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THE SECOND COHOMOLOGY GROUPS OF SIMPLE MODULES FOR G_2

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Abstract. The second cohomology groups for simple, simply connected algebraic group G_2 over an algebraically closed field of characteristic $p \geq 7$ with coefficients in the simple finite-dimensional modules are described.

Keywords: algebraic group, cohomology, simple module, Frobenius kernel.

1. Introduction

In [13, 12] D.I. Stewart calculated the second cohomology groups of simple modules for simple, simply connected algebraic groups SL_2 and SL_3 over an algebraically closed field of positive characteristic. In this paper we consider this question for G_2.

Let G be a simple and simply connected algebraic group of type G_2 defined over an algebraically closed field k of characteristic $p \geq h$ and G^1 is the first Frobenius kernel of G. Here h is the Coxeter number. All G-modules considered in this paper will be finite-dimensional.

By the main result the simple G-module $L(\lambda)^{(d)}$ $(d \geq 0)$ has non-vanishing second cohomology if and only if

$$\lambda \in \{(p-5)\lambda_1 + p\lambda^{'},\, 3\lambda_1 + (p-2)\lambda_2 + p\lambda^{'},\, |\, \lambda^{'} \in A(0)\} \cup$$
$$\{(p-2)\lambda_1 + \lambda_2 + p\lambda^{'},\, 4\lambda_1 + (p-4)\lambda_2 + p\lambda^{'},\, |\, \lambda^{'} \in A(\lambda_1)\} \cup$$
$$\{3\lambda_1 + (p-2)\lambda_2 + p\lambda^{'},\, |\, \lambda^{'} \in A(\lambda_2)\} \cup$$

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$\{(p-6)\lambda_1+\lambda_2, 4\lambda_1+(p-3)\lambda_2, (p-2)(\lambda_1+\lambda_2), p\lambda_2, (2p-5)\lambda_1+2\lambda_2,$
$(p+3)\lambda_1+(p-4)\lambda_2, (p+4)\lambda_1+(p-3)\lambda_2, 4\lambda_1+(2p-3)\lambda_2, (p-2)\lambda_1+(2p-2)\lambda_2\}$,
where

$A(0)=\{p^i(p-5)\lambda_1, p^i(3\lambda_1+(p-2)\lambda_2), p^i((p-2)\lambda_1+\lambda_2)+p^{i+1}\lambda_1, p^i(4\lambda_1+(p-4)\lambda_2)+p^{i+1}\lambda_1, p^i(3\lambda_1+(p-2)\lambda_2)+p^{i+1}\lambda_2,\ i\geq 0\}$,

$A(\lambda_1)=\{(p-6)\lambda_1, 4\lambda_1+(p-2)\lambda_2, 4\lambda_1+(2p-2)\lambda_2, (p+6)\lambda_1+(p-5)\lambda_2, (2p-3)\lambda_1+2\lambda_2,\}\cup\{\lambda_1+p\lambda_0\,|\,\lambda_0\in A(0)\}$ for $p>7$,

$A(\lambda_1)=\{4\lambda_1+5\lambda_2, 11\lambda_1+2\lambda_2, 13\lambda_1+2\lambda_2, 4\lambda_1+12\lambda_2,\}\cup\{\lambda_1+7\lambda_0\,|\,\lambda_0\in A(0)\}$ for $p=7$,

$A(\lambda_2)=\{(p-8)\lambda_1+\lambda_2, 6\lambda_1+(p-3)\lambda_2, 6\lambda_1+(2p-3)\lambda_2, (p+7)\lambda_1+(p-6)\lambda_2, (2p-2)\lambda_1+2\lambda_2,\}\cup\{\lambda_2+p\lambda_0\,|\,\lambda_0\in A(0)\}$ for $p>7$,

$A(\lambda_2)=\{5\lambda_1+4\lambda_2, 6\lambda_1+4\lambda_2, 6\lambda_1+11\lambda_2, 12\lambda_1+2\lambda_2,\}\cup\{\lambda_2+7\lambda_0\,|\,\lambda_0\in A(0)\}$ for $p=7$.

In Section 2 we introduce the basic notation. Section 3 is devoted to calculations of the formal characters of some useful Weyl modules with highest weights having a strong linkage relation with the zero or fundamental weight. (More detailed information on the strong linkage principle is contained in [1]). In Section 4 we will describe the first and second cohomology groups of restricted simple modules over G^1. In Section 5 we determine extensions of simple G-modules with fundamental and zero highest weights by the arbitrary simple modules. Finally, in Section 6 we formulate and prove the main Theorem 6.1.

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2. Notation

Let G, G^1 and k as above. Choose the root system

$$R=\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)$$

of G with the maximal short root $\widetilde{\alpha}_0=2\alpha_1+\alpha_2$ and the maximal root $\alpha_0=3\alpha_1+2\alpha_2$. Denote by B the Borel subgroup of G corresponding to negative roots, and by T the maximal torus in G. The Weyl group W of R acts on the character group $X(T)$ of T by $s_\alpha(\lambda)=\lambda-\langle\lambda,\alpha^\vee\rangle\alpha$, where $s_\alpha\in W$, α is the positive root and $\alpha^\vee$ is the coroot of α. If ρ is half the sum of the positive roots, then the dot action is given by $w\cdot\lambda=w(\lambda+\rho)-\rho$, where $w\in W$, $\lambda\in X(T)$.

The affine Weyl group W_p is the group generated by all $s_{\alpha,np}$ for $\alpha\in R^+$ and $n\in\mathbb{Z}$, where R^+ is the set of positive roots. We will use the dot action of W_p on $X(T)$:

$$s_{\alpha,np}\cdot\lambda=\lambda-\langle\lambda+\rho,\alpha^\vee\rangle\alpha+np\alpha.$$

Let

$$X_+(T)=\{\lambda\in X(T)\,|\,\langle\lambda,\alpha^\vee\rangle\geq 0 \text{ for all } \alpha\in S\}$$

is the set of dominant weights, where S is the set of simple roots, and

$$X_1(T)=\{\lambda\in X(T)\,|\,0\leq\langle\lambda,\alpha^\vee\rangle<p \text{ for all } \alpha\in S\}$$

is the set of restricted weights.

For any $\lambda \in X(T)$ there is an one-dimensional B-module k_λ and the induced G-module $H^0(\lambda) = \mathrm{Ind}_B^G(k_\lambda)$. $H^0(\lambda) \neq 0$ if and only if $\lambda \in X_+(T)$. If $V(\lambda)$ is the Weyl module with the highest weight λ over G, then $H^0(\lambda) \cong V(-w_0\lambda)^*$. So $H^0(\lambda)$ is the dual of the Weyl module with the highest weight $-w_0(\lambda)$. The simple G-module $L(\lambda)$ is the simple socle of $H^0(\lambda)$ and the unique simple quotient of $V(\lambda)$ ([5], 5.7).

We denote by $\chi(\lambda)$ the formal character of the Weyl module $V(\lambda)_C$ in zero characteristic, and by $\chi_k(\lambda)$ the formal character of the simple module $L(\lambda)$.

Let L be a rational G-module. Denote by $L^{(d)}$ the d-th Frobenius twist of L. Recall that there is a unique $d \geq 1$ such that Frobenius untwist $L^{(-d)}$ is a G-module on which G^1 acts nontrivially.

Let $\lambda = r\lambda_1 + s\lambda_2 \in X_+(T)$, where λ_1, λ_2 are the fundamental weights, and $r, s \in k$. Using the notation

$\alpha_1 = \varepsilon_1 - \varepsilon_2 = 2\lambda_1 - \lambda_2,$

$\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = -3\lambda_1 + 2\lambda_2,$

$\alpha_1 + \alpha_2 = -\varepsilon_1 + \varepsilon_3 = -\lambda_1 + \lambda_2,$

$2\alpha_1 + \alpha_2 = -\varepsilon_2 + \varepsilon_3 = \lambda_1,$

$3\alpha_1 + \alpha_2 = \varepsilon_1 - 2\varepsilon_2 + \varepsilon_3 = 3\lambda_1 - \lambda_2,$

$3\alpha_1 + 2\alpha_2 = -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3 = \lambda_2;$

$\lambda_1 = \widetilde{\alpha}_0 = -\varepsilon_2 + \varepsilon_3,$

$\lambda_2 = \alpha_0 = -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3,$

$\rho = \lambda_1 + \lambda_2 = -\varepsilon_1 - 2\varepsilon_2 + 3\varepsilon_3$

we have

$$\langle \lambda + \rho, \alpha^\vee \rangle = \begin{cases} r+1, & \text{if } \alpha = \alpha_1; \\ s+1, & \text{if } \alpha = \alpha_2; \\ r+3s+4, & \text{if } \alpha = \alpha_1 + \alpha_2; \\ 2r+3s+5, & \text{if } \alpha = 2\alpha_1 + \alpha_2; \\ r+s+2, & \text{if } \alpha = 3\alpha_1 + \alpha_2; \\ r+2s+3, & \text{if } \alpha = 3\alpha_1 + 2\alpha_2, \end{cases} \tag{1}$$

where $\{\varepsilon_1 = (1,0,0),\ \varepsilon_2 = (0,1,0),\ \varepsilon_3 = (0,0,1)\}$ is the orthonormal basis.

3. Characters of the Weyl modules

Let

$$V(\lambda)^0 = V(\lambda) \supset V(\lambda)^1 \supset V(\lambda)^2 \supset \cdots$$

be the Jantzen filtration of the Weyl module $V(\lambda)$ with following well-known summation formula [8], II.8.19:

$$\sum_{j>0} \chi(V(\lambda)^j) = \sum_{\alpha \in R_+} \sum_{0<np<\langle \lambda+\rho, \alpha^\vee \rangle} \nu_p(np)\chi(s_{\alpha,np} \cdot \lambda), \tag{2}$$

where ν_p denotes the p-adic valuation of the integers, that is, $\nu_p(p^n q) = n$ if $p \nmid q$, and $\chi(w \cdot \lambda) = (-1)^{l(w)} \chi(\lambda)$ for all $w \in W$.

Consider the following elements of W_p :

$w^0 = s_{\widetilde{\alpha}_0,p}$,

$w^1 = s_{\widetilde{\alpha}_0,p} \circ s_{\alpha_1,0}$,

$w^2 = s_{\widetilde{\alpha}_0,p} \circ s_{\alpha_1,0} \circ s_{\alpha_2,0}$,

$w^3 = s_{\widetilde{\alpha}_0,p} \circ s_{\alpha_1,0} \circ s_{\alpha_2,0} \circ s_{\alpha_1,0}$,

$w^4 = s_{\widetilde{\alpha}_0,p} \circ s_{\alpha_1,0} \circ s_{\alpha_2,0} \circ s_{\alpha_1,0} \circ s_{\alpha_2,0}$,

$w^5 = s_{\widetilde{\alpha}_0,p} \circ s_{\alpha_1,0} \circ s_{\alpha_2,0} \circ s_{\alpha_1,0} \circ s_{\alpha_2,0} \circ s_{\alpha_1,0}$,

where a symbol $\circ$ means a composition of the affine reflections.

Let C_i $(1 \le i \le 16)$ are the affine alcoves introduced by J.C. Jantzen in ([6], p.139-140), and

$\nu \in C_1 \cap X_+(T)$,

$\omega_i(\nu) = w^i \cdot \nu \in C_{i+2},\ i = 0, 1, 2, 3$,

$\beta_0(\nu) = w^3 \cdot \omega_0(\nu) \in C_6$,

$\omega_4(\nu) = w^4 \cdot \nu \in C_7$,

$\delta_0(\nu) = w^4 \cdot \omega_0(\nu) \in C_8$,

$\omega_5(\nu) = w^5 \cdot \nu \in C_9$,

$\delta_i(\nu) = w^4 \cdot \omega_i(\nu) \in C_{9+2i},\ i = 1, 2, 3$,

$\gamma_i(\nu) = w^5 \cdot \omega_i(\nu) \in C_{10+2i},\ 0, 1, 2$,

$\beta_1(\nu) = w^4 \cdot \beta_0(\nu) \in C_{16}$.

Proposition 3.1. *If* $\nu \in \{0,\ \lambda_1,\ \lambda_2\}$, *then*

(3) $$\chi(\nu) = \chi_k(\nu);$$

(4) $$\chi(\omega_0(\nu)) = \chi_k(\omega_0(\nu)) + \chi_k(\nu);$$

(5) $$\chi(\omega_i(\nu)) = \chi_k(\omega_i(\nu)) + \chi_k(\omega_{i-1}(\nu)),\ i = 1, 2, 3, 4, 5;$$

(6) $$\chi(\beta_0(\nu)) = \chi_k(\beta_0(\nu)) + \chi_k(\omega_3(\nu));$$

(7) $$\chi(\delta_0(\nu)) = \chi_k(\delta_0(\nu)) + \chi_k(\beta_0(\nu)) + \sum_{i=2}^{4} \chi_k(\omega_i(\nu));$$

(8) $$\chi(\delta_1(\nu)) = \chi_k(\delta_1(\nu)) + \chi_k(\delta_0(\nu)) + \sum_{i=1}^{5} \chi_k(\omega_i(\nu));$$

(9) $$\chi(\delta_2(\nu)) = \chi_k(\delta_2(\nu)) + \chi_k(\delta_1(\nu)) + \sum_{i=0}^{3} \chi_k(\omega_i(\nu));$$

$$\chi(\delta_3(\nu)) = \chi_k(\delta_3(\nu)) + \chi_k(\delta_2(\nu)) + \chi_k(\nu) + \sum_{i=0}^{2} \chi_k(\omega_i(\nu)); \tag{10}$$

$$\chi(\gamma_0(\nu)) = \chi_k(\gamma_0(\nu)) + \chi_k(\delta_0(\nu)) + \chi_k(\omega_5(\nu)) + \chi_k(\omega_5(\nu)); \tag{11}$$

$$\chi(\gamma_1(\nu)) = \chi_k(\gamma_1(\nu)) + \chi_k(\gamma_0(\nu)) + \chi_k(\delta_1(\nu)) +$$

$$\chi_k(\delta_0(\nu)) + \chi_k(\beta_0(\nu)) + \chi_k(\omega_5(\nu)) + \chi_k(\omega_4(\nu)) + \chi_k(\omega_3(\nu)); \tag{12}$$

$$\chi(\gamma_2(\nu)) = \chi_k(\gamma_2(\nu)) + \chi_k(\gamma_1(\nu)) + \chi_k(\delta_2(\nu)) +$$

$$\chi_k(\delta_1(\nu)) + \chi_k(\delta_0(\nu)) + \chi_k(\beta_0(\nu)) + \chi_k(\omega_3(\nu)) + \chi_k(\omega_2(\nu)); \tag{13}$$

$$\chi(\beta_1(\nu)) = \chi_k(\beta_1(\nu)) + \chi_k(\gamma_2(\nu)) +$$

$$\chi_k(\delta_3(\nu)) + \chi_k(\delta_2(\nu)) + \chi_k(\omega_0(\nu)) + \chi_k(\nu). \tag{14}$$

Доказательство. (3)–(6) is proved in ([11], Th.1.3., p.402). We will prove (7). The other cases can be proved in the same way.

According to (1), $\langle \delta_0(\nu) + \rho, \alpha^\vee \rangle > p$ for the four positive roots, except for α_1 and α_2. We have also:

$p < \langle \delta_0(\nu) + \rho, \alpha^\vee \rangle < 2p$ for $\alpha = 3\alpha_1 + \alpha_2, \alpha_0$;

$2p < \langle \delta_0(\nu) + \rho, \alpha^\vee \rangle < 3p$ for $\alpha = \alpha_1 + \alpha_2, \widetilde{\alpha}_0$.

Then, by the summation formula (2),

$$\sum_{j>0} \chi(V(\delta_0(\nu))^j) = \chi(s_{3\alpha_1+\alpha_2,p} \cdot \delta_0(\nu)) + \chi(s_{\alpha_0,p} \cdot \delta_0(\nu)) +$$

$$\chi(s_{\alpha_1+\alpha_2,p} \cdot \delta_0(\nu)) + \chi(s_{\alpha_1+\alpha_2,2p} \cdot \delta_0(\nu)) +$$

$$\chi(s_{\widetilde{\alpha}_0,p} \cdot \delta_0(\nu)) + \chi(s_{\widetilde{\alpha}_0,2p} \cdot \delta_0(\nu)) =$$

$$-\chi(\nu) + \chi(\beta_0(\nu)) + \chi(\omega_4(\nu)) + \chi(\omega_2(\nu)) - \chi(\omega_1(\nu)) + \chi(\omega_0(\nu)) =$$

$$\text{by } (3) - (6)$$

$$= -\chi_k(\nu) + \chi_k(\beta_0(\nu)) + \chi_k(\omega_3(\nu)) + \chi_k(\omega_4(\nu)) + \chi_k(\omega_3(\nu)) +$$

$$\chi_k(\omega_2(\nu)) + \chi_k(\omega_1(\nu)) - \chi_k(\omega_1(\nu)) - \chi_k(\omega_0(\nu)) + \chi_k(\omega_0(\nu)) + \chi_k((\nu)) =$$

$$\chi_k(\omega_2(\nu)) + 2\chi_k(\omega_3(\nu)) + \chi_k(\omega_4(\nu)) + \chi_k(\beta_0(\nu)).$$

According to the transition principle ([7], Theorem, p.297), the multiplicity of $L(\omega_3(\nu))$ in $V(\delta_0(\nu))$ is equal to the multiplicity of $L(\omega_3(\nu))$ in $V(\omega_4(\nu))$. By (5), this number is 1. Therefore using ([6], Satz 5, p.127), we get (7). □

4. The first and second cohomology of G^1

For the calculation of $H^1(G^1, L)$ and $H^2(G^1, L)$ for a simple G^1-module L we use, first, the general Andersen-Jantzen's formula for $p \geq h$ [3]

$$H^k(G^1, H^0(w \cdot 0 + p\nu))^{(-1)} \cong H^0(S^{(k-l(w))/2}(\mathbf{u}^*) \otimes k_\nu), \tag{15}$$

where $S(\mathbf{u}^*)$ is a symmetric algebra of $\mathbf{u}^*$, $\mathbf{u}$ is the maximal nilpotent subalgebra of $\mathbf{g}$ corresponding to the negative roots, and the long exact cohomological sequence of G^1-cohomology corresponding to the short exact sequence

$$0 \to L(\lambda) \to H^0(\lambda) \to M(\lambda) \to 0. \tag{16}$$

Second, using Lemma 4.1 below, we can find all non-vanishing first and second cohomology groups for simple G^1-modules.

We also use the linkage principle for G^1 ([8], II.6.17). There are only twelve restricted weights G^1-linked to 0:

$$0, \omega_4(0) = (p-2)\lambda_1 + \lambda_2, \delta_3(0) = 3\lambda_1 + (p-2)\lambda_2, \omega_3(0) = (p-5)\lambda_1 + 2\lambda_2,$$
$$\delta_2(0) = 4\lambda_1 + (p-3)\lambda_2, \omega_2(0) = (p-6)\lambda_1 + 2\lambda_2, \delta_1(0) = 4\lambda_1 + (p-4)\lambda_2,$$
$$\omega_1(0) = (p-6)\lambda_1 + \lambda_2, \delta_0(0) = 3\lambda_1 + (p-4)\lambda_2, \omega_0(0) = (p-5)\lambda_1,$$
$$\beta_0(0) = (p-3)\lambda_2, \beta_1(0) = (p-2)(\lambda_1 + \lambda_2).$$

For these weights we have the following expressions by the simple reflections s_{α_1} and s_{α_2} of the Weyl group W:

$\omega_4(0) = s_{\alpha_1} \cdot 0 + p\lambda_1 \in C_7$,

$\delta_3(0) = s_{\alpha_2} \cdot 0 + p\lambda_2 \in C_{15}$,

$\omega_3(0) = s_{\alpha_1} \circ s_{\alpha_2} \cdot 0 + p\lambda_1 \in C_5$,

$\delta_2(0) = s_{\alpha_2} \circ s_{\alpha_1} \cdot 0 + p\lambda_2 \in C_{13}$,

$\omega_2(0) = s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \cdot 0 + p\lambda_1 \in C_4$,

$\delta_1(0) = s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \cdot 0 + p\lambda_2 \in C_{11}$,

$\omega_1(0) = s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \cdot 0 + p\lambda_1 \in C_3$,

$\delta_0(\nu) = s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \cdot 0 + p\lambda_2 \in C_8$,

$\omega_0(0) = s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \cdot 0 + p\lambda_1 \in C_2$,

$\beta_0(0) = s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \cdot 0 + p\lambda_2 \in C_6$,

$\beta_1(0) = s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \circ s_{\alpha_1} \circ s_{\alpha_2} \cdot 0 + p(\lambda_1 + \lambda_2) \in C_{12}$.

Below we need the following

Lemma 4.1. *If* $\nu \in \{0, \lambda_1, \lambda_2\}$, *then*

(a) $\mathrm{Soc}_G M(\nu) = 0$;

(b) $\mathrm{Soc}_G M(\omega_0(\nu)) \cong L(\nu)$;

(c) $\mathrm{Soc}_G M(\omega_i(\nu)) \cong L(\omega_{i-1}(\nu)),\ i = 1, 2, 3, 4, 5$;

(d) $\mathrm{Soc}_G M(\beta_0(\nu)) \cong L(\omega_3(\nu))$;

(e) $\mathrm{Soc}_G M(\delta_0(\nu)) \cong L(\beta_0(\nu)) \oplus L(\omega_4(\nu)) \oplus L(\omega_2(\nu))$;

(f) $\mathrm{Soc}_G M(\delta_1(\nu)) \cong L(\delta_0(\nu)) \oplus L(\omega_5(\nu)) \oplus L(\omega_3(\nu)) \oplus L(\omega_1(\nu))$;

(g) $\mathrm{Soc}_G M(\delta_2(\nu)) \cong L(\delta_1(\nu)) \oplus L(\omega_2(\nu)) \oplus L(\omega_0(\nu))$;

(h) $\mathrm{Soc}_G M(\delta_3(\nu)) \cong L(\delta_2(\nu)) \oplus L(\omega_1(\nu)) \oplus L(\nu)$;

(i) $\mathrm{Soc}_G M(\beta_1(\nu)) \cong L(\gamma_2(\nu)) \oplus L(\delta_3(\nu)) \oplus L(\omega_0(\nu))$;

(j)$\mathrm{Soc}_G M(w^5 \cdot \omega_4(\nu)) \cong L(\gamma_3(\nu)) \oplus L(\delta_4(\nu)) \oplus L(\delta_2(\nu)) \oplus L(\omega_5(\nu)) \oplus L(\nu)$;

(k) $\mathrm{Soc}_G M(w^5 \cdot \delta_1(\nu)) \cong L(w^5 \cdot \delta_0(\nu)) \oplus L(w^4 \cdot \delta_1(\nu)) \oplus L(\gamma_5(\nu)) \oplus L(\delta_4(\nu)) \oplus L(\omega_5(\nu)) \oplus L(\omega_3(\nu)) \oplus L(\omega_1(\nu)) \oplus L(\nu)$;

(l) $\mathrm{Soc}_G M((w^4)^2 \cdot \delta_3(\nu)) \cong L((w^5)^2 \cdot \omega_5(\nu)) \oplus L((w^4)^2 \cdot \delta_2(\nu)) \oplus L(w^5 \cdot \gamma_3(\nu)) \oplus L(w^5 \cdot \delta_1(\nu)) \oplus L(\nu)$.

Доказательство. Since $\lambda^* = -w_0(\lambda) = \lambda$, we can use the formal character formulas from Proposition 3.1. Here w_0 is the longest element in W. By Proposition 3.1, the multiplicity of each nonzero composition factor of $V(\lambda)$ is 1. Then the existence of the symmetric bilinear contravariant form on $V(\lambda)$ provides the semi-simplicity of each layer $V(\lambda)^i/V(\lambda)^{i+1}$ of the Jantzen filtration for $V(\lambda)$ ([7], p. 298). Therefore the socle of $M(\lambda)$ is isomorphic to the second layer $V(\lambda)^1/V(\lambda)^2$. So, it remains to calculate $V(\lambda)^1/V(\lambda)^2$.

The cases $(a) - (d)$ are obvious.

(e) Let V be a rational G-module and L is a simple G-module. Denote by $[V : L]$ the multiplicity of L in V. By the summation formula (2),

$$\sum_{j>0} \chi(V(\delta_0(\nu))^j) = \chi_k(\omega_2(\nu)) + 2\chi_k(\omega_3(\nu)) + \chi_k(\omega_4(\nu)) + \chi_k(\beta_0(\nu)).$$

Hence for $\mu \in \{\omega_2(\nu),\, \omega_4(\nu),\, \beta_0(\nu)\}$ we have

$$[V(\delta_0(\nu))^i : L(\mu)] = \begin{cases} 1 & \text{for } i = 0,1; \\ 0 & \text{for } i \geq 2. \end{cases}$$

For $L(\omega_3(\nu))$ there are two possibilities:

(A) $[V(\delta_0(\nu))^i : L(\omega_3(\nu))] = \begin{cases} 2 & \text{for } i = 0,1; \\ 0 & \text{for } i \geq 2, \end{cases}$

(B) $[V(\delta_0(\nu))^i : L(\omega_3(\nu))] = \begin{cases} 1 & \text{for } i = 0,1,2; \\ 0 & \text{for } i \geq 3. \end{cases}$

By the transition principle, $[V(\delta_0(\nu)) : L(\omega_3(\nu)] = [V(\omega_4(\nu)) : L(\omega_3(\nu)] = 1$. From this it follows that the second possibility (B) holds. Thus

$$\mathrm{Soc}_G M(\delta_0(\nu)) \cong V(\delta_0(\nu))^1/V(\delta_0(\nu))^2 = L(\beta_0(\nu)) \oplus L(\omega_4(\nu)) \oplus L(\omega_2(\nu)).$$

(f) By the summation formula (2), $\sum_{j>0} \chi(V(\delta_1(\nu))^j) =$

$$\chi_k(\omega_1(\nu)) + 2\chi_k(\omega_2(\nu)) + \chi_k(\omega_3(\nu)) + 2\chi_k(\omega_4(\nu)) + \chi_k(\omega_5(\nu)) + \chi_k(\delta_0(\nu)).$$

Hence for $\mu \in \{\omega_1(\nu), \omega_3(\nu), \omega_5(\nu), \delta_0(\nu)\}$ we have

$$[V(\delta_0(\nu))^i : L(\mu)] = \begin{cases} 1 & \text{for } i = 0, 1; \\ 0 & \text{for } i \geq 2. \end{cases}$$

For $\mu \in \{\omega_2(\nu), \omega_4(\nu)\}$ there are two possibilities:

(C) $[V(\delta_1(\nu))^i : L(\mu)] = \begin{cases} 2 & \text{for } i = 0, 1; \\ 0 & \text{for } i \geq 2, \end{cases}$

(D) $[V(\delta_1(\nu))^i : L(\mu)] = \begin{cases} 1 & \text{for } i = 0, 1, 2; \\ 0 & \text{for } i \geq 3. \end{cases}$

According to the transition principle, $[V(\delta_1(\nu)) : L(\omega_2(\nu)] = [V(\delta_0(\nu)) : L(\omega_2(\nu)] = 1$ and $[V(\delta_1(\nu)) : L(\omega_4(\nu)] = [V(\delta_0(\nu)) : L(\omega_4(\nu)] = 1$. This shows that for each of these simple factors the second possibility (D) holds. Thus $\mathrm{Soc}_G M(\delta_1(\nu)) \cong V(\delta_1(\nu))^1/V(\delta_1(\nu))^2 = L(\delta_0(\nu)) \oplus L(\omega_5(\nu)) \oplus L(\omega_3(\nu)) \oplus L(\omega_1(\nu))$.

By same way as in above cases it is easy to show the G-module isomorphisms $(g) - (l)$. □

Proposition 4.2. *Let $p \geq 7$ and $\lambda \in X_1(T)$. Then $H^1(G^1, L(\lambda)) = 0$, except in the following cases:*

(a) $H^1(G^1, L(\omega_0(0))) \cong k$;

(b) $H^1(G^1, L(\omega_4(0))) \cong L(\lambda_1)^{(1)}$;

(c) $H^1(G^1, L(\delta_1(0))) \cong L(\lambda_1)^{(1)}$;

(d) $H^1(G^1, L(\delta_3(0))) \cong L(\lambda_2)^{(1)} \oplus k$.

Доказательство. First we prove that there exist only four simple restricted modules with non-vanishing first cohomology groups. These are modules mentioned in the proposition.

The Weyl group W has only two elements with the length 1, s_{α_1} and s_{α_1}. Then according to (15),

$$H^1(G^1, H^0(\lambda)) \cong \begin{cases} L(\lambda_1)^{(1)}, & \text{if } \lambda = s_{\alpha_1} \cdot 0 + p\lambda_1 = \omega_4(0); \\ L(\lambda_2)^{(1)}, & \text{if } \lambda = s_{\alpha_2} \cdot 0 + p\lambda_2 = \delta_3(0); \\ 0, & \text{in other cases,} \end{cases} \tag{17}$$

because $H^0(\lambda_i) = L(\lambda_i)$ for $i = 1, 2$ ([9], p.299).

The operation of G on $H^i(G^1, V)$ for G-module V is the following. If (V_j, f_j) is an injective resolution of V as a G-module then first G operates on the fixed point complex $(V_j^{G^1}, f_j)$, and then G operates on the cohomology space of this complex. Therefore the long exact sequence of G^1-cohomology corresponding to the short exact sequence of G-modules (16) can be regards as the long exact sequence of G-modules. So (16) yields the following exact sequence:

$$0 \to M(\lambda)^{G^1} \to H^1(G^1, L(\lambda)) \to H^1(G^1, H^0(\lambda)) \to H^1(G^1, M(\lambda)). \tag{18}$$

By Lemma 4.1, $M(\lambda)^{G^1} \neq 0$ only possibly for three modules: $H^0(\omega_0(0))$, $H^0(\delta_3(0))$, $H^0(\delta_1(0))$. The module $L(0)$ does not occur in the socle of $M(\beta_1(0))$. Therefore, it follows from (17) and from the exactness of (18), that $H^1(G^1, L(\lambda)) \neq 0$ only for the four cases listed in Proposition 4.2.

Now we will establish the G-module isomorphisms (a)–(e).

(a) $\lambda = \omega_0(0)$. By Lemma 4.1(b), $M(\lambda) \cong L(0) \cong k$, and from the exactness of (18) we have that $H^1(G^1, L(\lambda)) \cong M(\lambda)^{G^1} \cong k$, since $H^1(G^1, H^0(\lambda)) = 0$.

(b) $\lambda = \omega_4(0)$. By Lemma 4.1(c) (for $\nu = 0$), $M(\lambda) \cong L(\omega_3(0))$. Then, using the exact sequence (18), we get

$$H^1(G^1, L(\omega_4(0))) \cong H^1(G^1, H^0(\omega_4(0))) \cong L(\lambda_1)^{(1)}.$$

(c) $\lambda = \delta_1(0)$. By Lemma 4.1(f),

$$\mathrm{Soc}_G M(\lambda) \cong L(\delta_0(0)) \oplus L(\omega_5(0)) \oplus L(\omega_3(0)) \oplus L(\omega_1(0)).$$

Since $H^1(G^1, H^0(\lambda)) = 0$, from the exactness of (18) we have

$$H^1(G^1, L(\lambda)) \cong L(\omega_5(0))^G \cong L(p\lambda_1)^{G^1} = L(\lambda_1)^{(1)}.$$

(d) If $\lambda = \delta_3(0)$, then by Lemma 4.1(h),

$$\mathrm{Soc}_G M(\lambda) \cong L(\delta_2(0)) \oplus L(\omega_1(0)) \oplus L(0).$$

Using (17) and the exactness of (18) for each $\lambda \in \{\delta_2(0), \omega_1(0), 0\}$ it is easy to show that

$$H^1(G^1, M(\lambda)) \cong H^1(G^1, L(\delta_2(0))) \oplus H^1(G^1, L(\omega_1(0))) \oplus H^1(G^1, L(0)) = 0.$$

Therefore, the sequence of G-modules

$$0 \to k \to H^1(G^1, L(\lambda)) \to L(\lambda_2)^{(1)} \to 0$$

is exact. Since $\mathrm{Ext}^1_G(k, L(\lambda_2)) = 0$ ([8], II.2.14(4)), we get the required isomorphism of G-modules

$$H^1(G^1, L(\lambda)) \cong L(\lambda_2)^{(1)} \oplus k.$$

□

Remark. Proposition 4.2 improves the corresponding result in ([10], Table 1) to include the primes $p = 7$ and 11 if $\lambda = 0$.

Proposition 4.3. *Let $p \geq 7$ and $\lambda \in X_1(T)$. Then $H^2(G^1, L(\lambda)) = 0$, except in the following cases:*

(a) $H^2(G^1, k) \cong L(\lambda_2)^{(1)}$;

(b) $H^2(G^1, L(\omega_1(0))) \cong k$;

(c) $H^2(G^1, L(\omega_3(0))) \cong L(\lambda_1)^{(1)}$;

(d) $H^2(G^1, L(\delta_0(0))) \cong L(\lambda_1)^{(1)}$;

(e) $H^2(G^1, L(\delta_2(0))) \cong L(\lambda_1)^{(1)} \oplus k \oplus L(\lambda_2)^{(1)}$;

(f) $H^2(G^1, L(\beta_1(0))) \cong L(\lambda_2)^{(1)} \oplus k \oplus k$.

Доказательство. We use the same arguments as in the proof of Proposition 4.2.

The Weyl group W has exactly two elements with the length 2, $s_{\alpha_1} \circ s_{\alpha_2}$ and $s_{\alpha_2} \circ s_{\alpha_1}$. Also there is the identity element with length 0. Then by (15),

$$H^2(G^1, H^0(\lambda)) \cong \begin{cases} L(\lambda_1)^{(1)}, & \text{if } \lambda = s_{\alpha_1} \circ s_{\alpha_2} \cdot 0 + p\lambda_1 = \omega_3(0); \\ L(\lambda_2)^{(1)}, & \text{if } \lambda = s_{\alpha_2} \circ s_{\alpha_1} \cdot 0 + p\lambda_2 = \delta_2(0), 0; \\ 0, & \text{in other cases.} \end{cases} \tag{19}$$

The proof of Proposition 4.2 shows that the composition factors of $H^1(G^1, M(\lambda))$ do not occur in $H^1(G^1, H^0(\lambda))$. Therefore, the cohomological exact sequence corresponding to the short exact sequence (16) yields:

$$0 \to H^1(G^1, M(\lambda)) \to H^2(G^1, L(\lambda)) \to H^2(G^1, H^0(\lambda)) \xrightarrow{\pi} H^2(G^1, M(\lambda)). \tag{20}$$

By Lemma 4.1 and Propositions 4.2, $H^1(G^1, M(\lambda)) \neq 0$ only for four modules: $H^0(\omega_1(0))$, $H^0(\delta_0(0))$, $H^0(\delta_2(0))$, and $H^0(\beta_1(0))$. Therefore, from (19) and from the exactness of (20) it follows that there are at most only six non-vanishing cases listed in Proposition 4.3.

Now we will prove all G-module isomorphisms of Proposition 4.3.

(a) It is evident.

(b) $\lambda = \omega_1(0)$. By Lemma 4.1(c), $M(\lambda) \cong L(\omega_0(0))$. Then using the exact sequence (20) and Proposition 4.2, we get

$$H^2(G^1, L(\omega_1(0))) \cong H^1(G^1, L(\omega_0(0))) \cong k,$$

because $H^2(G^1, H^0(\lambda)) = 0$.

(c). If $\lambda = \omega_3(0)$, then by Lemma 4.1(c), $M(\lambda) \cong L(\omega_2(0))$.

Therefore, from the exact sequence (20) and Proposition 4.2 we have

$$H^2(G^1, L(\lambda)) \cong H^2(G^1, H^0(\lambda)) \cong L(\lambda_1)^{(1)}.$$

(d) $\lambda = \delta_0(0)$. By Lemma 4.1(e), the socle of $M(\lambda)$ isomorphic to

$$L(\beta_0(0)) \oplus L(\omega_4(0)) \oplus L(\omega_2(0)).$$

Since $H^2(G^1, H^0(\lambda)) = 0$, from the exact sequence (20) we have that

$$H^2(G^1, L(\lambda)) \cong H^1(G^1, M(\lambda)).$$

Then, according to Proposition 4.2, $H^2(G^1, L(\lambda)) \cong L(\lambda_1)^{(1)}$.

(e) $\lambda = \delta_2(0)$. In this case by Lemma 4.1(g),

$$\mathrm{Soc}_G M(\lambda) \cong L(\delta_1(0)) \oplus L(\omega_2(0)) \oplus L(\omega_0(0)).$$

By Propositions 4.2,

$$H^1(G^1, M(\lambda)) \cong H^1(G^1, L(\delta_1(0))) \oplus H^1(G^1, \omega_0(0))) \cong L(\lambda_1)^{(1)} \oplus k,$$

and then by (19) the sequence

$$0 \to L(\lambda_1)^{(1)} \oplus k \to H^2(G^1, L(\lambda)) \to L(\lambda_2)^{(1)} \xrightarrow{\pi} H^2(G^1, M(\lambda))$$

is exact. Using (19), Proposition 4.2 and the exactness of (20) for each $\lambda \in \{\delta_1(0), \omega_2(0), \omega_0(0)\}$ we get

$$H^1(G^1, M(\lambda)) \cong H^2(G^1, L(\delta_1(0))) \oplus H^2(G^1, L(\omega_2(0))) \oplus H^2(G^1, L(\omega_0(0))) = 0.$$

Therefore the map π is zero. From this we have

$$H^2(G^1, L(\lambda)) \cong (L(\lambda_1)^{(1)} \oplus k) \oplus L(\lambda_2)^{(1)},$$

because $\mathrm{Ext}^1_G(L(\lambda_1), L(\lambda_2)) = \mathrm{Ext}^1_G(k, L(\lambda_2)) = 0$ ([8], II.2.14(4)).

(f) $\lambda = \beta_1(0)$:

$$H^2(G^1, L(\lambda)) \cong H^1(G^1, M(\lambda)) \cong$$
$$H^1(G^1, L(\delta_3(0)) \oplus H^1(G^1, L(\omega_0(0)) \cong L(\lambda_2)^{(1)} \oplus k \oplus k.$$

□

5. Extensions of simple modules

Consider the Lyndon-Hochschild-Serre spectral sequence applied to $G^1 \lhd G$ ([8],I.6.6.(3)). For each G-module L, the spectral sequence

$$E_2^{nm} = H^n(G, H^m(G^1, L)^{(-1)}) \Rightarrow H^{n+m}(G, L) \tag{21}$$

gives

$$H^i(G, L) = \bigoplus_{n+m=i} E_\infty^{nm}, \tag{22}$$

where E_∞^{nm} is the limiting term of the spectral sequence.

Lemma 5.1. *Let $p \geq 7$. For the simple G-module $L = L(\lambda)$ with $\lambda = \lambda_0 + p\lambda'$, $\lambda_0 \in X_1(T)$, $\lambda' \in X_+(T)$, one has*

(a) $E_2^{01} = E_\infty^{01}$;

(b) $E_2^{10} = E_\infty^{10}$;

(c)$H^1(G, L) = E_2^{01} \oplus E_2^{10}$.

Доказательство. It is evident that $L(\lambda_0)^{G^1} = 0$ whenever $\lambda_0 \neq 0$. Then by Proposition 4.2, $E_2^{n-2,m+1} = E_2^{n+2,m-1} = 0$ for $(n,m) = (0,1),(1,0)$, whenever $E_2^{nm} \neq 0$. For any first quadrant spectral sequence $E_3^{01} = E_\infty^{01}$ and $E_2^{10} = E_\infty^{10}$. So, (a) and (b) hold.

(c) follows from (22). □

Lemma 5.2. *Let $p \geq 7$. For the simple G-module $L = L(\lambda)$ with $\lambda = \lambda_0 + p\lambda'$, $\lambda_0 \in X_1(T)$, $\lambda' \in X_+(T)$, one has*

(a) $E_2^{20} = E_\infty^{20}$;

(b) $E_2^{11} = E_\infty^{11}$;

(c) $E_2^{02} = E_\infty^{02}$.

(d)$H^2(G, L) = E_2^{20} \oplus E_2^{11} \oplus E_2^{02}$.

Доказательство. From the definition of E_3^{nm} it follows that $E_2^{nm} = E_3^{nm}$ if

$$E_2^{n-2,m+1} = E_2^{n+2,m-1} = 0, \quad \text{whenever } E_2^{nm} \neq 0. \tag{23}$$

If $E_2^{nm} \neq 0$, then $H^m(G^1, L) \neq 0$. Therefore, according to Propositions 4.2 and 4.3, (23) holds for all $(n,m) = (2,0), (1,1), (0,2)$.

Since $E_3^{20} = E_\infty^{20}$ and $E_3^{11} = E_\infty^{11}$ for any first quadrant spectral sequence (a) and (b) hold.

It remains to check that $E_3^{02} = E_4^{02}$. Since E_4^{02} is cohomology of

$$E_3^{-3,2} \to E_3^{02} \to E_3^{30},$$

it is enough to show that $E_2^{30} = 0$ whenever $E_2^{02} \neq 0$.

Suppose that $E_2^{02} \neq 0$ and $E_2^{30} \neq 0$. The latter implies $H^0(G^1, L(\lambda)) \neq 0$ so we may write $L(\lambda) = L(\lambda^{'})^{(1)}$. Then by Proposition 4.3,

$$E_2^{02} = H^0(G, H^2(G^1, k)^{(-1)} \otimes L(\lambda^{'})) = \mathrm{Hom}_G(L(\lambda_2), L(\lambda^{'})).$$

Hence $L(\lambda^{'}) \cong L(\lambda_2)$. But, then $E_2^{30} = H^3(G, H^0(G^1, k)^{(-1)} \otimes L(\lambda^{'})) = H^3(G, L(\lambda^{'})) = H^3(G, L(\lambda_2)) = 0$. Thus, $E_3^{02} = E_4^{02} = E_\infty^{02}$.

(d) follows from (22) and from the statements (a) –(c). □

Proposition 5.3. *Let $p \geq 7$ and $A(0) = \{\, p^i\omega_0(0),\ p^i\delta_3(0),\ p^i\omega_4(0) + p^{i+1}\lambda_1,$ $p^i\delta_1(0) + p^{i+1}\lambda_1, p^i\delta_0(0) + p^{i+1}\lambda_2,\ i \geq 0\}$. Then*

$$H^1(G, L(\lambda)) \cong \begin{cases} k, & \text{if } \lambda \in A(0); \\ 0, & \text{in other cases.} \end{cases}$$

Доказательство. Let $L = L(\lambda)$ and $\lambda = \lambda_0 + p\lambda^{'}$, where $\lambda_0 \in X_1(T)$, $\lambda^{'} \in X_+(T)$. By lemma 5.1(c), $H^1(G, L) = E_2^{01} \oplus E_2^{10}$. Suppose $E_2^{01} \neq 0$. Then by Proposition 4.2, we have $L(\lambda_0)^{G^1} = 0$ and so $E_2^{10} = 0$.

By (21)

$$E_2^{01} = H^0(G, H^1(G^1, L(\lambda))^{(-1)}) =$$
$$H^0(G, H^1(G^1, L(\lambda_0))^{(-1)} \otimes L(\lambda^{'})) = \mathrm{Hom}_G(H^1(G^1, L(\lambda_0))^{(-1)^*}, L(\lambda^{'})).$$

So, if $E_2^{01} \neq 0$, then $\mathrm{Hom}_G(H^1(G^1, L(\lambda_0))^{(-1)}, L(\lambda^{'})) \neq 0$.

Then using Proposition 4.2 we have that $H^1(G, L(\lambda)) = k$ if

$$\lambda \in B(0) = \{\omega_0(0), \delta_3(0), \omega_4(0) + p\lambda_1, \delta_1(0) + p\lambda_1, \delta_3(0) + p\lambda_2\}.$$

Now, we suppose that $E_2^{10} \neq 0$. According to (21),

$$E_2^{10} = H^1(G, H^0(G^1, L(\lambda))^{(-1)}) = H^1(G, H^0(G^1, L(\lambda_0))^{(-1)} \otimes L(\lambda^{'})) \neq 0.$$

From this it follows that $\lambda_0 = 0$, and $H^1(G, L(p\lambda^{'})) = E_2^{10} = H^1(G, L(\lambda^{'}))$. Since $E_2^{10} \neq 0$, then $\lambda^{'}$ must be one of $\lambda^{'} \in \{p^i\mu \mid \mu \in B(0), i > 0\}$. □

Remark. Proposition 5.3 improves the corresponding result in ([10], Theorem in p. 1945) to include the primes $p = 7$ and 11 if $\lambda = 0$.

We introduce a set $A(\lambda) \subset X_+(T)$ as defined in the following table.

Table 1

λ, p	$A(\lambda)$
0, $p \geq 7$	$p^i\omega_0(0), p^i\delta_3(0), p^i\omega_4(0) + p^{i+1}\lambda_1, p^i\delta_1(0) + p^{i+1}\lambda_1,$ $p^i\delta_3(0) + p^{i+1}\lambda_2,\ i \geq 0$
λ_1, $p > 7$	$\omega_0(\lambda_1), \delta_3(\lambda_1), \omega_4(\lambda_1) + p\lambda_1, \delta_1(\lambda_1) + p\lambda_1, \delta_3(\lambda_1) + p\lambda_2,\ \lambda_1 + p^i\omega_0(0),$ $\lambda_1 + p^i\delta_3(0),\ \lambda_1 + p^i\omega_4(0) + p^{i+1}\lambda_1,\ \lambda_1 + p^i\delta_1(0) + p^{i+1}\lambda_1,$ $\lambda_1 + p^i\delta_3(0) + p^{i+1}\lambda_2,\ i \geq 1$
λ_1, $p = 7$	$4\lambda_1 + 5\lambda_2,\ 11\lambda_1 + 2\lambda_2,\ 13\lambda_1 + 2\lambda_2,\ 4\lambda_1 + 12\lambda_2,\ \lambda_1 + p^i\omega_0(0),$ $\lambda_1 + p^i\delta_3(0),\ \lambda_1 + p^i\omega_4(0) + p^{i+1}\lambda_1,\ \lambda_1 + p^i\delta_1(0) + p^{i+1}\lambda_1,$ $\lambda_1 + p^i\delta_3(0) + p^{i+1}\lambda_2,\ i \geq 1$
λ_2, $p > 7$	$\omega_0(\lambda_2), \delta_3(\lambda_2), \omega_4(\lambda_2) + p\lambda_1, \delta_1(\lambda_2) + p\lambda_1, \delta_3(\lambda_2) + p\lambda_2,\ \lambda_2 + p^i\omega_0(0),$ $\lambda_2 + p^i\delta_3(0),\ \lambda_2 + p^i\omega_4(0) + p^{i+1}\lambda_1,\ \lambda_2 + p^i\delta_1(0) + p^{i+1}\lambda_1,$ $\lambda_2 + p^i\delta_3(0) + p^{i+1}\lambda_2,\ i \geq 1$
λ_2, $p = 7$	$5\lambda_1 + 4\lambda_2,\ 6\lambda_1 + 4\lambda_2,\ 6\lambda_1 + 11\lambda_2,\ 12\lambda_1 + 2\lambda_2,\ \lambda_2 + p^i\omega_0(0),$ $\lambda_2 + p^i\delta_3(0),\ \lambda_2 + p^i\omega_4(0) + p^{i+1}\lambda_1,\ \lambda_2 + p^i\delta_1(0) + p^{i+1}\lambda_1,$ $\lambda_2 + p^i\delta_3(0) + p^{i+1}\lambda_2,\ i \geq 1$

Proposition 5.4. *Let $p \geq 7$, $\lambda \in \{0, \lambda_1, \lambda_2\}$ and $\mu \in X_+(T)$. Then*

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \begin{cases} k, & \text{if } \mu \in A(\lambda); \\ 0, & \text{in other cases.} \end{cases}$$

Доказательство. Since $\mathrm{Ext}^1_G(L(0), L(\mu)) = H^1(G, L(\mu))$ then the part of the claim respect to $\mathrm{Ext}^1_G(L(0), L(\mu))$ follows from Proposition 5.3.

Now, for $\lambda \in \{\lambda_1, \lambda_2\}$ we use the following well-known facts.

For the decompositions $\lambda = \lambda_0 + p\lambda'$, $\mu = \mu_0 + p\mu'$, $\lambda_0, \mu_0 \in X_1(T)$, $\lambda', \mu' \in X_+(T)$ and for $p > 2$ one has ([2], Theorem 5.2; [14],(1.4))

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong$$

$$(24) \qquad \begin{cases} \mathrm{Hom}_G(L(\mu'), \mathrm{Ext}^1_{G_1}(L(\lambda_0), L(\mu_0))^{(-1)} \otimes L(\lambda')); \text{ if } \lambda_0 \neq \mu_0; \\ \mathrm{Ext}^1_G(L(\lambda'), L(\mu')) \text{ if } \lambda_0 = \mu_0. \end{cases}$$

Moreover, for $p \geq 15$,

$$(25) \qquad [\mathrm{Ext}^1_{G_1}(L(\lambda_0), L(\mu_0)) : L(\nu)^{(1)}]_G = \dim \mathrm{Ext}^1_G(L(\lambda_0), L(\mu_0 + p\nu)),$$

where ν satisfies the condition $\mu_0 + p\nu \leq 2p\rho + w_0 \cdot \lambda_0$.

For $\lambda_0 \in \{\lambda_1, \lambda_2\}$, $\langle \lambda_0 + \rho, \widetilde{\alpha}_0^\vee \rangle \leq 13$, and then the Jantzen's generic decomposition pattern condition $\langle \lambda_0 + \rho, \widetilde{\alpha}_0^\vee \rangle \leq p(p-h+2)$ is satisfied if $p \geq 7$. Therefore each non-vanishing extension $\mathrm{Ext}_{G^1}(L(\lambda_0), L(\mu_0))$ is a completely reducible as G-module. So, (25) is valid for $p \geq 7$.

Provided $\lambda \not\leq \mu$, we have ([4],3.11 b))

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(L(\lambda), M(\mu)).$$

Let $p > 7$. From the linkage principle for G and Lemma 4.1 follows that $\mathrm{Ext}^1_G(L(\lambda_0), L(\mu_0 + p\nu)) \neq 0$ only in the following five cases:

$\mu_0 + p\nu = \omega_3(\lambda_0)$,

$\mu_0 + p\nu = \delta_3(\lambda_0)$,

$\mu_0 + p\nu = w^5 \cdot \omega_4(\lambda_0) = \omega_4(\lambda_0) + p\lambda_1$,

$\mu_0 + p\nu = w^5 \cdot \delta_1(\lambda_0) = \delta_1(\lambda_0) + p\lambda_1$,

$\mu_0 + p\nu = (w^4)^2 \cdot \delta_3(\lambda_0) = \delta_3(\lambda_0) + p\lambda_2$.

If $p = 7$ then $\lambda_1 \in \overline{C}_1$ and $\lambda_2 \notin \overline{C}_1$. Therefore in this case we can't use the results of Lemma 4.1. The additional calculations show that $L(\lambda_0)$ occurs in the socle of $M(\mu_0 + p\nu)$ only in the following cases:

$$\mu_0 + p\nu = 4\lambda_1 + 5\lambda_2,\ 11\lambda_1 + 2\lambda_2,\ 13\lambda_1 + 2\lambda_2,\ 4\lambda_1 + 12\lambda_2$$

for $\lambda_0 = \lambda_1$, and

$$\mu_0 + p\nu = 5\lambda_1 + 4\lambda_2,\ 6\lambda_1 + 4\lambda_2,\ 6\lambda_1 + 11\lambda_2,\ 12\lambda_1 + 2\lambda_2$$

for $\lambda_0 = \lambda_2$.

Finally, using (24)and (25) we get all non-vanishing groups $\mathrm{Ext}_G(L(\lambda), L(\mu))$ with $\lambda \in \{\lambda_1, \lambda_2\}$ as claimed. □

Remark. Proposition 5.4 improves the corresponding result in ([10], Theorem in p. 1945) to include the primes $p = 7$ and 11 if $\lambda \in \{0,\ \lambda_1,\ \lambda_2\}$.

6. The main theorem

Define the following sets of simple G-modules:

$M_1 = \{L(\lambda_0) \otimes L(\lambda^{'})^{(1)} \mid H^1(G^1, L(\lambda_0))^{(-1)} \neq 0$ and $\mathrm{Ext}^1_G(H^1(G^1, L(\lambda_0))^{(-1)^*}, L(\lambda^{'})) \neq 0\}$,

$M_2 = \{L(\lambda_0) \otimes L(\lambda^{'})^{(1)} \mid H^2(G^1, L(\lambda_0))^{(-1)} \neq 0$ and $\mathrm{Hom}_G(H^2(G^1, L(\lambda_0))^{(-1)^*}, L(\lambda^{'})) \neq 0\}$.

Here, the set M_1 is determined explicitly by Proposition 4.2 and 5.4 and the set M_2 is determined explicitly by Proposition 4.3 and Lemma 4.1.

The main result of this paper is

Theorem 6.1. *Let G be the simple, simply connected algebraic group of type G_2 over an algebraically closed field k of characteristic $p \geq 7$, and $L(\lambda)^{(d)}$ any Frobenius twist ($d \geq 0$) of the simple G-module $L(\lambda)$ with highest weight λ. Then*

$$H^2(G, L(\lambda)^{(d)}) \cong \begin{cases} k, & \text{if } L(\lambda) \in \bigcup_{i=1}^2 M_i \setminus \{L(\beta_1(0))\}; \\ k \oplus k, & \text{if } L(\lambda) = L(\beta_1(0)); \\ 0, & \text{in other cases.} \end{cases}$$

Доказательство. Let $L = L(\lambda)$, where $\lambda = \lambda_0 + p\lambda^{'}$ with $\lambda_0 \in X_1(T)$, $\lambda^{'} \in X_+(T)$. By Lemma 5.2(d), $H^2(G, L) = E_2^{20} \oplus E_2^{11} \oplus E_2^{02}$. Suppose $H^2(G, L) \neq 0$.

First assume that $E_2^{11} \neq 0$. We will show that $E_2^{20} = E_2^{02} = 0$. Since

$$E_2^{11} = \mathrm{Ext}^1_G(H^1(G^1, L(\lambda_0))^{(-1)^*}, L(\lambda^{'})) \tag{26}$$

and $H^1(G^1, L) \neq 0$ by assumption, then from Propositions 4.2 and 5.4 it follows that $\lambda^{'}$ must be one of the weights listed in the table 1, and λ_0 must be one of $\omega_0(0), \omega_4(0), \delta_1(0), \delta_3(0)$. In this case

$$E_2^{20} = H^2(G, H^0(G^1, L(\lambda_0))^{(-1)} \otimes L(\lambda^{'})) = \begin{cases} H^2(G, L(\lambda^{'})), \text{ if } \lambda_0 = 0, \\ 0, \text{ in other cases.} \end{cases} \tag{27}$$

Since $\lambda_0 \neq 0$, we have $E_2^{20} = 0$.

Similarly, by Proposition 4.3, $E_2^{02} = 0$.

Thus if $E_2^{11} \neq 0$, then $H^2(G, L) \cong k$. Using (26) and Propositions 4.2, 5.4, we get the simple G-modules with non-vanishing second cohomology group. They are coincide with the set M_1.

Now, let $E_2^{11} = 0$ and $E_2^{02} \neq 0$. Then from

$$E_2^{02} = \mathrm{Hom}_G(H^2(G^1, L(\lambda_0))^{(-1)^*}, L(\lambda^{'})) \tag{28}$$

and from Proposition 4.3 it follows that

$$\lambda_0 \in \{0, \omega_1(0),\ \omega_3(0),\ \delta_0(0),\ \delta_2(0),\ \beta_1(0)\}.$$

If $\lambda_0 \neq 0$, then by (27) $E_2^{20} = 0$. Suppose $\lambda_0 = 0$, then by (28), $L(\lambda^{'}) \cong L(\lambda_2)$ and, hence, $E_2^{20} = H^2(G, L(\lambda_2)) = H^1(G, M(\lambda_2)) = 0$. Therefore $H^2(G, L) = E_2^{02} \cong k$, except the case when $\lambda = \beta_1(0)$. In this case $H^2(G, L) = E_2^{02} \cong \mathrm{Hom}_G(H^2(G^1, L(\beta_1(0)))^{(-1)^*}, k) \cong k \oplus k$. Thus we have proved that $\lambda \in M_1 \cup M_2$ as required.

Finally, let $E_2^{11} = E_2^{02} = 0$ and $E_2^{20} \neq 0$. By (27) $\lambda_0 = 0$, and $E_2^{20} = H^2(G, L(\lambda^{'}))$. Thus

$$H^2(G, L(\lambda)) = H^2(G, L(\lambda^{'})^{(1)}),$$

and $L(\lambda)$ is the Frobenius twist of one of the modules listed in $\bigcup_{i=1}^{2} M_i$ with $H^2(G, L(\lambda)) \neq 0$. Proof of Theorem 6.1 is finished. □

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