of M, we have $N_M(S) = M$. It follows that N is nilpotent. Set $K = O_{r'}(N)\Phi(N)$. Then N/K is an elementary abelian r-group. Since $(M/K)/(N/K) \cong G$, we have $K = 1$ by the minimality of M. Thus, N is elementary abelian. The natural action of G on N gives rise to a split extension $N \rtimes G$. By [6, Lemma 10], we have $\omega(N \rtimes G) \subseteq \omega(M)$. Hence, if two primes are connected in $\Gamma(N \rtimes G)$, they are connected in $\Gamma(M) = \Gamma(G)$ as well. Thus, we may assume that $M = N \rtimes G$ is a split extension. Consider N as a vector space over a field $\mathbb{F}_r$ of r elements. Suppose that the action of G on N is not absolutely irreducible. Let E be a finite splitting field for G of characteristic r and consider an irreducible submodule W of the reducible EG-module $N \otimes_{\mathbb{F}_r} E$. It remains to show that $\Gamma(W \rtimes G) = \Gamma(G)$. If $g \in G$ is an element of prime order other than r which fixes a nonzero vector of W then g has an eigenvalue 1 considered as a linear transformation of W. But then it also has an eigenvalue 1 considered as a linear transformation of N. This implies that every prime connected with r in $\Gamma(W \rtimes G)$ is also connected with r in $\Gamma(N \rtimes G) = \Gamma(G)$. The proof is complete. □

The case where G is a Lie-type group acting on an equicharacteristic module is usually treated separately. We consider this case for $G = \mathrm{L}_n^\varepsilon(p)$. With the following assertion, we generalize and give a more conceptual proof of the main result of [13]:

Theorem 11. *Let $G = \mathrm{L}_n^\varepsilon(p)$ be a simple group which acts absolutely irreducibly on a vector space W over a field of characteristic p. Then $\Gamma(W \rtimes G) \neq \Gamma(G)$ in each of the following cases:*

(1) *either n or p is odd* (*except when* $G = \mathrm{L}_7(2)$, $\mathrm{L}_3(3)$, $\mathrm{U}_3(3)$);
(2) $p = 2$ *and* $n = 4, 6, 10, 16$ (*except when* $G = \mathrm{L}_6(2)$, $\mathrm{U}_4(2)$).

Proof. In order to show that $\Gamma(W \leftthreetimes G) \neq \Gamma(G)$, it is sufficient to find a p-maximal prime element order $r \in \omega(G)$ such that $pr \in \omega(W \leftthreetimes G)$.

Let n be odd. By Lemma 1, we may take $r = p_{[\varepsilon(n-1)]}$, unless $G = \mathrm{L}_7(2)$ or $\mathrm{L}_3^\varepsilon(2^l - \varepsilon)$, and then apply Corollary 9(*ii*). If $G = \mathrm{L}_3^\varepsilon(p)$, with $p = 2^l - \varepsilon$ prime, then $(3, p - \varepsilon) = 3$, unless $G = \mathrm{L}_3(3)$, $\mathrm{U}_3(3)$, and we may take $r = p_{[\varepsilon n]}$ and apply Corollary 9(*i*).

Let p be odd and n even. Then by Lemma 1 we may take $r = p_{[\varepsilon n]}$, unless $G = \mathrm{L}_2^\varepsilon(2^l - \varepsilon)$, and apply Corollary 9(*i*). If $G = \mathrm{L}_2^\varepsilon(p)$, we take $r = 2$ and use Corollary 9(*iv*).

Suppose that $p = 2$ and $n = 4, 6, 10, 16$. Let λ be the highest weight of W considered as a module for G. Since $G = \mathrm{L}_n^\varepsilon(q)$ with $q = p$ prime, it follows that λ is p-restricted. Let $k = \delta(\lambda)$. Then $0 \leqslant k \leqslant n-1$. Since both $p_{[\varepsilon n]}$ and $p_{[\varepsilon(n-1)]}$ exist (unless $G = \mathrm{L}_6(2)$, $\mathrm{U}_4(2)$) and either $(n, k) > 1$, or $(n-1, k) > 1$, or $(n-1, k-1) > 1$, the claim follows by Theorem 8. □

We remark that if $G = \mathrm{L}_n^\varepsilon(p)$, $(n, p - \varepsilon) = 1$, and V is the natural n-dimensional module for G then

- $\Gamma(V \leftthreetimes G) = \Gamma(G)$ if $G = \mathrm{L}_7(2)$, $\mathrm{L}_3(3)$, $\mathrm{U}_3(3)$, $\mathrm{U}_4(2)$;
- $\Gamma((\wedge^2 V) \leftthreetimes G) = \Gamma(G)$ if $G = \mathrm{L}_6(2)$;
- $\Gamma((\wedge^3 V) \leftthreetimes G) = \Gamma(G)$ if $G = \mathrm{L}_8(2)$, $\mathrm{U}_8(2)$;

where $\wedge^i V$ is the ith exterior power of V.

If $G = \mathrm{L}_n^\varepsilon(q)$, where q is nonprime, then there exists an example of an equicharacteristic module V for $n = 4$ and $\varepsilon = +$ such that $\omega(V \leftthreetimes G) = \omega(G)$; in particular, $\Gamma(V \leftthreetimes G) = \Gamma(G)$, see [11]. The situation in the general case of nonprime q (where one may assume n to be even by Corollary 9(*ii*)) remains largely unexplored, but can be reduced to certain number-theoretic problems using the ideas presented above.

References

[1] N. Bourbaki, *Éléments de mathématique. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées*, Actualités Scientifiques et Industrielles, N 1364, Hermann, Paris, 1975.

[2] M. Hagie, *The prime graph of a sporadic simple group*, Commun. Algebra, **31**:9 (2003), 4405–4424.

[3] J.C. Jantzen, *Representations of algebraic groups. Second edition*, Mathematical Surveys and Monographs, 107, American Mathematical Society, Providence, RI, 2003.

[4] F. Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math., **4** (2001), 135–169.

[5] V.D. Mazurov, *The set of orders of elements in a finite group*, Algebra and Logic, **33**:1 (1994), 81–89.

[6] V.D. Mazurov, A.V. Zavarnitsine, *Element orders in coverings of symmetric and alternating groups*, Algebra and Logic, **38**:3 (1999), 159–170.

[7] V.D. Mazurov, A.V. Zavarnitsine, *On element orders in coverings of the simple groups* $\mathrm{L}_n(q)$ *and* $\mathrm{U}_n(q)$, Proceedings of the Steklov Institute of Mathematics, **1** (2007), 145–154.

[8] R. Steinberg, *Lectures on Chevalley Groups*, Yale University, New Haven, 1968.

[9] A.V. Vasiliev, E.P. Vdovin, *An Adjacency Criterion for the Prime Graph of a Finite Simple Group*, Algebra and Logic, **44**:6 (2005), 381–406.

[10] A.V. Zavarnitsine, *On recognition of finite groups by the prime graph*, Algebra and Logic, **45**:4 (2006), 220–231.
[11] A.V. Zavarnitsine, *Exceptional action of the simple groups* $\mathrm{L}_4(q)$ *in the defining characteristic*, Siberian Electronic Math. Reports, **5** (2008), 68–74.
[12] A.V. Zavarnitsine, *Properties of element orders in covers for* $\mathrm{L}_n(q)$ *and* $\mathrm{U}_n(q)$, Sib. Math. J., **49**:2 (2008), 246–256.
[13] A.V. Zavarnitsine, *Uniqueness of the prime graph of* $\mathrm{L}_{16}(2)$, Siberian Electronic Math. Reports, **7** (2010), 119–122.

Andrei V. Zavarnitsine
Group Theory Lab.
Sobolev Institute of Mathematics
4, Koptyug av.
630090, Novosibirsk, Russia
E-mail address: zav@math.nsc.ru