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SOME TOPICS IN GRAPH THEORY RELATED WITH GROUP THEORY

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Abstract. We discuss some concepts concerning graphs (mainly concepts of symmetrical extensions of graphs by graphs and of k-contractibility, k a positive integer, for graphs), which are related with the group theory and seem to be of interest.

Keywords: vertex-symmetric graph, symmetrical extension, k-contractibility.

1. Introduction

In this paper some concepts concerning graphs are considered. The considered concepts are related with group theory (roughly speaking, with the concept of an extension of a group by a group and the concept of a finitely presented group) and seem to be of interest.

The graphs considered in this paper are undirected graphs without loops or multiple edges. Recall some standard terminology concerning such graphs. For a graph Γ, $V(\Gamma)$ is the vertex set of Γ and $E(\Gamma)$ is the edge set of Γ. A graph Γ is locally finite if the valency of any vertex of Γ is finite. A path of a graph Γ is a sequence $x_0, ..., x_l$ of vertices of Γ such that $\{x_i, x_{i+1}\} \in E(\Gamma)$ for all $0 \leq i < l$, where l is an arbitrary non-negative integer (called the length of the path). We regard an isomorphism φ of a graph Γ_1 onto a graph Γ_2 as a bijection $V(\Gamma_1)$ onto $V(\Gamma_2)$ such that $\{x, y\} \in E(\Gamma_1)$ if and only if $\{\varphi(x), \varphi(y)\} \in E(\Gamma_2)$. Thus, we regard automorphisms of a graph Γ as permutations on $V(\Gamma)$, and the group $Aut(\Gamma)$ of all automorphisms of Γ as a permutation group on $V(\Gamma)$. A graph Γ admitting a vertex-transitive group of automorphisms is called vertex-symmetric. For a vertex-transitive group G of automorphisms of a graph Γ, an imprimitivity system σ of

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G is a partition of $V(\Gamma)$ (elements of σ are called blocks) such that $g(X) \in \sigma$ for any $X \in \sigma$ and any $g \in G$. If σ is an imprimitivity system of a vertex-transitive group G of automorphisms of a graph Γ, then the quotient graph of Γ over σ (i.e. the graph whose vertices are blocks of σ and edges are pairs $\{X, Y\}$ of blocks such that $X \neq Y$ and $\{x, y\} \in E(\Gamma)$ for some $x \in X, y \in Y$) is denoted by Γ/σ, and the group of automorphisms of Γ/σ induced by G is denoted by G^σ.

For a group G and $M \subseteq G$ satisfying $1 \notin M = M^{-1}$, define $\Gamma_{G,M}$ to be the graph with $V(\Gamma_{G,M}) = G$ and $E(\Gamma_{G,M}) = \{\{x, xg\} : x \in G, g \in M\}$. (In the case $G = \langle M \rangle$ the connected graph $\Gamma_{G,M}$ is therefore the Cayley graph of the group G with respect to the generating set M.) The natural action of G on $\Gamma_{G,M}$ by automorphisms is defined by $g : x \mapsto gx$ for any $g \in G$ and any $x \in V(\Gamma_{G,M}) = G$.

2. Symmetrical extensions of graphs by graphs

Let Γ and Δ be graphs, and let G be a vertex-transitive group of automorphisms of Γ. Define a connected graph $\tilde{\Gamma}$ to be a *G-symmetrical extension of* Γ *by* Δ if there exist a vertex-transitive group $\tilde{G}$ of automorphisms of $\tilde{\Gamma}$ and an imprimitivity system σ of $\tilde{G}$ (on $V(\tilde{\Gamma})$) such that the following assertions (i) and (ii) hold:

(i) there exists an isomorphism φ of $\tilde{\Gamma}/\sigma$ onto Γ such that $\varphi\tilde{G}^\sigma\varphi^{-1} = G$;

(ii) blocks of σ generate in $\tilde{\Gamma}$ subgraphs isomorphic to Δ.

If $\tilde{G}$ and σ can be chosen such that, in addition, blocks of σ are orbits of a normal subgroup of $\tilde{G}$, $\tilde{\Gamma}$ is a *normal G-symmetrical extension of* Γ *by* Δ.

It is obvious that if a G-symmetrical extension of Γ by Δ exists then Γ is connected and Δ is vertex-symmetric (and, on the other hand, that if Δ is vertex-symmetric then a normal G-symmetrical extension of Γ by Δ exists for any connected graph Γ and any vertex-transitive group G of automorphisms of Γ).

For graphs Γ and Δ, define a graph $\tilde{\Gamma}$ to be a (*normal*) *symmetrical extension of* Γ *by* Δ if $\tilde{\Gamma}$ is a (*normal*) *G-symmetrical extension of* Γ *by* Δ for some vertex-transitive group G of automorphisms of Γ. For a positive integer q and for a graph Γ and a vertex-transitive group G of automorphisms of Γ, define a graph $\tilde{\Gamma}$ to be a (*normal*) *G-symmetrical q-extension of* Γ if $\tilde{\Gamma}$ is a (normal) G-symmetrical extension of Γ by Δ for some graph Δ with $|V(\Delta)| = q$. Finally, for a graph Γ and a positive integer q, define a graph $\tilde{\Gamma}$ to be a (*normal*) *symmetrical q-extension of* Γ if $\tilde{\Gamma}$ is a (normal) G-symmetrical q-extension of Γ for some vertex-transitive group G of automorphisms of Γ.

In a sense, a symmetrical extension $\tilde{\Gamma}$ of a graph Γ by a graph Δ can be seen as Γ whose vertices, however, have an inner structure of Δ (or, more figuratively, are “molecules"of form Δ) and these inner structures are consistent with the structure of Γ in such a way that all vertices of the whole graph $\tilde{\Gamma}$ (which are “atoms") are indistinguishable from each other in $\tilde{\Gamma}$ (i.e. the hole graph $\tilde{\Gamma}$ admits a vertex-transitive group of automorphisms mapping “molecules"to “molecules"). >From this point of view, symmetrical extensions of grids by finite graphs or even T-symmetrical extensions of grids by finite graphs, where T are the groups of translations of grids, relate to crystallography of particles with inner structures and are of special interest.

Symmetrical extensions of graphs by graphs arise naturally in different contexts and are related with some well known constructions. Just to mention some of them:

(1) It is easy to see that if $\tilde{G}$ is a group generated by a set $\tilde{M}$ such that $1 \notin \tilde{M} = \tilde{M}^{-1}$ and if N is a normal subgroup of $\tilde{G}$ and $G := \tilde{G}/N$ (thus $\tilde{G}$ is an extension of the group G by the group N), then the Cayley graph $\Gamma_{\tilde{G},\tilde{M}}$ of $\tilde{G}$ with respect to $\tilde{M}$ is a normal G-symmetrical extension of $\Gamma_{G,M}$ by $\Gamma_{N,\tilde{M}\cap N}$, where $M = \{gN : g \in \tilde{M} \setminus N\}$ and G acts on $\Gamma_{G,M}$ in the natural way.

(2) If Γ is a connected graph admitting a vertex-transitive group of automorphisms G and if $\psi : V(\tilde{\Gamma}) \to V(\Gamma)$ is a covering of Γ by a connected graph $\tilde{\Gamma}$ such that the corresponding covering group $\tilde{G} := \{g \in Aut(\tilde{\Gamma}) : \psi g \in G\psi\}$ of G is a vertex-transitive group of automorphisms of $\tilde{\Gamma}$, then $\sigma := \{\psi^{-1}(x) : x \in V(\Gamma)\}$ is an imprimitivity system of $\tilde{G}$ (on $V(\tilde{\Gamma})$) and ψ induces an isomorphism φ of $\tilde{\Gamma}/\sigma$ onto Γ such that $\varphi\tilde{G}^{\sigma}\varphi^{-1}$ is a vertex-transitive subgroup of the group G. Thus $\tilde{\Gamma}$ is a $\varphi\tilde{G}^{\sigma}\varphi^{-1}$-symmetrical extension of Γ by Δ where Δ is the subgraph of the graph $\tilde{\Gamma}$ generated by a block of σ.

(3) For vertex-symmetric graphs Γ and Δ, any “standard"product (such as the Cartesian, strong, direct or lexicographic product) of Γ and Δ is a symmetrical extensions of Γ by Δ in the case it is connected.

We are mainly interested in symmetrical extensions of infinite locally finite graphs Γ by finite graphs Δ. For certain important classes of connected infinite locally finite vertex-symmetric graphs, there were obtained descriptions of the following form: graphs from the class are exactly G-symmetrical extensions of some explicitly given graphs Γ by finite graphs, where G are also explicitly given groups of automorphisms of graphs Γ. (Such type descriptions were obtained, for example, for the following classes of connected infinite locally finite vertex-symmetric graphs: graphs with polynomial growth (see [1]), graphs with recurrent symmetric random walk (see [2]), graphs with vertex-transitive bounded automorphism group (see [3]).) In this connection, investigation of such G-symmetrical extensions of graphs Γ by finite graphs could give a more detailed description of these important classes of graphs.

It is easy to give an example of a connected infinite locally finite vertex-symmetric graph Γ such that, for any positive integer q, there are (up to isomorphism) only finitely many symmetrical q-extensions of Γ. (For example, the 1-dimensional grid has this property.) On the other hand, the following result can be proved (the proof will be published elsewhere).

Theorem 1. *There exists a group G with a finite generating set $M = M^{-1}$, $1 \notin M$, such that, for any integer $q > 1$, the Cayley graph $\Gamma_{G,M}$ of G with respect to M has infinitely many pairwise non-isomorphic G-symmetrical q-extensions, where G acts on $\Gamma_{G,M}$ in the natural way. Moreover, the group G can be chosen to be a group whose subgroups different from* 1 *and G are of order p where p is some fixed prime number.*

It seems that even for some concrete rather simple connected infinite locally finite graphs Γ and finite graphs Δ a description (up to isomorphism) of all symmetrical extensions of Γ by Δ can be a problem (for example, for Γ to be a d-dimensional greed, $d > 1$, and Δ to be the complete graph K_q of order q, q a large power of 2). Moreover, it seems that for some concrete rather simple connected infinite locally finite graphs Γ and finite graphs Δ the question on the finiteness of the set of symmetrical extensions of Γ by Δ (considered up to isomorphism) can be a problem. In this connection the following questions are of interest.

Let Γ be a connected locally finite graph, G a vertex-transitive group of automorphisms of Γ and q a positive integer. Suppose that there are only finitely many G-symmetrical q-extensions of Γ. Is it true that in this case there are only finitely many symmetrical q-extensions of Γ?

Let Γ be a connected locally finite graph and q a positive integer. Suppose that there are only finitely many symmetrical extensions of Γ by the complete graph K_q of order q. Is it true that in this case there are only finitely many symmetrical q-extensions of Γ?

As it was mentioned before, T-symmetrical extensions of grids by finite graphs, where T are the groups of translations of grids, are to be of special interest. They relate to crystallography of particles with inner structures and to string theory (where the the space-time has extra compactified dimensions). It can be proved that, for such type extensions, the corresponding imprimitivity systems σ (see the definition of symmetrical extensions in the beginning of this Section) are uniquely determined. Moreover, a stronger result can be proved (see Theorem 2 below; the proof will be published elsewhere). To formulate this result, recall (see [3]) that an automorphism g of a connected graph Σ is called bounded, if there exists a number bounding above the distance between v and $g(v)$ for every vertex v of Σ. The normal subgroup of $Aut(\Sigma)$ consisting of all bounded automorphisms of Σ is denoted by $Aut_0(\Sigma)$. It follows easily from [2, Corollary 2 (i)] that, if Σ is locally finite, then the set $Tor(Aut_0(\Sigma))$ of all bounded automorphisms of Σ of finite order is a (normal) subgroups of $Aut(\Sigma)$.

Theorem 2. *Let Σ be a connected locally finite graph, H a vertex-transitive group of automorphisms of Σ and τ an imprimitivity system of H with finite blocks. Suppose that H^τ is a fixed-point-free and torsion-free group. Then blocks of τ are $Tor(Aut_0(\Sigma))$-orbits.*

Let Γ be a grid and T the group of translations of the grid Γ. If $\tilde{\Gamma}$ is a T-symmetrical extension of Γ by some finite graph Δ, then, by the definition, there exists a vertex-transitive group $\tilde{T}$ of automorphisms of $\tilde{\Gamma}$ and an imprimitivity system σ of $\tilde{T}$ with finite blocks such that $\tilde{T}^\sigma = T$ and blocks of σ generate in $\tilde{\Gamma}$ subgraphs isomorphic to Δ. By Theorem 2 applied to $\tilde{\Gamma}$, $\tilde{T}$ and σ (as Σ, H and τ, respectively), we get that blocks of σ are $Tor(Aut_0(\tilde{\Gamma}))$-orbits. Thus σ and hence Δ are uniquely determined by $\tilde{\Gamma}$ as a T-symmetrical extension of the grid Γ by some finite graph. Moreover, we can put $\tilde{T} = Aut_0(\tilde{\Gamma})$.

The following question concerning T-symmetrical extensions of grids is important: Is it true that, for any positive integer d and any finite graph Δ, there are only finitely many T-symmetrical extensions of the d-dimensional grid by Δ?

3. Graphs with k-contractible closed paths

We start with a translation of a part of Remark 2 on page 51 of [4]:

"Let Γ be a graph and k be a positive integer. For paths $X : x_0, ..., x_l$ and $X' : x'_0, ..., x'_{l'}$ of Γ put $X \overset{k}{\longleftrightarrow} X'$ if there exist non-negative integers m and m' such that $x_i = x'_i$ for all $0 \le i \le m$ and $x_{l-i} = x'_{l'-i}$ for all $0 \le i \le m'$ and such that $l - k \le m + m' \le l$ and $l' - k \le m + m' \le l'$. Let t be a positive integer. We say that a path X' of Γ is (k,t)-homotopic to a path X'' of Γ if there exists a sequence $X_0 = X', X_1, ..., X_n = X''$ of paths of Γ (where n is some positive integer) such that $X_j \overset{k}{\longleftrightarrow} X_{j+1}$ for all $0 \le j < n$ and such that the length of X_j is not greater than t for all $0 \le j \le n$. A path of Γ (k,t)-homotopic to a path of length 0 is called (k,t)-contractible. Furthermore, if f is a function from the set of non-negative integers into itself such that any closed path X of Γ is $(k, f(|X|))$-contractible, where $|X|$ is the length of X, then we say that the graph Γ has the property of (k,f)-contractibility. Note that a locally finite Cayley graph of a group G has the property of (k,f)-contractibility for some f if and only if G is finitely presented [i.e. G has a presentation with a finite set of generators and a finite set of relations among those generators], and has the property of (k,f)-contractibility for some recursive function f if and only if the finitely presented group G has a solvable word problem."

For locally finite Cayley graphs of groups, these concepts are now well developed in the geometric group theory. But it seems interesting to consider also a more general case of connected locally finite vertex-symmetric graphs.

For short, we say that paths X and X' of a graph Γ are *k-homotopic*, where k is a positive integer, if X and X' are (k,t)-homotopic for some positive integer t. Next, for a positive integer k, we say that a closed path X of a graph Γ is *k-contractible* if X is (k,t)-contractible for some positive integer t, and we say that a graph Γ has the *property of k-contractibility* if any closed path X of Γ is k-contractible. (Of course, Γ has the property of k-contractibility if and only if the following holds: considering Γ topologically and gluing 2-dimensional discs along boundaries to all closed paths of Γ of lengths not greater than k, we get a topological space with trivial fundamental group.) In addition, we say that a graph Γ has the *property of recursive k-contractibility* if Γ has the property of (k,f)-contractibility for some recursive function f.

In this terminology, a locally finite Cayley graph of a group G has the property of k-contractibility if and only if G is finitely presented, and has the property of recursive k-contractibility if and only if G is finitely presented and has a solvable word problem. It is well known that there are complicated finitely presented groups with complicated locally finite Cayley graphs. In particular, there are many finitely presented groups with unsolvable word problem (see, for example,[5]). Nevertheless, there obviously are only countably many finitely presented groups (among continually many finitely generated groups) and only countably many locally finite Cayley graphs of finitely presented groups (among continually many locally finite Cayley graphs of finitely generated groups). But I do not see arguments why, for any positive integer k, it should be only countably many connected locally finite vertex-symmetric graphs with the property of k-contractibility or even with the property of recursive k-contractibility.

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