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ON A QUESTION OF DIRAC ON CRITICAL AND VERTEX CRITICAL GRAPHS

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Abstract. We give a construction which for any N provides a graph on $n > N$ vertices which is vertex-critical with respect to being 4-chromatic, has at least cn^2 edges that are non-critical (i.e., the removal of any one does not change the chromaticity) and has at most Cn critical edges for some fixed positive constants c and C.

Thus for any $\varepsilon > 0$ we get 4-vertex-critical graphs in which less than an ε-proportion of the edges are non-critical.

Keywords: critical graph, vertex-criticality, critical edge, Dirac problem

1. Introduction

A graph is k-critical if it is k-chromatic but any proper subgraph is $(k-1)$-colourable; that is, it becomes $(k-1)$-colourable upon the removal of any edge or vertex. A graph is k-vertex-critical if it is k-chromatic, but becomes $(k-1)$-colourable upon the removal of any vertex. Clearly k-critical graphs are k-vertex-critical, but the converse statement is not necessarily true, and one would like to see how far a k-vertex-critical graph can be from being k-critical. When $k = 3$, it cannot be far: the 3-vertex-critical graphs are exactly the 3-critical graphs, that is, odd cycles. Where an edge in a k-vertex-critical graph is *critical* if its removal makes the graph $(k-1)$-colourable, Dirac asked the following question (see [2], [4]).

Problem 1 (Dirac)**.** *For all integers $k \geq 4$, do there exist k-vertex-critical graphs without any critical edges?*

Tommy Jensen, Mark Siggers On a question of Dirac on Critical and Vertex Critical Graphs.

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In [1] Brown exhibited the first known example of such a graph, for $k = 5$. In [3], Jensen showed that for each $k \geq 5$ there are in fact infinitely many examples, and in [5], Lattanzio showed further constructions for odd values of $k \geq 5$. But the case $k = 4$ remains stubbornly unanswered.

Given a graph G with edge set $E = E(G)$ and vertex set $V = V(G)$, let $E_* = E_*(G) \subset E$ be the set of critical edges of G. The question of Dirac asks if there are k-vertex-critical graphs for which E_* is empty.

We consider the set E_* in the case that $k = 4$ with regards to the supposition that if it is necessarily non-empty, then perhaps it also satisfies some stronger properties, such as being large. Anything with we can say about E_* will hopefully help in eventually proving whether or not it is empty.

Our first result is to show that E^*, as would be expected of any empty set, is not necessarily large.

Theorem 1. *There exist $c_1, c_2 > 0$ such that for infinitely many $n > 1$ there is a 4-vertex-critical graph H on n vertices such that $|E(H)| > c_1 n^2$ but $|E_*(H)| < c_2 n$.*

2. Proof of Theorem 1

The proof is constructive, and depends on a fixed integer $m \geq 3$. We start with the graph B, a simple blowup of the triangle.

Construction 1 ($B = B(m)$)**.** *Let B be the complete tripartite graph with three disjoint sets S_1, S_2 and S_3 of $2m$ vertices each.*

The following is clear.

Lemma 1. *The only 3-colouring of $B = B(m)$ ($m \geq 3$), up to permutation of the colours is $\phi(v) = i$ if $v \in S_i$. Further, this remains the only colouring of B on the removal of any edge.*

Доказательство. Indeed, assume that some vertex v_1 of S_1 has $\phi(v_1) = 1$. Then for any other vertex v_1' in S_1, there are adjacent common neighbours v_2 and v_3 of v_1 and v_1'. So $\phi(v_2)$ and $\phi(v_3)$ are different and different from 1, forcing $\phi(v_1') = 1$. Without loss of generality, we may say that $\phi(v_2) = 2$ and $\phi(v_3) = 3$, and the same argument as above allows us to conclude that ϕ is monochromatic on both S_2 and S_3.

This argument works until at least $m - 2$ edges have been removed from B, so as $m \geq 3$ works with the removal of any one edge. □

The idea of the construction is simple. We construct another graph G containing the vertex set $S_1 \cup S_2 \cup S_3$ as an independent set, and having relatively few (constant times m) vertices and edges. Any 3-colouring of G will disagree with the unique 3-colouring of B on exactly one vertex of B. Removing any vertex of B, we then get a good colouring, while removing any edge of B will not change things. Thus we can get a vertex-critical graph in which none of the $12m^2$ edges in B are critical.

We construct the graph G in a couple of stages. We begin with an auxiliary graph $T = T(m)$. The notation $[m]$ is used to denote the set $\{1, \ldots, m\}$.

Construction 2 ($T = T(m)$)**.** *(See Figure 1.) For $m > 3$, construct $T(m)$ from the path $x_0 x_1 \ldots x_{2m+1}$, and the independent vertices $y_1, \ldots, y_m$, by adding the edges $y_i x_{2i-1}$ and $y_i x_{2i}$ for $i \in [m]$. Let $Y = \{y_1, \ldots y_m\}$ and $Y^+ = \{x_0, x_{2m+1}\} \cup Y$.*

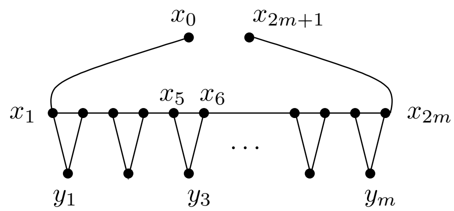

Рис. 1. The graph $T(m)$

Lemma 2. *The graph $T = T(m)$ satisfies the following properties.*

(1) *Any 3-colouring ϕ of Y^+ with $\phi(x_0) = \phi(x_{2m+1}) = c_0$ can be extended to a proper 3-colouring of T if and only if there is some $i \in [m]$ for which $\phi(y_i) = c_0$.*
(2) *Any 3-colouring ϕ of Y^+ with $\phi(x_0) \neq \phi(x_{2m+1})$ can be extended to a proper 3-colouring of T.*
(3) *For any vertex $v \in V(T)$, any 3-colouring ϕ of $Y^+ \setminus \{v\}$ can be extended to a proper 3-colouring of $T \setminus v$.*

Доказательство. We prove item (i) and in doing so also prove item (ii). Before we address item (i) directly though, we make an observation about a 3-colouring ϕ of $Y^+ \setminus \{x_{2m+1}\} \cup \{x_{2i}\}$ for some $i \in [m]$. Observe that if $\phi(y_{i+1}) \neq \phi(x_{2i})$ the only possible good extension of ϕ to x_{2i+1} and $x_{2(i+1)}$ has $\phi(x_{2(i+1)}) = \phi(x_{2i})$. On the other hand, if $\phi(y_{i+1}) \neq \phi(x_{2i})$ then ϕ can be extended to x_{2i+1} and $x_{2(i+1)}$ in two ways, which differ, in particular, on $x_{2(i+1)}$.

Now let ϕ be a 3-colouring of $Y^+ \setminus \{x_{2m+1}\}$, with $\phi(x_0) = 1$. We show that ϕ can be extended to a 3-colouring of T with $\phi(x_{2m+1}) = 1$ if and only if $1 \in \phi(Y)$. This is equivalent to item (i).

If $1 \notin \phi(Y)$, then if follows by induction from the observation, that $\phi(x_{2i}) = 1$ for all $i \in [m]$, and so $\phi(x_{2m+1}) \neq \phi(x_{2m}) = 1$.

On the other hand, let i be the smallest integer such that $\phi(y_i) = 1$. Then as we saw above, there is a good extension of ϕ to $\{x_1, \ldots, x_{2(i-1)}\}$ with $\phi(x_{2(i-1)}) = 1$, and two extensions of ϕ to $\{x_1, \ldots, x_{2i}\}$ that differ on x_{2i}. If one of them has $\phi(x_{2i}) = \phi(y_{i+1})$, then it extends in two ways to $\{x_{2i+1}, x_{2(i+1)}\}$. Otherwise, they both have $\phi(x_{2i}) \neq \phi(y_{i+1})$ so extend respectively to colourings including $\{x_{2i+1}, x_{2(i+1)}\}$, which differ on $x_{2(i+1)}$. By induction, ϕ extends to at least two colourings on $T \setminus x_{2m+1}$ which differ on x_{2m}. At least one of these extends to a colouring of T with $\phi(x_{2m+1}) = 1$, as needed.

Item (ii) has been proved in our proof of item (i). Item (iii) follows from items (i) and (ii) by assigning the appropriate colour to the missing vertex v.

□

Construction 3 ($T' = T'(m)$)**.** *Let $T'(m)$ be the graph constructed from $T(m)$ by adding a leaf z_i adjacent to y_i for each $i \in [m]$. Let $Z = \{z_1, \ldots, z_m\}$ and $Z^+ = Z \cup \{x_0, x_{2m+1}\}$.*

The following is clear.

Lemma 3. *Lemma 2 holds for T' in place of T if we replace Y^+ with Z^+ and $\phi(y_i) = c_0$ with $\phi(z_i) \neq c_0$, in item (i).*

Construction 4 ($G = G(m)$)**.** *Fix $m > 2$ and for $i \in [3]$ let T_i be a copy of $T(m)$ and T_i' be a copy of $T'(m)$. For $i \in [3]$ let S_i be the union of the copy of Y in T_i and the copy of Z in T_i'. Construct $G = G(m)$ from the star with center v_0 and leaves v_1, v_2 and v_3, and from the graphs T_i and T_i' for $i = 1, 2, 3$, as follows.*

- *Identify v_i with the copies of x_0 in T_i and T_i' for $i = 1, 2, 3$.*
- *Identify v_{i+1} with the copies of x_{2m+1} in T_i and T_i' for $i = 1, 2, 3$.*

As $T(m)$ has $3m + 2$ vertices and $3m + 1$ edges, and $T'(m)$ has $4m + 2$ vertices and $4m + 1$ edges, one can check that $G(m)$ has $n = 21m + 4$ vertices and $21m + 6$ edges.

Lemma 4. *The following are true of G:*

1. *Under any 3-colouring of G at least one of the sets S_i is not monochromatic.*
2. *For any vertex $v_* \in S_1 \cup S_2 \cup S_3$, there is a 3-colouring of $G \setminus v_*$ under which $S_i \setminus \{v_*\}$ is monochromatic for each $i = 1, 2, 3$.*

Доказательство. First we show statement (i). Indeed, if ϕ is a 3-colouring of G then some two of the vertices v_1, v_2, v_3, all being adjacent to v_0, must have the same colour. Assume, without loss of generality, that $\phi(v_1) = \phi(v_2) = 1$. As v_1, v_2 are identified with x_0 and x_{2m+1} in T_1 and T_1', the copy Y_1 of Y in T_1 contains some vertex of colour 1, and the copy Z_1 of Z in T_1' contains some vertex of colour different from 1. So S_1 contains vertices of two different colours.

For statement (ii), let $v_* \in S_1$. We define a 3-colouring ϕ of $G \setminus v_*$ that extends the 3-colouring of $(S_1 \cup S_2 \cup S_3) \setminus \{v_*\}$ which is defined by $\phi(v) = i$ if $v \in S_i$. As S_1 is the union of the copy Y_1 of Y in T_1 and the copy Z_1 of Z in T_1', we have two (very similar) cases to deal with.

Case $v_* \in Y_1$: We extend ϕ to $V(G) \setminus \{v_*\}$ by extending it first to the v_i, and then extending it, using Lemmas 2 and 3, to the graphs T_i and T_i'.

- Let $\phi(v_0) = 1$, $\phi(v_1) = \phi(v_2) = 2$, and $\phi(v_3) = 3$.
- The colouring ϕ is so far defined on the copy of Y^+ in T_1 except the vertex v_* that has been removed from the copy Y_1 of Y. By Lemma 2(iii), it can thus be extended to $T_1 \cap (G \setminus v_*)$.
- The copies x_0 and x_{2m+1} in T_2 are v_2 and v_3 respectively, so $\phi(x_0) = 2 \neq 3 = \phi(x_{2m+1})$. By Lemma 2(ii), therefore ϕ, which is so far defined only on the copy of Y^+ in T_2, can be extended to a good colouring of T_2.
- The copies of x_0 and x_{2m+1} in T_3 are v_3 and v_1 respectively, so ϕ can similarily be extended to a good colouring of T_3.
- The copies of x_0 and x_{2m+1} in T_1' are v_1 and v_2 respectively, so $\phi(x_0) = 2 = \phi(x_{2m+1})$. The copy Z_1 of Z is a subset of S_1 so $\phi(v) = 1$ for all $v \in Z_1$. Thus by Lemma 3(i), ϕ can be extended to a good colouring of T_1'.
- The copies of x_0 and x_{2m+1} in T_2' are v_2 and v_3 respectively, so $\phi(x_0) = 2 \neq 3 = \phi(x_{2m+1})$. Thus by Lemma 3(ii), ϕ can be extended to a good colouring of T_2'. Similarily it can be extended to T_3'.

We have thus extended ϕ to a good 3-colouring of $G \setminus v_*$ in the case that v_* is in $Y_1 \subset S_1$.

Case $v_* \in Z_1$ is very similar: In this case we start by defining ϕ on the v_i as follows. Let $\phi(v_0) = 3$, $\phi(v_1) = \phi(v_2) = 1$, and $\phi(v_3) = 2$.

From here, the proof that ϕ extends to T_2, T_3, T_2' and T_3' is the same, as again $\phi(x_2) \neq \phi(x_3)$ and $\phi(x_3) \neq \phi(x_1)$. The argument that it extends to T_1' is the same as the argument for T_1 in the previous case, except using Lemma 2(iii), and the argument that it extends to T_1 is the same as the argument for T_1' in the previous case, except using Lemma 2(i).

□

Now we are ready to prove the Theorem. For $m \geq 3$ let H' be the graph we get from $B = B(m)$ and $G = G(m)$ by identifying, for $i = 1, 2, 3$, the set S_i of B with the set S_i of G. By Lemmas 1 and 4(i), H' is not 3-colourable. Let H be a 4-critical subgraph that we get from H' by removing vertices as long as this does not make the graph 3-colourable. By Lemma 4(ii), $V(H)$ must contain all of $S_1 \cup S_2 \cup S_3$. Further, by Lemma 1, none of the edges of B are critical, so $E_*(H) \subset E(H) \setminus E(B) \subset E(G)$.

Now, as we observed just before Lemma 4 G has $21m + 4$ vertices and $21m + 6$ edges. So H has $n \leq 21m + 4$ vertices and $|E(H)| \geq |E(B)| = 12m^2 > \frac{12}{21^2}n^2$ edges, of which at most $|E(G)| = 21m + 6 = n + \varepsilon$ are in $E_*(H)$. Thus the Theorem is proved with $c_1 = 12/21^2$ and $c_2 = 1 + \varepsilon$.

3. Concluding Remarks

Our aim is to discover properties of the set E_* of critical edges in a 4-critical graph H. In our construction, we showed, as must be true if Dirac's Question is to be answered positively, that E_* may be small. In our construction, E_* was connected, and even a spanning subgraph of H. We cannot see how to avoid this.

If Dirac's Question has a negative answer for $k = 4$, it does not seem unlikely that one could further say that the graph E_* of critical edges is connected, or even spanning. We think it would be interesting to investigate even questions such as this.

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