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ON THE STRUCTURE OF PICARD GROUP FOR MOEBIUS LADDER

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ABSTRACT. The notion of the Picard group of a graph (also known as Jacobian group, sandpile group, critical group) was independently given by many authors. This is a very important algebraic invariant of a finite graph. In particular, the order of the Picard group coinsides with the number of spanning trees for a graph. The latter number is known for the simplest families of graphs such as Wheel, Fan, Prism, Ladder and Moebius ladder graphs. At the same time the structure of the Picard group is known only in several cases. The aim of this paper is to determine the structure of the Picard group of the Moebius ladder graphs.

Keywords: Graph, Picard group, Abelian group, Chebyshev polynomial.

1. Introduction

We define a Meobius ladder M_n of order n as the circulant graph $C_{2n}(1, n)$. In this case, M_n can be considered as a regular $2n$-gon whose n pairs of opposite vertices are joint by an edge. One can also realize M_n as a ladder with n steps on the Moebius band.

The aim of the present paper is to find the structure of the Picard group of the Moebius ladder M_n.

The notion of the Picard group of a graph (also known as Jacobian group, sandpile group, critical group) was independently given by many authors ([1], [2], [3], [4]). This is a very important algebraic invariant of a finite graph. In particular, the order of the Picard group coinsides with the number of spanning trees for a

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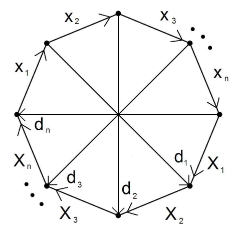

FIGURE 1. Moebius Ladder

graph. The latter number is known for the simplest families of graphs such as Wheel, Fan, Prism, Ladder and Moebius ladder graphs [5]. At the same time the structure of the Picard group is known only in several cases (see [6] for references).

Following Baker-Norine [2] we define the the Picard group (or the Jacobian) of a graph as follows.

Let G be a graph. Throughout this paper we suppose that G is finite, connected multigraph without loops. Let $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. Denote by $Div(G)$ a free Abelian group on $V(G)$. We refer to elements of $Div(G)$ as divisors on G. Each element $D \in Div(G)$ can be uniquely presented as $D = \sum_{x \in V(G)} D(x)(x)$, $D(x) \in \mathbb{Z}$. We define the degree of D by the formula $deg(D) = \sum_{x \in V(G)} D(x)$. Denote by $Div^0(G)$ the subgroup of $Div(G)$ consisting of divisors of degree zero.

Let f be a $\mathbb{Z}$-valued function on $V(G)$. We define the divisor of f by the formula

$$div(f) = \sum_{x \in V(G)} \sum_{xy \in E(G)} (f(x) - f(y))(x).$$

The divisor $div(f)$ can be naturally identified with the graph-theoretic Laplacian of f. Divisors of the form $div(f)$, where f is a $\mathbb{Z}$-valued function on $V(G)$, are called principal divisors. Denote by $Prin(G)$ the group of principal divisors of G. It is easy to see that every principal divisor has a degree zero, so that $Prin(G)$ is a subgroup of $Div^0(G)$.

The Picard group (or Jacobian) of G is defined to be quotient group

$$Jac(G) = Div^0(G)/Prin(G).$$

By making use of the Kirchhoff Matrix-Tree theorem [7] one can show that $Jac(G)$ is a finite Abelian group of order t_G, where t_G is number of spanning trees of G. Moreover, any finite Abelian group is the Picard group of some graph.

For a fixed base point $x_0 \in V(G)$ we define the Abel-Jacobi map $S_{x_0} : G \to Jac(G)$ by the formula $S_{x_0}(x) = [(x) - (x_0)]$, where $[d]$ is an equivalence class of divisor d.

We endow each edge of G by two possible orientations. Since G has no loops it is well-defined procedure. Let $\vec{E} = \vec{E}(G)$ be the set of oriented edges of G. For $e \in \vec{E}$ we denote initial vertex $o(e)$ and terminus vertex $t(e)$, respectively. We define the flow of e by the formula $\omega(e) = [t(e) - o(e)]$. We note that

$$\omega(e) = [(t(e) - x_0) - (o(e) - x_0)] = [[t(e) - x_0] - [o(e) - x_0]] = S_{x_0}(t(e)) - S_{x_0}(o(e))$$

does not depend of the choice of initial point x_0. By virtue of Lemma 1.8 in [2] (see also [4]) the Picard group $Jac(G)$ is an Abelian group generated by flows $\omega(e), e \in \vec{E}$, whose defining relations are given by the following two Kirchhoff laws.

(I) The flow through each vertex of G is equal to zero. It means that

$$\sum_{e\in\vec{E},t(e)=x} \omega(e) = 0 \text{ for all } x \in V(G).$$

(II) The flow along each closed orientable walk W in G is equal to zero. That is

$$\sum_{e\in W} \omega(e) = 0.$$

Recall that the closed orientable walk in G is a sequence of orientable edges $e_i \in \vec{E}(G)$, $i = 1, \ldots, n$ such that $t(e_i) = o(e_{i+1})$ for $i = 1, \ldots, n-1$ and $t(e_n) = o(e_1)$.

2. Preliminary

Let $a_1, a_2, \ldots, a_m \in \mathbb{Z}$. Denote by $GCD(a_1, a_2, \ldots, a_m) = (a_1, a_2, \ldots, a_m)$ the greatest common divisor of $a_1, a_2, \ldots, a_m$ in the ring of integers $\mathbb{Z}$. We will use the following evident properties of GCD.

(i) $(a, a+b) = (a, b) = (a, a-b)$,
(ii) $(a, b, c) = (a, (b, c))$,
(iii) $(k\,a, k\,b) = k(a, b)$.

The Chebyshev polynomials of the first and of the second kind are defined by

$$T_n(x) = \cos(n \arccos(x)) \text{ and } U_{n-1}(x) = \sin(n \arccos(x))/\sin(\arccos(x)),$$

respectively. Recall the following basic properties of these polynomials.

1° $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, $T_0(x) = 1$, $T_1(x) = x$,
2° $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, $U_0(x) = 1$, $U_1(x) = 2x$.

In this paper we will be mainly interesting in particular values of Chebyshev polynomials at the point $x = 2$. In this case $T_n(2) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2$ and $U_{n-1}(2) = ((2+\sqrt{3})^n - (2-\sqrt{3})^n)/(2\sqrt{3})$.

We will use the following version of the fundamental theorem of finite Abelian group (see, for instance [8], p. 344).

Theorem A. Let $\mathcal{A}$ be a finite Abelian group generated by $x_1, x_2, \ldots, x_n$ and satisfying the system of relations

$$\sum_{j=1}^{n} a_{ij}x_j = 0,\ i = 1, \ldots, m,$$

where $A = \{a_{ij}\}$ is an integer $m \times n$ matrix. Set $d_j,\ j = 1, \ldots, r = \min(n, m)$, for the greatest common divisor of all $j \times j$ minors of A. Then,

$$\mathcal{A} \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2/d_1} \oplus \mathbb{Z}_{d_3/d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r/d_{r-1}}.$$

The latter decomposition is known as the Smith Normal Form. See ([9], Ch. 3.22) for details of calculations.

3. Main result

Consider the Moebius ladder M_n as graph shown on the Fig. 1 with vertices labeled by $1, 2, \dots, 2n$. Denote by d_i, $i = 1, \dots, n$ the flow along orientable edge $(i,\, i{+}n)$ with initial vertex i and terminal vertex $i{+}n$. We also denote by x_i and X_i, $i = 1, \dots, n$ the flows along orientable edges $(i-1,\, i)$ and $(n+i-1,\, n+i)$, respectively. For simplicity, we identify vertices 0 and $2n$.

By the first Kirchhoff law we have the following equations

$$\text{(1)} \qquad \begin{cases} d_i = x_i - x_{i+1},\ i = 1, \dots, n-1,\ d_n = x_n - X_1, \\ d_i = X_{i+1} - X_i,\ i = 1, \dots, n-1,\ d_n = x_1 - X_n. \end{cases}$$

Applying the second Kirchhoff law for the closed walks $W_i = (i, n+i, n+i+1, i+1)$ we get the following system of equations

$$\text{(2)} \qquad \begin{cases} x_{i+1} + d_{i+1} - X_{i+1} - d_i = 0,\ i = 1, \dots, n-1 \\ X_1 - d_1 - x_1 - d_n = 0. \end{cases}$$

One can easy check that the last equation in system (2) is a consequence of the first $n-1$ equations.

Excluding d_i from system (1) and putting them in (2) we get the following relations between X_i and x_i.

$$\text{(3)} \qquad \begin{pmatrix} X_1 \\ X_2 \\ \dots \\ X_{n-1} \\ X_n \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 & \dots & -1 & -1 & -1 \\ -1 & 3 & -1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 3 & -1 \\ 2 & 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_{n-1} \\ x_n \end{pmatrix}.$$

Substituting these identities into system (1) we obtain that the Picard group $Jac(M_n)$ is an Abelian group generated by $x_1, x_2, \dots, x_n$ satisfying the following relations

$$\text{(4)} \qquad \begin{pmatrix} 1 & -5 & 5 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -5 & 5 & -1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -5 & 5 & -1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 2 & -9 & 6 \\ 6 & -9 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & \dots & 1 & 0 & 6 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} = 0.$$

Now we reduce the number of generators of the group $Jac(M_n)$ from n to 3. Namely, we will show that the group $Jac(M_n)$ is generated by x_1, x_2, x_3 satisfying three linear equations. To do this we note that the generators $x_1, x_2, \dots, x_n$ satisfy the following recursive relation.

$$x_j - 5x_{j+1} + 5x_{j+2} - x_{j+3} = 0,\ j = 1, 2, \dots, n-3.$$

The characteristic polynomial for the above equation is

$$1 - 5q + 5q^2 - q^3 = 0.$$

The roots of this polynomial are $q_1 = 1$, $q_{2,3} = 2 \pm \sqrt{3}$. Hence, the general solution of recursion is given by $x_j = C_1 + C_2 q^j + C_3 q^{-j}$, where $q = 2 + \sqrt{3}$ and C_1, C_2, C_3 are constants dependant only of initial values x_1, x_2, x_3. As a result, we obtained $x_4, x_5, \ldots, x_n$ as linear combinations of x_1, x_2 and x_3 whose coefficients can be found explicitly. Substituting the obtained equations into the last three lines of system (4) we get

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0. \end{cases} \tag{5}$$

We note that $T_n(2) = (q^n + q^{-n})/2$ and $U_{n-1}(2) = (q^n - q^{-n})/(2\sqrt{3})$. By straightforward calculations we obtain the following explicit formulae for a_{ij}, $i, j = 1, 2, 3$.

$$\begin{aligned} &a_{11} = \frac{3}{2}T - \frac{5}{2}U, && a_{12} = -2T + 3U, && a_{13} = \frac{1}{2}T - \frac{1}{2}U, \\ &a_{21} = \frac{11}{2}T - \frac{19}{2}U, && a_{22} = -7T + 12U, && a_{23} = \frac{3}{2}T - \frac{5}{2}U, \\ &a_{31} = 2T - \frac{7}{2}U - \frac{n}{2}, && a_{32} = -\frac{5}{2}T + \frac{9}{2}U + 2n, && a_{33} = \frac{1}{2}T - U - \frac{n}{2}, \end{aligned}$$

where $T = 1 + T_n(2)$ and $U = U_{n-1}(2)$.

Now we are able to proof the following lemma.

Lemma 1. *Let d_1 be the greatest common divisor of a_{ij}, $i, j = 1, 2, 3$. Then*

$$d_1 = GCD(n, T, U)/GCD(2, n).$$

Proof of Lemma 1. We have

$$\begin{aligned} d_1 &= GCD(a_{ij}) = GCD(a_{11}, a_{12}, a_{13}, a_{21}, a_{22} - 4a_{12}, a_{23} - 3a_{13}, a_{31}, a_{32}, a_{33}) \\ &= GCD(a_{11}, a_{12}, a_{13}, a_{21}, T, -U, a_{31}, a_{32}, a_{33}) \\ &= GCD(T, U, \frac{1}{2}(T - U), -\frac{1}{2}(U + n), \frac{1}{2}(-T + U) + 2n, \frac{1}{2}(T - n)) \\ &= GCD(T, U, \frac{1}{2}(T - U), \frac{1}{2}(T - n) - \frac{1}{2}(U + n) + \frac{1}{2}(-T + U), \\ &\quad \frac{1}{2}(-T + U) + 2n, \frac{1}{2}(T - n)) \\ &= GCD(T, U, -n, \frac{1}{2}(T - U), \frac{1}{2}(-T + U) + 2n, \frac{1}{2}(T - n)) \\ &= GCD(T, U, n, \frac{1}{2}(T - U), \frac{1}{2}(T - n)). \end{aligned}$$

From the main recursive relations for the Chebyshev polynomials 1° and 2° we have the following properties. The numbers $T = 1 + T_n(2)$ and $U = U_{n-1}(2)$ are of the same parity as n. Moreover, if n is even then $\frac{T-n}{2}$ is odd.

Let us consider two cases: n is odd and n is even. In the first case, we have

$$\begin{aligned} d_1 &= GCD(T, U, n, \frac{1}{2}(T-U), \frac{1}{2}(T-n)) = GCD(T, U, n, T-U, T-n) \\ &= GCD(T, U, n) = GCD(n, T, U)/GCD(2, n). \end{aligned}$$

In the second case,

$$d_1 = GCD(T, U, n, \frac{1}{2}(T-U), \frac{1}{2}(T-n)).$$

Since $\dfrac{T-n}{2}$ is odd, we have

$$\begin{aligned} d_1 &= GCD(\frac{1}{2}T, \frac{1}{2}U, \frac{1}{2}n, \frac{1}{2}(T-U), \frac{1}{2}(T-n)) = GCD(\frac{1}{2}n, \frac{1}{2}T, \frac{1}{2}U) \\ &= GCD(n, T, U)/2 = GCD(n, T, U)/GCD(2, n). \end{aligned}$$

Now our aim is to find the greatest common divisor of two-by-two minors of matrix $A = \{a_{ij}\}_{i,j=1,2,3}$. Denote by m_{ij} the two-by-two minor of A obtained by removing i-th row and j-th column of A. By direct calculations we have

$$\begin{aligned} &m_{11} = -m_{22} = \frac{1}{2}((n+1)T) - nU, \quad m_{12} = -m_{23} = (-\frac{1}{2} - 2n)T + \frac{7n}{2}U, \\ &m_{13} = \frac{1}{2}(1+15n)T - 13nU, \;\; m_{21} = \frac{1}{2}T - \frac{n}{2}U, \;\; m_{31} = -m_{32} = m_{33} = T. \end{aligned}$$

We assert that the following lemma is true.

Lemma 2. *Let d_2 be the greatest common divisor of m_{ij}, $i, j = 1, 2, 3$. Then*

$$d_2 = GCD(T, nU)/GCD(2, n).$$

Proof of Lemma 2. By virtue of explicit formulae for m_{ij} we have

$$\begin{aligned} d_2 &= GCD(m_{11}, m_{12}, m_{13}, m_{21}, m_{33}) \\ &= GCD(m_{11}, m_{12} + m_{21}, m_{13}, m_{21}, m_{33}) \\ &= GCD(m_{11}, m_{12} + m_{21} + 2nm_{33}, \\ &\quad m_{13} - 7nm_{33} + m_{11} - (n+1)m_{33}, m_{21}, m_{33}) \\ &= GCD(\frac{1}{2}((n+1)T) - nU, 3nU, -14nU, \frac{1}{2}T - \frac{n}{2}U, T) \\ &= GCD(\frac{1}{2}((n+1)T) - nU, nU, \frac{1}{2}T - \frac{n}{2}U, T). \end{aligned}$$

Now, let us consider the case $n = 2m$ is even. Then

$$\begin{aligned} d_2 &= GCD(\frac{1}{2}((2m+1)T) - 2mU,\, 2mU,\, \frac{1}{2}T - \frac{2m}{2}U,\, T) \\ &= GCD(\frac{1}{2}T,\, 2mU,\, \frac{1}{2}T - mU,\, T) \\ &= GCD(\frac{1}{2}T,\, 2mU,\, -mU) \\ &= GCD(\frac{1}{2}T,\, mU) = GCD(\frac{1}{2}T,\, \frac{2m}{2}U) \\ &= GCD(T,\, nU)/2 = GCD(T,\, nU)/GCD(2,n). \end{aligned}$$

On the other hand, if $n = 2m+1$ is odd, then both T and U are odd. We have

$$\begin{aligned} d_2 &= GCD((m+1)T - (2m+1)U,\, (2m+1)U,\, \frac{1}{2}T - \frac{2m+1}{2}U,\, T) \\ &= GCD((2m+1)U,\, \frac{1}{2}T - \frac{2m+1}{2}U,\, T) \\ &= GCD((2m+1)U,\, T - (2m+1)U,\, T) \\ &= GCD(T,\, (2m+1)U) = GCD(T,\, nU). \end{aligned}$$

Let d_3 be the determinant of matrix $a_{ij},\ i,j = 1,2,3$. By the Kirchhoff Matrix-Tree Theorem d_3 coincides with the number of spanning trees of Moebius ladder $M(n)$. This number is well known and was independently calculated by many authors (J. Sedláček, J.W. Moon, N. Biggs and others) [5]. We represent the result as follows.

Lemma 3. *Let d_3 be the determinant of matrix $a_{ij},\ i,j = 1,2,3$. Then d_3 is given by the formula*

$$d_3 = nT,$$

where $T = 1 + T_n(2)$, and $T_n(2) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2$ is the Chebyshev polynomial of the first kind.

By the fundamental theorem of finite Abelian groups (Theorem A) we have the following decomposition for the Picard group of $M(n)$

$$Jac(M(n)) = \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2/d_1} \oplus \mathbb{Z}_{d_3/d_2}.$$

Taking into account Lemma 1, Lemma 2 and Lemma 3, we have the following theorem.

Theorem 1. *The Picard group of Moebius ladder $M(n)$ has the following presentation*

$$Jac(M(n)) = \mathbb{Z}_{\frac{(n,T,U)}{(2,n)}} \oplus \mathbb{Z}_{\frac{(T,nU)}{(n,T,U)}} \oplus \mathbb{Z}_{\frac{(2,n)nT}{(T,nU)}},$$

where $(l,m,n) = GCD(l,m,n)$, $T = 1 + T_n(2)$, $U = U_{n-1}(2)$ and
$T_n(x) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/2,\ U_{n-1}(x) = ((2+\sqrt{3})^n + (2-\sqrt{3})^n)/(2\sqrt{3})$ are Chebyshev polynomials of the first and second kind, respectively.

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