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SIMPLE ALGORITHM FOR FINDING A SECOND HAMILTON CYCLE

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Abstract. A classical theorem of C.A.B. Smith states that for every edge e of a cubic graph G, the number of Hamilton cycles containing e in G is an even number. Tutte proved Smith's theorem using a nonconstructive parity argument. Thomason later invented the lollipop algorithm and provided a first constructive proof.

We describe a simple algorithm based on Tutte's proof, thus providing an alternative constructive proof of Smith's theorem. Also this algorithm is exponential in the worst case.
Keywords: Smith Theorem, cubic graph, Hamilton cycle, lollipop algorithm, parity argument.

1. Introduction

The following theorem of Smith was published by Tutte in 1946.

Theorem. (C.A.B. Smith, [7]) If G is a cubic graph, then the number of Hamilton cycles of G through any given edge is an even number.

The proof of Smith's Theorem presented by Tutte in [7] is a nonconstructive parity argument. The same is true for Bondy's simplified version of the proof in [1]. The ingeneous algorithmic proof by Thomason [6] is however independent of Tutte's proof.

We now present a simple algorithmic proof based on Tutte's original argument.

For any two subgraphs X, Y of a graph G define their *symmetric difference* $X\Delta Y$ as the subgraph of G induced by the edges $E(X \cup Y) \setminus E(X \cap Y)$.

Proof of Theorem. Assume that G is a cubic graph and e is an edge of G. The conclusion of the theorem is obvious if G contains no Hamilton cycle through e. So

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we assume that H is a Hamilton cycle through e in G. Since G is cubic, the length of H is even.

The following is a recursive construction of a sequence $\{F_0^n, F_1^n, F_2^n\}$ for $n = 0, 1, 2, \ldots, N$, each member of the sequence being a set of three distinct subgraphs of G. Inductively along the recursion, each $\{F_0^n, F_1^n, F_2^n\}$ is a 1-factorization of G such that $e \in F_i^n$ if $n + i \equiv 1 \pmod 3$.

Let $F_0^0 = G - E(H)$ be the 1-factor of G consisting of the chords of H. Let F_1^0 be the 1-factor of H that contains e, and let $F_2^0 = H - E(F_1^0)$ be the other 1-factor of H.

Assume $n \geq 1$, $n + i \equiv 1 \pmod 3$, $n + j \equiv 2 \pmod 3$, and $n + k \equiv 0 \pmod 3$, where $\{i, j, k\} = \{0, 1, 2\}$. Let $C := F_i^{n-1} \cup F_j^{n-1}$. Then C is a 2-factor of G, and $e \in C$ follows from $e \in F_j^{n-1}$.

Let C_n be the component of C that contains e, and define $\{F_0^n, F_1^n, F_2^n\}$ as follows.

$$\begin{aligned} F_i^n &= F_i^{n-1} \Delta C_n \\ F_j^n &= F_j^{n-1} \Delta C_n \\ F_k^n &= F_k^{n-1}. \end{aligned}$$

Let v be any vertex of G and assume that e_1, e_2, e_0 are the three edges incident to v in G, indexed with $e_\ell \in F_\ell^{n-1}$ for $\ell = 0, 1, 2$. If v is not a vertex of C_n, then $e_\ell \in F_\ell^n$ for $\ell = 0, 1, 2$ clearly follows. If v belongs to C_n, then $e_i, e_j \in C_n$ holds by construction of C and C_n, and we deduce $e_i \in F_j^n$, $e_j \in F_i^n$, and $e_k \in F_k^n$. Hence $\{F_0^n, F_1^n, F_2^n\}$ is again a 1-factorization of G. From $e \in C_n$ and $e \notin F_i^{n-1}$ we deduce $e \in F_i^n$.

If C_n is a Hamilton cycle of G, then the construction terminates with $N = n$ and $H' = C_n$.

Having described the construction of the sequence $\{F_0^n, F_1^n, F_2^n\}$, we show that the construction terminates, and that H' is distinct from H. It is then clear from the termination criterion that H' is a Hamilton cycle through e.

Consider a graph Γ whose vertices are all 1-factorizations $\{F_0, F_1, F_2\}$ of G with adjacencies defined as follows. Assume that $F = \{F_0, F_1, F_2\}$ is a vertex of Γ, where the indices are chosen so that $e \in F_0$. Let $i \in \{1, 2\}$ and let C^i be the component of the 2-factor $F_0 \cup F_i$ that contains e. Let $F' = \{F_i \Delta C^i, F_0 \Delta C^i, F_j\}$, where $j = 3 - i$. If F' and F are distinct, then they are adjacent in Γ.

If $F' = \{F_0', F_1', F_2'\}$, $F_0' = F_i \Delta C^i$, $F_i' = F_0 \Delta C^i$, and $F_j' = F_j$ then $F_0 = F_i' \Delta C^i$, $F_1 = F_0' \Delta C^i$, and $F_j = F_j'$, so that adjacency is indeed symmetrically determined. In particular it follows that the maximal degree of Γ is at most 2.

If C^i is a Hamilton cycle in G, then $C^i = F_0 \cup F_i$ implies $F_i \Delta C^i = F_0$ and $F_0 \Delta C^i = F_1$, hence $F = F'$, so that F has degree at most 1 in Γ. Conversely, if F has degree less than 2 in Γ, then $F_i = F_0 \Delta C^i$ for at least one $i \in \{1, 2\}$, hence $C^i = F_0 \Delta F_i$ implies that C^i is a Hamilton cycle. In particular F has degree zero in Γ if and only if both $F_0 \cup F_1$ and $F_0 \cup F_2$ are Hamilton cycles of G.

It is clear from the construction that $\{F_0^{n-1}, F_1^{n-1}, F_2^{n-1}\}$ and $\{F_0^n, F_1^n, F_2^n\}$ are adjacent in Γ for all $n = 1, 2, \ldots, N$ if $N > 0$. It also follows easily that $\{F_0^n, F_1^n, F_2^n\}$ is distinct from $\{F_0^{n-2}, F_1^{n-2}, F_2^{n-2}\}$ for $n = 2, 3, \ldots, N$ if $N > 1$. Hence the finite sequence $(\{F_0^0, F_1^0, F_2^0\}, \{F_0^1, F_1^1, F_2^1\}, \ldots, \{F_0^N, F_1^N, F_2^N\})$ describes a path

P in Γ (possibly of length zero if $N = 1$ and $\{F_0^0, F_1^0, F_2^0\} = \{F_0^N, F_1^N, F_2^N\}$). The termination criterion of the recursion ensures that starting from the endvertex $\{F_0^0, F_1^0, F_2^0\}$, the endvertex $\{F_0^N, F_1^N, F_2^N\}$ is reached at the last step.

If P is nontrivial, its distinct ends correspond to 1-factorizations of G in which the union of two of the 1-factors is a Hamilton cycle containing e. If P is trivial, its single vertex corresponds to a 1-factorization in which the union of the 1-factor containing e with each of the other 1-factors is also such a Hamilton cycle. It follows that the number of Hamilton cycles of G containing e is even. □

2. Complexity

The following open problem was first raised by Chrobak and Poljak, and it was also mentioned by Papadimitriou [5].

Problem (Chrobak and Poljak [3]) Is there a polynomial algorithm such that given any G and H as input, where G is cubic and H a Hamilton cycle of G, the algorithm finds a second Hamilton cycle?

An ingeneous algorithmic proof by Thomason [6] is independent of Tutte's proof. As in the proof given above, Thomason's Lollipop Algorithm proves Smith's Theorem by constructing for any Hamilton cycle H in G with $e \in H$ a second Hamilton cycle $\mathrm{lol}(H)$, where $\mathrm{lol}(H) \neq H$ and $e \in \mathrm{lol}(H)$, and so that lol is an involution defined on the set of all Hamilton cycles through e in G. Thus the number of such Hamilton cycles is even. It was proved by Cameron [2], based on examples first constructed by Krawczyk [4], that the worst case performance of the Lollipop Algorithm is exponential.

We will verify that the construction given in the proof in Section 1 also has worst case exponential behavior.

A *Möbius ladder* is a cubic graph obtained from a cycle $v_1 v_2 \dots v_{2n} v_1$ by adding all diagonal edges $v_i v_{n+i}$, for $i = 1, 2, \dots, n$.

Proposition. For every integer $m > 0$ there exists a cubic graph G of order $4m$ containing an edge e and a Hamilton cycle H through e, such that the algorithm of Section 1 when given G, e and H as input finds a second Hamilton cycle through e in no fewer than 2^{m-1} steps.

Proof of Proposition. Let m be a positive integer, let G be the Möbius ladder of order $4m$ with Hamilton cycle $H = v_1 v_2 \dots v_{4m} v_1$, and let $e = v_1 v_{4m}$. Denote the edges of H by $e_s = v_s v_{s+1}$ for $s = 1, \dots, 4m-1$ and $e_0 = e$, and the chords of H in G by $c_s = v_s v_{2m+s}$ for $s = 1, 2, \dots, 2m$.

It is not difficult by induction on $n \geq 0$ to determine the structure of the cycle C_n that appears in step n of the construction. If n is an odd number, then C_n is a 4-cycle through the chords c_1 and c_{2m} of H. If n is even, C_n contains exactly two chords c_s and c_{2m+1-s} of H, for some $s \in \{2, \dots, m\}$, while the 1-factor F_k^n contains the chords c_t and c_{2m+1-t} of H for all $t \in \{1, 2, \dots, s-1\}$.

Hence C_n is a cycle $v_1 v_2 \dots v_s v_{2m+s} v_{2m+s-1} \dots v_{2m+1-s} v_{4m+1-s} v_{4m+2-s} \dots \dots v_{4m} v_1$.

This may be deduced from the following more detailed statement.

Claim. For every $s \in \{1, 2, \dots, m\}$ the chords c_s and c_{2m+1-s} of H belong to the 1-factor F_ℓ^n given by $\ell \equiv \lfloor 2^{-s} n + \frac{1}{2} \rfloor (-1)^{s-1}$ (modulo 3).

Before proving the Claim, we observe how it implies our Proposition.

The cycle C_n contains all vertices of G only if it contains the chord c_m. But $c_m \in C_n$ implies that c_m belongs either to $F_i^{n-1} \cap F_j^n$ or to $F_j^{n-1} \cap F_i^n$, where $i \neq j$. By the Claim, this is only possible if $\lfloor 2^{-m}n + \frac{1}{2} \rfloor \neq 0$, which does not occur unless $n \geq 2^{m-1}$. Therefore the algorithm needs at least 2^{m-1} steps to terminate. It is not hard to see that this bound is exact.

Proof of Claim. By induction on $n \geq 0$.

For $n = 0$ the 1-factor F_0^0 consists of the chords of H. Thus the claim holds, since $\lfloor \frac{1}{2} \rfloor (-1)^{s-1} = 0$ for all s.

Assume $n > 0$ and the claim is true for $n-1$ in place of n. Assume that $i, j, k \in \{0, 1, 2\}$, $\{F_0^n, F_1^n, F_2^n\}$, C_n and C are defined as in the n'th step of the algorithm.

In the remaining part of this proof all congruences are taken modulo 3.

We write $n = q2^{p-1}$, where p and q are positive integers and q is odd.

By hypothesis, $c_s \in F_\ell^{n-1}$ holds if and only if $(-1)^{s-1}\ell \equiv \lfloor 2^{-s}(n-1) + \frac{1}{2} \rfloor$, which is equivalent to

$$\begin{aligned} (-1)^{s-1}(n+\ell) &\equiv \lfloor 2^{-s}(n-1) - (-1)^s n + \frac{1}{2} \rfloor \\ &= \lfloor q(1-(-2)^s)2^{p-1-s} - 2^{-s} + \frac{1}{2} \rfloor. \end{aligned}$$

If $s < p$ holds, then $q(1-(-2)^s)2^{p-1-s}$ is an integer divisible by 3, since $1-(-2)^s \equiv 1 - 1^s = 0$. Hence

$$\lfloor q(1-(-2)^s)2^{p-1-s} - 2^{-s} + \frac{1}{2} \rfloor = q(1-(-2)^s)2^{p-1-s} + \lfloor -2^{-s} + \frac{1}{2} \rfloor \equiv \lfloor -2^{-s} + \frac{1}{2} \rfloor = 0.$$

So in this case ℓ satisfies $n + \ell \equiv 0$, which implies $\ell = k$, and in particular $c_s \notin C = F_i^{n-1} \cup F_j^{n-1}$.

For $s = p$, we have

$$q(1-(-2)^s)2^{p-1-s} - 2^{-s} + \frac{1}{2} = \frac{q(1-(-2)^p)+1}{2} - 2^{-p}.$$

Here $q(1-(-2)^p)+1$ is an even integer, and

$$\frac{q(1-(-2)^p)+1}{2} \equiv -(q(1-(-2)^p)+1) \equiv -1.$$

Hence it follows that

$$\lfloor q(1-(-2)^s)2^{p-1-s} - 2^{-s} + \frac{1}{2} \rfloor \equiv -1 + \lfloor -2^{-s} \rfloor = -1 - 1 \equiv 1.$$

Then $c_p \in C$ follows from $n + \ell \not\equiv 0$, and $c_p \in C_n$ follows from $c_s \notin C$ for all $s < p$. We deduce from the symmetry that C_n is induced by the edges e_h for $h \in \{0, \ldots, p-1\} \cup \{2m-p+1, \ldots, 2m+p-1\} \cup \{4m-p+1, \ldots, 4m-1\}$, together with c_p and c_{2m-p}.

The 1-factorization $\{F_0^n, F_1^n, F_2^n\}$ is constructed so that c_s and c_{2m-s} become edges of F_k^n for all $s < p$, and so that c_p and c_{2m-p} belong to F_i^n if they are in F_j^{n-1} and to F_j^n if they are in F_i^{n-1}, whereas every chord c_s for $s > p$ satisfies $c_s \in F_\ell^n$ if and only if $c_s \in F_\ell^{n-1}$ for every $\ell = 0, 1, 2$. We will show that this agrees with the statement of the Claim.

First assume $s < p$. Then $c_s, c_{2m-s} \in F_k^n$, where $n + k \equiv 0$. Now

$$(-1)^{s-1}n + \lfloor 2^{-s}n + \frac{1}{2} \rfloor = \lfloor \frac{n(1-(-2)^s)}{2^s} + \frac{1}{2} \rfloor = q(1-(-2)^s)2^{p-1-s} + \lfloor \frac{1}{2} \rfloor \equiv 0.$$

Therefore $(-1)^{s-1}(n+k) \equiv 0$ implies $(-1)^{s-1}k \equiv \lfloor 2^{-s}n + \frac{1}{2} \rfloor$, as desired.

Next assume $s = p$. Above we found $c_p, c_{2m-p} \in C_n$ and $c_p, c_{2m-p} \in F_\ell^{n-1}$ where $(-1)^{p-1}(n+\ell) \equiv 1$. By construction of F_i^n, F_j^n it follows that $c_p, c_{2m-p} \in F_\ell^n$ holds when $(-1)^{p-1}(n+\ell) \equiv 2$. Since

$$(-1)^{p-1}n + \lfloor 2^{-p}n + \frac{1}{2} \rfloor = \lfloor \frac{(1-(-2)^p)q+1}{2} \rfloor = \frac{(1-(-2)^p)q+1}{2} \equiv$$
$$\equiv -((1-(-2)^p)q+1) \equiv 2,$$

we conclude that $c_p, c_{2m-p} \in F_\ell^n$ is satisfied when $(-1)^{p-1}\ell \equiv \lfloor 2^{-p}n + \frac{1}{2} \rfloor$.

Finally assume $s > p$. Then

$$\lfloor 2^{-s}(n-1) + \frac{1}{2} \rfloor = \lfloor \frac{q - 2^{-p+1} + 2^{s-p}}{2^{s-p+1}} \rfloor = \lfloor \frac{q + 2^{s-p}}{2^{s-p+1}} \rfloor = \lfloor 2^{-s}n + \frac{1}{2} \rfloor.$$

Since $c_s, c_{2m-s} \in F_\ell^n$ is equivalent to $c_s, c_{2m-s} \in F_\ell^{n-1}$, it follows that $c_s, c_{2m-s} \in F_\ell^n$ is equivalent as well to $\ell \equiv \lfloor 2^{-s}n + \frac{1}{2} \rfloor (-1)^{s-1}$.

This proves the Claim by induction, and hence the Proposition. □

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