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GLIVENKO THEOREM FOR N^*-EXTENSIONS

S.P.ODINTSOV

Abstract. Logic N^* was defined semantically via combination of Kripke frames for intuitionistic logic with Routley's $*$-operator, which is used to interpret the negation operation. In this notice, we find out the least logic in the class of N^*-extensions, which satisfy Glivenko's theorem, and describe the Kripke semantics of this logic.

Keywords: Routley semantics, Glivenko theorem

The logic N^* was introduced in [1] as a logical framework for investigation of the well founded semantics of logic programs with negation. Main features of the semantical Kripke style definition of this logic are as follows: positive connectives are defined as in intuitionistic logic whereas the negation is interpreted via Routley's $*$-operator [3]. The algebraic semantics of N^* in terms of so called Heyting-Ockham algebras was defined in [2]. The aim of this short notice is to describe the least N^*-extension L, which satisfies Glivenko theorem, i.e., such that for every formula φ, we have $\neg\neg\varphi \in L$ if and only if φ belongs to classical propositional logic Cl.

First we recall the definition of the logic N^*. Fix propositional language $\mathcal{L} = \langle \vee, \wedge, \to, \neg \rangle$. By a logic in this language we mean a set of formulas closed under the rules of substitution, *modus ponens* $\frac{\varphi, \varphi\to\psi}{\psi}$, and contraposition $\frac{\varphi\to\psi}{\neg\psi\to\neg\varphi}$.

The logic N^* is the least logic in the language $\mathcal{L}$ containing the axioms of positive logic:

$P1)$ $p \to (q \to p)$; $P2)$ $(p \wedge q) \to p$;
$P3)$ $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$; $P4)$ $(p \wedge q) \to q$;
$P5)$ $(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$; $P6)$ $p \to (p \vee q)$;
$P7)$ $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$; $P8)$ $q \to (p \vee q)$;

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and the following axioms for negation:

$$N1)\ (\neg p \wedge \neg q) \to \neg(p \vee q); \qquad N2)\ \neg(p \to p) \to q;$$
$$N3)\ \neg(p \wedge q) \to (\neg p \vee \neg q); \qquad N4)\ \neg\neg(p \to p).$$

In [2], we have used the axiom $\neg((p \to p) \to \neg(q \to q))$ instead of $N4$, but to check that these axioms are equivalent we have to apply the contraposition rule to the following positive tautologies:

$$((p \to p) \to \neg(q \to q)) \to \neg(q \to q) \text{ and } \neg(p \to p) \to ((q \to q) \to \neg(p \to p)).$$

We use $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \to \psi) \wedge (\psi \to \varphi)$ and as was noted in [2] one can define in N^* the intuitionistic negation $-\varphi$ as $\varphi \to \neg(p_0 \to p_0)$. Recall also that N^* is closed under the replacement of equivalents.

Definition 0.1. *A* Routley frame *is a triple $\mathcal{W} = \langle W, \leq, * \rangle$ such that: (i) W is a non empty set (of worlds); (ii) $\leq$ is a partial ordering on W; (iii) $* : W \to W$ is a function such that $y^* \leq x^*$ whenever $x \leq y$.*

An Routley model *$\mathcal{M} = \langle W, \leq, R, v \rangle$ is a Routley frame $\mathcal{W} = \langle W, \leq, R \rangle$ augmented with a valuation function $v : Prop \longrightarrow 2^W$ satisfying the persistency condition:*

$$u \in v(p) \ \& \ u \leq w \ \Rightarrow \ w \in v(p). \tag{1}$$

We say in this case that $\mathcal{M}$ is a model over *$\mathcal{W}$.*

The validity of formulas at worlds of $\mathcal{M}$ is defined by induction as follows:

- $\mathcal{M}, w \models p \ \Leftrightarrow \ w \in v(p)$;
- $\mathcal{M}, w \models \varphi \wedge \psi \ \Leftrightarrow \ \mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$;
- $\mathcal{M}, w \models \varphi \vee \psi \ \Leftrightarrow \ \mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$;
- $\mathcal{M}, w \models \varphi \to \psi \ \Leftrightarrow \ \forall w'(w \leq w' \ \Rightarrow \ (\mathcal{M}, w' \models \varphi \Rightarrow \mathcal{M}, w' \models \psi))$;
- $\mathcal{M}, w \models \neg\varphi \ \Leftrightarrow \mathcal{M}, w^* \not\models \varphi$.

In what follows we will write $w \models \varphi$ instead of $\mathcal{M}, w \models \varphi$ if it does not lead to a confusion. For a set of formulas we write $\mathcal{M}, w \vDash \Sigma$ or $w \vDash \Sigma$ if $\mathcal{M}, w \vDash \psi$ for all $\psi \in \Sigma$.

As usual, a formula φ is valid in a Routley model if it is valid at every world of this model, and it is valid in a Routley frame if it is valid in any model over this frame. It is known that the analog of condition (1) above holds for any formula φ:

$$\mathcal{M}, u \models \varphi \ \& \ u \leq w \ \Rightarrow \ \mathcal{M}, w \models \varphi. \tag{2}$$

Moreover N^* is complete for this semantics, i.e. $\varphi \in N^*$ iff φ is valid in all Routely frames.

Completeness proofs for N^* and for its normal extensions can be obtained via the method of canonical models. Let S be some logic extending N^*, i.e., some set of formulas containing N^* and closed under substitution, contraposition rule, and *modus ponens*. First we say that a set of formulas Γ is a *theory* wrt S (S-*theory*) if it contains S and is closed under *modus ponens* and a *prime* S-*theory* if it additionally is non-trivial (does not equal to the set of all formulas) and satisfies the disjunction property: $\alpha \vee \beta \in \Gamma \Rightarrow \alpha \in \Gamma$ or $\beta \in \Gamma$.

Let $\Sigma \cup \{\varphi\}$ be a set of formulas. A relation $\Sigma \vdash_S \varphi$ means that φ can be obtained from elements of S and Σ using the rule of *modus ponens*.

For a class of frames $\mathcal{K}$, a relation $\Sigma \vDash_{\mathcal{K}} \varphi$ means that for every frame $\mathcal{W} \in \mathcal{K}$, every model $\mathcal{M}$ over $\mathcal{W}$, and very world x of $\mathcal{W}$, if $\mathcal{M}, x \vDash \Sigma$, then $\mathcal{M}, x \vDash \varphi$.

Lemma 0.2 (Extension lemma)**.** *For any extension S of N^*, any set of formulas Σ and formula φ, if $\Sigma \nvdash_S \varphi$, then there is a prime S-theory $\Gamma \supseteq \Sigma$ such that $\Gamma \nvdash_S \varphi$.*

Definition 0.3 (Canonical model)**.** *Let S be an extension of N^*. The* canonical S-frame *is the triple $\mathcal{W}^S = \langle W^S, \leq^S, *^S \rangle$ such that: W^S is the set of all prime theories wrt S; $\Gamma \leq^S \Delta$ iff $\Gamma \subseteq \Delta$; $\Gamma^{*^S} := \{\alpha \mid \neg\alpha \notin \Gamma\}$ for $\Gamma, \Delta \in W^S$.*

The canonical S-model *$\mathcal{M}^S$ is the canonical S-frame $\mathcal{W}^S$ together with the valuation function v^S such that $\Gamma \in v^S(p)$ iff $p \in \Gamma$.*

Proposition 0.4. [2] *For every normal N^*-extension S, the canonical model $\mathcal{M}^S$ is a Routley model.*

Lemma 0.5 (Canonical model lemma)**.** [2] *Let S be an N^*-extension. In the canonical S-model $\mathcal{M}^S$, for every $\Gamma \in W^S$ and every φ,*

$$\mathcal{M}^S, \Gamma \models \varphi \iff \varphi \in \Gamma.$$

Let us define the logic

$$G^* := N^* + \{\neg\neg(p \vee \neg p), \neg\neg((p \wedge \neg p) \to q), \neg\neg((p \to q) \to ((p \to \neg q) \to \neg p))\}.$$

Theorem 0.6. (1) *For every formula φ, we have $\varphi \in Cl$ iff $\neg\neg\varphi \in G^*$.*

(2) *If $L \in \mathcal{E}N^*$ is such that $\varphi \in Cl$ iff $\neg\neg\varphi \in L$ for every φ, then $G^* \subseteq L$.*

Доказательство. 1. Axioms $P1$–$P8$ together with $p \vee \neg p$, $(p \wedge \neg p) \to q$, and $(p \to q) \to ((p \to \neg q) \to \neg p))$ form the standard axiomatics of classical logic. Let $\varphi \in Cl$ and $\psi_1, \dots, \psi_n = \varphi$ be its inference from partial cases of the above mentioned axioms with the help of modus ponens. Consider the sequence $\neg\neg\psi_1, \dots, \neg\neg\psi_n$ and check that each element of this sequence belongs to G^*. If ψ_i is one of axioms $P1$–$P8$, then $\neg\neg\psi_i \in G^*$ by axiom $N4$ and replacement rule. If ψ_i is one of negation axioms, then its double negation belongs to G^* by definition. So it remains to check that $\neg\neg\alpha$, $\neg\neg(\alpha \to \beta) \in G^*$ implies $\neg\neg\beta \in G^*$. In fact, we have $(\neg\neg\alpha \wedge \neg\neg(\alpha \to \beta)) \to \neg\neg\beta \in N^*$. Indeed, consider a model $\mathcal{M} = \langle W, \leq, *, v\rangle$ and $x \in W$. If $x \vDash \neg\neg\alpha$ and $x \vDash \neg\neg(\alpha \to \beta)$, then $x^{**} \vDash \alpha$ and $x^{**} \vDash \alpha \to \beta$. Consequently, $x^{**} \vDash \beta$, and so $x \vDash \neg\neg\beta$.

The inverse implication is obvious.

2. This follows from the fact that G^* is axiomatized over N^* via double negations of classical tautologies.

□

We have thus proved that G^* is the least logic in the class of N^*-extensions satisfying Glivenko theorem. Now we show that this logic has a natural semantic characterization.

Let $\mathcal{W} = \langle W, \leq, * \rangle$ be a Routley frame. We say that $\mathcal{W}$ is a G^*-frame if for every $x \in W$ the following holds: (i) the world x^{**} is maximal wrt $\leq$; (ii) $x^{**} = x^{***}$. The class of all G^*-frames is denoted by $\mathcal{G}^*$. These frames characterize the logic G^*.

Theorem 0.7. *For every set of formulas Σ and formula φ, we have $\Sigma \vdash_{G^*} \varphi$ iff $\Sigma \vDash_{\mathcal{G}^*} \varphi$.*

Доказательство. Let $\mathcal{W} = \langle W, \leq, * \rangle$ be a G^*-frame. Since every world of the form x^{**} is maximal and $x^{**} = x^{***}$, validity of formulas in this world will be checked according to classical rules. In particular, every classical tautology is valid in the world x^{**}. A formula $\neg\neg\varphi$ is valid in $\mathcal{W}$ iff φ is valid at x^{**} for every $x \in W$.

Consequently, if $\varphi \in Cl$, then $\neg\neg\varphi$ is true in $\mathcal{W}$. The logic G^* is axiomatized modulo N^* via double negations of classical tautologies, therefore all elements of G^* are valid in all G^*-frames. This fact implies the direct implication.

To prove the completeness we check that the canonical frame for the logic G^* is a G^*-frame. It follows by definition that

$$\Gamma^{**} = \{\varphi \mid \neg\neg\varphi \in \Gamma\} \text{ and } \Gamma^{***} = \{\varphi \mid \neg\neg\neg\varphi \notin \Gamma\}.$$

The first equality and Theorem 0.6 immediately imply the following

Lemma 0.8. *For every $\Gamma \in W^{G^*}$, the theory Γ^{**} is a classical complete theory.*

Check that Γ^{**} is maximal in W^{G^*} wrt inclusion. Assume that $\Gamma^{**} \subset \Gamma_1 \in W^{G^*}$, $\varphi \in \Gamma_1 \setminus \Gamma^{**}$, and $\psi \notin \Gamma_1$. By Lemma 0.8 we have $\varphi \vee (\varphi \to \psi) \in \Gamma^{**}$. According to our choice $\varphi \notin \Gamma^{**}$, consequently, by disjunction property $\varphi \to \psi \in \Gamma^{**} \subset \Gamma_1$. We obtain in this way $\psi \in \Gamma_1$, which contradicts the choice of ψ. We have thus proved that Γ^{**} is maximal.

Now we prove that $\Gamma^{**} = \Gamma^{***}$. Let $\varphi \in \Gamma^{**}$. If $\neg\neg\neg\varphi \in \Gamma$, then $\neg\varphi \in \Gamma^{**}$. Since Γ^{**} is a classical theory by Lemma 0.8, the inclusion $\{\varphi, \neg\varphi\} \subseteq \Gamma^{**}$ means that Γ^{**} is trivial, a contradiction. Consequently, $\neg\neg\neg\varphi \notin \Gamma$ and $\varphi \in \Gamma^{***}$. We have thus proved that $\Gamma^{**} \subseteq \Gamma^{***}$.

Assume now $\varphi \in \Gamma^{***}$. This means that $\neg\neg\neg\varphi \notin \Gamma$, whence $\neg\varphi \notin \Gamma^{**}$. Since Γ^{**} is a complete classical theory, we obtain $\varphi \in \Gamma^{**}$. Thus, $\Gamma^{***} \subseteq \Gamma^{**}$.

We have thus proved that the canonical frame $\mathcal{W}^{G^*}$ is a G^*-frame. Now the completeness proof can be finished in the standard way. If $\Sigma \nvdash_{G^*} \varphi$, then by extension lemma there is a prime G^*-theory Γ such that $\Sigma \subseteq \Gamma$ and $\varphi \notin \Gamma$. By canonical model lemma we have $\mathcal{M}^{G^*}, \Gamma \models \Sigma$ and $\mathcal{M}^{G^*}, \Gamma \not\models \varphi$. Thus, the relation $\Sigma \models_{\mathcal{G}^*} \varphi$ is refuted on the G^*-frame $\mathcal{W}^{G^*}$.

□

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Sergei Pavlovich Odintsov
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
E-mail address: odintsov@math.nsc.ru