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ON s-SEMIPERMUTABLE AND WEAKLY s-PERMUTABLE SUBGROUPS OF FINITE GROUPS

LI CHANGWEN

Abstract. Let H be a subgroup of a finite group G. H is said to be s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$; H is called weakly s-permutable in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. We fix in every non-cyclic Sylow subgroup P of G a subgroup D with $1 < |D| < |P|$ and study the structure of G under the assumption that every subgroup H of P with $|H| = |D|$ is either s-semipermutable or weakly s-permutable in G. Some recent results are generalized and unified.

Keywords: s-semipermutable; weakly s-permutable; p-nilpotent; the generalized Fitting subgroup.

1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation, as in Huppert [1]. G denotes always a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$ and G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$.

Let $\mathfrak{F}$ be a class of groups. We call $\mathfrak{F}$ a formation provided that (i) if $G \in \mathfrak{F}$ and $H \trianglelefteq G$, then $G/H \in \mathfrak{F}$, and (ii) if G/M and G/N are in $\mathfrak{F}$, then $G/(M \cap N)$ is in $\mathfrak{F}$ for any normal subgroups M, N of G. A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. In this paper, the symbol $\mathfrak{U}$ denotes the class

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of all supersolvable groups. Clearly, $\mathfrak{U}$ is a saturated formation ([1], p. 713, Satz 8.6).

Two subgroups H and K of G are said to be permutable if $HK = KH$. A subgroup H of G is said to be s-permutable (or s-quasinormal, π-quasinormal) in G [2] if H permutes with every Sylow subgroup of G; H is said to be s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$; H is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. It is clear that both s-semipermutable subgroup and weakly s-permutable subgroup are the generalization of s-permutability. There are examples showing that there exist weakly s-permutable subgroups that are not s-semipermutable, and in general the converse is also false. The aim of this article is to unify and improve some earlier results using s-semipermutable and weakly s-permutable subgroups.

In the literature, usually there are the assumptions on either the minimal subgroups (and cyclic subgroups of order 4 when $p = 2$) or the maximal subgroups of some kinds of subgroups of G when investigating the structure of G, such as in [3]-[5], etc. In the nice paper [6], Skiba gave the following unified result.

Theorem A. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and let G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ possesses a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P either of order $|H| = |D|$ or of order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathfrak{F}$.*

In this paper, we extend Theorem A in the following way.

Theorem B. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and let G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ possesses a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P either of order $|H| = |D|$ or of order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or weakly s-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathfrak{F}$.*

The following result relating p-nilpotency of a group is the main technical tool in the proof of Theorem B.

Theorem C. (i.e., Theorem 3.2) *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If P possesses a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P either of order $|H| = |D|$ or of order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or weakly s-permutable in G, then G is p-nilpotent.*

2. PRELIMINARIES

Lemma 2.1. *Suppose that H is an s-semipermutable subgroup of a group G and N is a normal subgroup of G. Then*

(a) *H is s-semipermutable in K whenever $H \leq K \leq G$.*

(b) *If H is p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*

(c) *If $H \leq O_p(G)$, then H is s-permutable in G.*

Proof. (a) is [3, Property 1], (b) is [3, Property 2], and (c) is [3, Lemma 3].

Lemma 2.2. ([6], Lemma 2.10) *Let H be a weakly s-permutable subgroup of a group G.*

(a) *If $H \leq K \leq G$, then H is weakly s-permutable in K.*

(b) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is weakly s-permutable in G/N.*

(c) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-permutable in G/N.*

(d) *Suppose H is a p-group for some prime p and H is not s-permutable in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.*

Lemma 2.3. ([6], Lemma 2.11) *Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly s-permutable in G. Then some maximal subgroup of N is normal in G.*

Lemma 2.4. *Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is s-semipermutable in G. Then some maximal subgroup of N is normal in G.*

Proof. By Lemma 2.3 and Lemma 2.1(c).

Lemma 2.5. ([1], III, 5.2 and IV, 5.4) *Suppose G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then*

(a) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*

(b) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(c) *The exponent of P is p or 4.*

Lemma 2.6. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every subgroup of prime order or order 4(when P is a nonabelian 2-group) of P is either s-semipermutable or weakly s-permutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. By Lemma 2.1 and 2.2, it is easy to see that G is a minimal non-p-nilpotent group. By Lemma 2.5, $G = P \rtimes Q$. Let $x \in P$. Then the order of x is p or 4. By the hypothesis, $< x >$ is either s-semipermutable or weakly s-permutable in G. If $< x >$ is s-semipermutable in G, then $< x >$ is s-permutable in G by

Lemma 2.1. If $< x >$ is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G =< x > T$ and $< x > \cap T \leq< x >_{sG}$. Hence $P = P \cap G = P\cap < x > T =< x > (P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$. If $P \cap T \leq \Phi(P)$, then $< x >= P \trianglelefteq G$. It follows that G is p-nilpotent, a contraction. If $P = P \cap T$, then $T = G$ and so $< x >=< x >_{sG}$ is s-permutable in G. For any element x in P, now we have $< x > Q$ is a proper subgroup of G, then $< x > Q =< x > \times Q$. This implies that $G = P \times Q$, a contradiction.

Lemma 2.7. ([7], A, 1.2) *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*
(a) $U \cap VW = (U \cap V)(U \cap W)$.
(b) $UV \cap UW = U(V \cap W)$.

Lemma 2.8. ([4], Theorem 3.3) *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divising $|G|$. If every maximal subgroup of P is s-semipermutable in G, then G is p-nilpotent.*

Lemma 2.9. ([8], Lemma 2.2.) *If P is an s-permutable p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.10. *Let G be a group and $N \trianglelefteq G$.*
(a) *If $N \trianglelefteq G$, then $F^*(N) \leq F^*(G)$.*
(b) *If $G \neq 1$, then $F^*(G) \neq 1$. In fact, $F^*(G)/F(G) = Soc\,(F(G)C_G(F(G))/F(G))$.*
(c) *$F^*(F^*(G)) = F^*(G) \geq F(G)$. If $F^*(G)$ is Solvable, then $F^*(G) = F(G)$.*

3. MAIN RESULTS

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is either s-semipermutable or weakly s-permutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G has a unique minimal normal subgroup N and G/N is p-nilpotent. Moreover $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. Consider G/N. We will show that G/N satisfies the hypothesis of the theorem. Let M/N be a maximal subgroup of PN/N. It is easy to see $M = P_1N$ for some maximal subgroup P_1 of P. It follows that $P_1 \cap N = P \cap N$ is a Sylow p-subgroup of N. If P_1 is s-semipermutable in G, then M/N is s-semipermutable in G/N by Lemma 2.1. If P_1 is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG}$. So $G/N = M/N \cdot TN/N = P_1N/N \cdot TN/N$. Since $(|N : P_1 \cap N|, |N : T \cap N|) = 1$, we have $(P_1 \cap N)(T \cap N) = N = N \cap G = N \cap P_1T$. By Lemma 2.7, $(P_1N) \cap (TN) = (P_1 \cap T)N$. It follows that $(P_1N/N) \cap (TN/N) = (P_1N \cap TN)/N = (P_1 \cap T)N/N \leq (P_1)_{sG}N/N \leq (P_1N/N)_{sG}$. Hence M/N is weakly s-permutable in G/N. Therefore, G/N satisfies the hypothesis of the theorem.

The choice of G yields that G/N is p-nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by step (1). since $G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$ is p-nilpotent, hence G is p-nilpotent, a contradiction.

(3) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (1) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G)\cap M$ is normalized by N and M, $O_p(G)\cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P\cap M)$. Furthermore $P\cap M < P$, thus there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Hence $P = NP_1$. By the hypothesis, P_1 is either s-semipermutable or weakly s-permutable in G. If we assume P_1 is s-semipermutable in G, then P_1M_q is a group for $q \neq p$. Hence $P_1 < M_p, M_q | q \in \pi(M), q \neq p >= P_1M$ is a group. Then $P_1M = M$ or G by maximality of M. If $P_1M = G$, then $P = P \cap P_1M = P_1(P \cap M) = P_1$, a contradiction. If $P_1M = M$, then $P_1 \leq M$. Therefore, $P_1 \cap N = 1$ and N is of prime order. Then the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction. Therefore we may assume P_1 is weakly s-permutable in G. Then there is a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = N \leq O^p(G)$ since N is the unique minimal normal subgroup of G. Since $|G : T|$ is a number of p-power, $O^p(G) \leq T$. Hence, $P_1 \cap T \leq (P_1)_{sG} \leq O^p(G) \cap P_1 \leq T \cap P_1$, and so $P_1 \cap T = (P_1)_{sG} = O^p(G) \cap P_1$. Consequently, $G = PO^p(G)$ implies that $(P_1)_{sG} \trianglelefteq G$ by Lemma 2.9. By the minimality of N, we have $(P_1)_{sG} = N$ or $(P_1)_{sG} = 1$. If $(P_1)_{sG} = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. Thus $P_1 \cap T = (P_1)_{sG} = 1$, and so $|T|_p = p$. Then T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'} \lhd \lhd G$ and $T_{p'}$ is a p'-Hall subgroup of G. It follows that $T_{p'}$ is the normal p-complement of G, a contradiction.

(4) The final contradiction.

If P has a maximal subgroup P_1 which is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G = P_1T$ and $P_1\cap T \leq (P_1)_{sG} \leq O_p(G) = 1$. Then $P_1 \cap T = 1$. Hence $|T|_p = p$. Therefore, T is p-nilpotent. Thus G is p-nilpotent, a contradiction. Now we may assume that all maximal subgroups of P are s-semipermutable in G. Then G is p-nilpotent by Lemma 2.8, a contradiction.

Theorem 3.2. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If P possesses a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P either of order $|H| = |D|$ or of order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either s-semipermutable or weakly s-permutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, Lemma 2.1 and 2.2 guarantee that $G/O_{p'}(G) \in C$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

(2) $|D| > p$.
By Lemma 2.6.

(3) $|P : D| > p$.
By Theorem 3.1.

(4) P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is s-semipermutable in G.

Assume that $H \leq P$ such that $|H| = |D|$ and H is weakly s-permutable in G. Then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap K \leq H_{sG}$. By Lemma 2.2, we may assume G has a normal subgroup M such that $|G : M| = p$ and $G = HM$. Since $|P : D| > p$ by Step (3), M satisfies the hypotheses of the theorem. The choice of G yields that M is p-nilpotent. It is easy to see that G is p-nilpotent, contrary to the choice of G.

(5) If $N \leq P$ and N is minimal normal in G, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 2.4, N has a maximal subgroup which is normal in G, contrary to the minimality of N.

(6) Suppose that $N \leq P$ and N is minimal normal in G. Then G/N is p-nilpotent.

If $|N| < |D|$, G/N satisfies the hypotheses of the theorem by Lemma 2.1. Thus G/N is p-nilpotent by the minimal choice of G. So we may suppose that $|N| = |D|$ by Step (5). We will show that every cyclic subgroup of P/N of order p or order 4(when P/N is a non-abelian 2-group) is s-semipermutable in G/N. Let $K \leq P$ and $|K/N| = p$. By Step (2), N is non-cyclic, so are all subgroups containing N. Hence there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course, $|N| = |D| = |L|$. Since L is s-semipermutable in G by the hypotheses, $K/N = LN/N$ is s-semipermutable in G/N by Lemma 2.1 (b). If $p = 2$ and P/N is non-abelian, take a cyclic subgroup X/N of P/N of order 4. Let K/N be maximal in X/N. Then K is maximal in X and $|K/N| = 2$. Since X is non-cyclic and X/N is cyclic, there is a maximal subgroup L of X such that N is not contained in L. Thus $X = LN$ and $|L| = |K| = 2|D|$. By the hypotheses, L is s-semipermutable in G. By Lemma 2.1, $X/N = LN/N$ is s-semipermutable in G/N. Hence G/N satisfies the hypotheses. By the minimal choice of G, G/N is p-nilpotent.

(7) $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By Step (6), G/N is p-nilpotent. It is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. Hence $O_p(G)$ is an elementary abelian p-group. On the other hand, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that

$O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is p-nilpotent. Then G can be written as $G = N(M \cap P)M_{p'}$, where $M_{p'}$ is the normal p-complement of M. Pick a maximal subgroup S of $M_p = P \cap M$. Then $NSM_{p'}$ is a subgroup of G with index p. Since p is the minimal prime in $\pi(G)$, we know that $NSM_{p'}$ is normal in G. Now by step (3) and the induction, we have $NSM_{p'}$ is p-nilpotent. Therefore, G is p-nilpotent, a contradiction.

(8) The minimal normal subgroup L of G is not p-nilpotent.

If L is p-nilpotent, then it follows from the fact that $L_{p'}$ char $L \trianglelefteq G$ that $L_{p'} \leq O_{p'}(G) = 1$. Thus L is a p-group. Then $L \leq O_p(G) = 1$ by step (7), a contradiction.

(9) G is a non-abelian simple group.

Suppose that G is not a simple group. Take a minimal normal subgroup L of G. Then $L < G$. If $|L|_p > |D|$, then L is p-nilpotent by the minimal choice of G, contrary to step (8). If $|L|_p \leq |D|$. Take $P_* \geq L \cap P$ such that $|P_*| = p|D|$. Hence P_* is a Sylow p-subgroup of P_*L. Since every maximal subgroup of P_* is of order $|D|$, every maximal subgroup of P_* is s-semipermutable in G by hypotheses, thus in P_*L by Lemma 2.1. Now applying Theorem 3.1, we get P_*L is p-nilpotent. Therefore, L is p-nilpotent, contrary to step (8).

(10) The final contradiction.

Suppose that H is a subgroup of P with $|H| = |D|$ and Q is a Sylow q-subgroup with $q \neq p$. Then $HQ^g = Q^gH$ for any $g \in G$ by the hypotheses that H is s-semipermutable in G. Since G is simple by step (9), $G = HQ$ from [1, VI, 4.10], the final contradiction.

Corollary 3.3. *Suppose that G is a group. If every non-cyclic Sylow subgroup P of G possesses a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P either of order $|H| = |D|$ or of order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either s-semipermutable or weakly s-permutable in G, then G has a Sylow tower of supersolvable type.*

Theorem 3.4. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and let G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of E possesses a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P either of order $|H| = |D|$ or of order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either s-semipermutable or weakly s-permutable in G. Then $G \in \mathfrak{F}$.*

Proof. Since P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either s-semipermutable or weakly s-permutable in G by hypotheses, thus in E by Lemma 2.1 and 2.2. Applying Corollary 3.3, we conclude that E has a Sylow tower of supersolvable type. Let q be the maximal prime divisor of $|E|$ and Q a Sylow q-subgroup of E. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathfrak{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \leq O_q(G)$, H is either s-permutable or weakly s-permutable in G by Lemma 2.1. Since s-permutability implies weakly s-permutability and $F^*(Q) = Q$ by Lemma 2.10, we get $G \in \mathfrak{F}$ by applying Theorem A.

Proof of Theorem B: We distinguish two cases:

Case 1. $\mathfrak{F} = \mathfrak{U}$.

Let G be a minimal counterexample.

(1) Every proper normal subgroup N of G containing $F^*(E)$ (if it exists) is supersolvable.

If N is a proper normal subgroup of G containing $F^*(E)$, then $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 2.10, $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup P of $F^*(E \cap N) = F^*(E)$, P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either s-semipermutable or weakly s-permutable in G by hypotheses, thus in N by Lemma 2.1 and 2.2. So N and $N \cap H$ satisfy the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable.

(2) $E = G$.

If $E < G$, then $E \in \mathfrak{U}$ by Step (1). Hence $F^*(E) = F(E)$ by Lemma 2.10. It follows that every Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 2.1 and 2.2, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is weakly s-permutable in G. Applying Theorem A for the special case $\mathfrak{F} = \mathfrak{U}$, $G \in \mathfrak{U}$, a contradiction.

(3) $F^*(G) = F(G) < G$.

If $F^*(G) = G$, then $G \in \mathfrak{U}$ by Theorem 3.4, contrary to the choice of G. So $F^*(G) < G$. By Step (1), $F^*(G) \in \mathfrak{U}$ and $F^*(G) = F(G)$ by Lemma 2.10.

(4) The final contradiction.

Since $F^*(G) = F(G)$, each non-cyclic Sylow subgroup of $F^*(G)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is weakly s-permutable in G by Lemma 2.1. Applying Theorem A, $G \in \mathfrak{U}$, a contradiction.

Case 2. $\mathfrak{F} \neq \mathfrak{U}$.

By hypotheses, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either s-semipermutable or weakly s-permutable in G, thus in E by Lemma 2.1 and 2.2. Applying Case 1, $E \in \mathfrak{U}$. Then $F^*(E) = F(E)$ by Lemma 2.10. It follows that each Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 2.1, each non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is weakly s-permutable in G. Applying Theorem A, $G \in \mathfrak{F}$.

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Li Changwen
School of Mathematical Science,
Xuzhou Normal University,
630090, Xuzhou, 221116, China
E-mail address: lcwxz@xznu.edu.cn