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CONDITIONS FOR NON-SYMMETRIC RELATIONS OF SEMI-ISOLATION

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Abstract. We consider necessary and sufficient conditions for non-symmetric relations of semi-isolation in terms of colorings for neighborhoods of types, quasi-neighborhoods, and the existence of limit models. We show that, for any type p in a small theory, its non-symmetry of isolation is equivalent to the non-symmetry of semi-isolation (where a realization $\bar{a}$ of p isolates a realization $\bar{b}$ of p and $\bar{b}$ does not semi-isolates $\bar{a}$) and is equivalent to the existence of a limit model over p. We generalize the Tsuboi theorem on the absence of Ehrenfeucht unions of pseudo-superstable theories and the Kim theorem on the absence of Ehrenfeucht supersimple theories for unions of pseudo-supersimple theories. We also present a survey of results related to non-symmetric semi-isolation.

Keywords: relation of semi-isolation, (p, q)-preserving formula, Ehrenfeucht theory, powerful type, quasi-neighborhood, coloring of a structure, strict order property, limit model.

The non-symmetry of the semi-isolation is a key property in the study of Ehrenfeucht theories. In this paper, we consider new necessary and sufficient conditions for non-symmetric relations of semi-isolation in terms of colorings of neighborhoods for types as well as in terms of quasi-neighborhoods. In addition, using these notions, we obtain some known propositions on the non-symmetry of the semi-isolation, collected in [1, Chapter 1], and a criterion for the existence of a limit model over a type in terms of non-symmetry of the relation of semi-isolation, the Tsuboi theorem [2, 3] on non-representability of Ehrenfeucht theory without non-identical definable dense orders (and, in particular, without the strict order property) as a union of countably

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categorical theories. We prove a generalization of the Tsuboi theorem [3] on non-representability of Ehrenfeucht theory as a union of pseudo-superstable theories and of the Kim theorem [4] on absence of Ehrenfeucht supersimple theories. This generalization includes the Baldwin — Lachlan theorem on the number of countable models for ω_1-categorical theories [5], the Lachlan theorem [6] on non-existence of Ehrenfeucht theories in the class of superstable theories; analogous results were proven by A. Pillay [7, 8] for normal and 1-based theories and by E. Hrushovski [9] for theories admitting finite codings.

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We denote infinite structures (i. e., models of elementary theories) by $\mathfrak{M}, \mathfrak{N}, \ldots$, possibly with indices; and we use corresponding Latin letters $M, N, \ldots$ to denote their universes. The type of a tuple $\bar{a}$ in $\mathfrak{M}$ over a set $A \subseteq M$ will be denoted by $\mathrm{tp}_{\mathfrak{M}}(\bar{a}/A)$ or by $\mathrm{tp}(\bar{a}/A)$ if the structure is given. If $A = \varnothing$, we write $\mathrm{tp}(\bar{a})$ instead of $\mathrm{tp}(\bar{a}/\varnothing)$. We denote by $S^n(T)$ and by $S^n(\varnothing)$ the set of n-types of a theory T. The set of all types of T over the empty set is denoted by $S(T)$ and by $S(\varnothing)$.

In what follows, we consider only complete theories T without finite models.

1. Semi-isolation and (p, q)-preserving formulas

Definition (A. Pillay [7]). Let $\mathfrak{M}$ be a model of a theory T, $\bar{a}$ and $\bar{b}$ be tuples in $\mathfrak{M}$, and let A be a subset of M. We say that the tuple $\bar{a}$ *semi-isolates* the tuple $\bar{b}$ over the set A if there exists a formula $\varphi(\bar{a}, \bar{y}) \in \mathrm{tp}(\bar{b}/A\bar{a})$ for which $\varphi(\bar{a}, \bar{y}) \vdash \mathrm{tp}(\bar{b}/A)$ holds. In this case we say that the formula $\varphi(\bar{a}, \bar{y})$ (with parameters in A) *witnesses that $\bar{b}$ is semi-isolated over $\bar{a}$ with respect to A.*

Similarly, a tuple $\bar{a}$ *isolates* a tuple $\bar{b}$ over A if there exists a formula $\varphi(\bar{a}, \bar{y}) \in \mathrm{tp}(\bar{b}/A\bar{a})$ for which $\varphi(\bar{a}, \bar{y}) \vdash \mathrm{tp}(\bar{b}/A)$ and $\varphi(\bar{a}, \bar{y})$ is a principal (i. e., isolating) formula. In this case we say that the formula $\varphi(\bar{a}, \bar{y})$ (with parameters in A) *witnesses that $\bar{b}$ is isolated over $\bar{a}$ with respect to A.*

If $\bar{a}$ (semi-)isolates $\bar{b}$ over $\varnothing$, we simply say that $\bar{a}$ *(semi-)isolates* $\bar{b}$; and if a formula $\varphi(\bar{a}, \bar{y})$ witnesses that $\bar{a}$ (semi-)isolates $\bar{b}$ over $\varnothing$ then we say that $\varphi(\bar{a}, \bar{y})$ *witnesses that $\bar{a}$ (semi-)isolates $\bar{b}$.*

Notice that if $\bar{a}$ (semi-)isolates $\bar{b}$ over A by means of a formula $\varphi(\bar{a}, \bar{y})$ and $\bar{b} = \bar{b}^1\bar{b}^2$ then $\bar{a}$ (semi-)isolates $\bar{b}^1$ and $\bar{b}^2$ over A by means of the formulas $\exists \bar{y}^2 \varphi(\bar{a}, \bar{y}^1, \bar{y}^2)$ and $\exists \bar{y}^1 \varphi(\bar{a}, \bar{y}^1, \bar{y}^2)$ respectively.

The following notion proposed by B. S. Baizhanov generalizes the notion of p-stability introduced in [10] (see also B. S. Baizhanov, B. Sh. Kulpeshov [11]).

Definition. Let $p \rightleftharpoons p(\bar{x})$ and $q \rightleftharpoons q(\bar{y})$ be some (may be incomplete) types over a set $A \subseteq M$ in a model $\mathfrak{M}$ of a theory T. A formula $\varphi(\bar{x}, \bar{y})$ with parameters in A is said to be *(p, q)-preserving*, a *$(p \to q)$-formula*, or a *$(q \leftarrow p)$-formula* if, for any realization $\bar{a}$ of p, $\varphi(\bar{a}, \bar{y}) \vdash q(\bar{y})$ holds. A formula $\varphi(\bar{x}, \bar{y})$ is called a *$(p \leftrightarrow q)$-formula* if $\varphi(\bar{x}, \bar{y})$ is both a $(p \to q)$-formula and a $(p \leftarrow q)$-formula. If $p = q$ then a (p, q)-preserving formula is called *p-preserving* or a *$(p \to p)$-formula.*

Lemma 1.1. *A formula $\varphi(\bar{x}, \bar{y})$ with parameters in A is (p, q)-preserving if and only if for any formula $\theta(\bar{y})$ (with parameters in A) satisfying $q(\bar{y}) \vdash \theta(\bar{y})$ there* *exists a formula $\theta'(\bar{x})$ (with parameters in A) such that $p(\bar{x}) \vdash \theta'(\bar{x})$ and*

$$\mathfrak{M} \models \forall \bar{x}, \bar{y}\, ((\theta'(\bar{x}) \wedge \varphi(\bar{x}, \bar{y})) \to \theta(\bar{y}))\,. \quad (1)$$

Proof. Suppose that a formula $\varphi(\bar{x}, \bar{y})$ is (p, q)-preserving. Consider an arbitrary formula $\theta(\bar{y})$ (with parameters in A) such that $q(\bar{y}) \vdash \theta(\bar{y})$. Assuming that there are no formulas $\theta'(\bar{x})$ (with parameters in A) such that $p(\bar{x}) \vdash \theta'(\bar{x})$ and $\mathfrak{M} \models \forall \bar{x}, \bar{y}\, ((\theta'(\bar{x}) \wedge \varphi(\bar{x}, \bar{y})) \to \theta(\bar{y})))$, by Compactness Theorem we obtain a realization $\bar{a}$ of p such that $\mathfrak{M} \models \exists \bar{y}\, (\varphi(\bar{a}, \bar{y}) \wedge \neg\theta(\bar{y}))$. It contradicts the condition of (p, q)-preservation for the formula $\varphi(\bar{x}, \bar{y})$.

Now assume that for any formula $\theta(\bar{y})$ such that $q(\bar{y}) \vdash \theta(\bar{y})$ there exists a formula $\theta'(\bar{x})$ such that $p(\bar{x}) \vdash \theta'(\bar{x})$ and the condition (1) holds but the formula $\varphi(\bar{x}, \bar{y})$ is not (p, q)-preserving, i. e., for some realization $\bar{a}$ of p, $\varphi(\bar{a}, \bar{y}) \not\vdash q(\bar{y})$ holds. This means that the formula $\varphi(\bar{a}, \bar{y})$ is consistent with some formula $\neg\theta(\bar{y})$ such that $q(\bar{y}) \vdash \theta(\bar{y})$. Then for any formula $\theta'(\bar{x})$ with $p(\bar{x}) \vdash \theta'(\bar{x})$ we have $\mathfrak{M} \models \exists \bar{x}, \bar{y}\, (\theta'(\bar{x}) \wedge \varphi(\bar{x}, \bar{y}) \wedge \neg\theta(\bar{y}))$, which is impossible by the assumption. □

The following example shows that the existence of a (p, q)-preserving formula does not necessarily imply the completeness of the types p and/or q.

Example 1.2. Consider the language Σ consisting of unary predicate symbols $Q^{(1)}, Q_n^{(1)}, P^{(1)}, R_n^{(1)}$, $n \in \omega$, and one binary predicate symbol $S^{(2)}$. Construct a structure $\mathfrak{M}$ of language Σ with a universe $M = N \cup \{a_n \mid n \in \omega\} \cup \{b_n \mid n \in \omega\}$, where N is the set of naturals. We interpret the predicate symbols as follows: $Q(\mathfrak{M}) = N$, $\neg Q(\mathfrak{M}) = \{a_n \mid n \in \omega\} \cup \{b_n \mid n \in \omega\}$, $P(\mathfrak{M}) = \{a_n \mid n \in \omega\}$, $\mathfrak{M} \models Q_n(k) \iff k = n$, $\mathfrak{M} \models \forall x \forall y (S(x, y) \to (\neg Q(x) \wedge Q(y))$, $\mathfrak{M} \models R_n(a_j) \iff j = n$, $\mathfrak{M} \models R_n(b_j) \iff j = n$, $\mathfrak{M} \models S(a_n, k) \iff n \leq k$, $\mathfrak{M} \models S(b_n, k) \iff n \leq k$.

For a non-principal and incomplete type $p(x)$ consisting of formulas $\theta(x)$ which are deducible from the set $\{\neg Q(x)\} \cup \{\neg R_n(x) \mid n \in \omega\}$ and for the complete type $q(y)$, consisting of formulas $\theta'(y)$ which are deducible from the set $\{Q(y)\} \cup \{\neg Q_n(y) \mid n \in \omega\}$, the formula $S(x, y)$ is (p, q)-preserving. □

The following statement is obvious.

Lemma 1.3. *A formula $\varphi(\bar{a}, \bar{y})$ witnesses that $\bar{a}$ semi-isolates $\bar{b}$ iff $\varphi(\bar{x}, \bar{y})$ is $(\mathrm{tp}(\bar{a}), \mathrm{tp}(\bar{b}))$-preserving and $\mathfrak{M} \models \varphi(\bar{a}, \bar{b})$.*

Lemma 1.4 (A. Pillay [7]). *The relation of semi-isolation with respect to A forms a preorder (i. e. a reflexive and transitive relation) on the set of tuples in the model $\mathfrak{M}$.*

Proof. Notice that any tuple $\bar{a} = \langle a_1, \ldots, a_n \rangle$ semi-isolates itself with respect to A by the formula $\bigwedge\limits_{i=1}^{n} (a_i \approx y_i)$ $\Big(\bigwedge\limits_{i=1}^{n} (x_i \approx y_i)$ is a $(\mathrm{tp}(\bar{a}) \leftrightarrow \mathrm{tp}(\bar{a}))$-formula$\Big)$, i. e., the relation of semi-isolation is reflexive.

Assume now that, for tuples $\bar{a}$, $\bar{b}$, and $\bar{c}$, a formula $\varphi(\bar{a}, \bar{y})$ (with parameters in A) witnesses that $\bar{b}$ is semi-isolated over $\bar{a}$ and a formula $\psi(\bar{b}, \bar{z})$ (with parameters in A) witnesses that $\bar{c}$ is semi-isolated over $\bar{b}$. The formula $\exists \bar{y}\, (\varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{y}, \bar{z}))$. is $(\mathrm{tp}(\bar{a}), \mathrm{tp}(\bar{c}))$-preserving and, moreover, it witnesses that $\bar{a}$ semi-isolates $\bar{c}$ with respect to A. Thus, the relation of semi-isolation is transitive. □

The set of (p,q)-preserving formulas $\varphi(\bar{x},\bar{y})$ (with parameters from A, in a model $\mathfrak{M}$) is denoted by $\mathrm{Pres}_{p,q}(\bar{x},\bar{y})$ and the set of p-preserving formulas $\varphi(\bar{x},\bar{y})$ is denoted by $\mathrm{Pres}_p(\bar{x},\bar{y})$.

Let $p(\bar{x})$ and $q(\bar{y})$ be complete types over a set A, $\mathfrak{M}\models p(\bar{a})$, $\mathfrak{M}\models q(\bar{b})$, and $\mathfrak{M}\models\varphi(\bar{a},\bar{b})$. We say that the formula $\varphi(\bar{x},\bar{y})$ is *(non-)symmetric with respect to (p,q)-preservation* if $\varphi(\bar{x},\bar{y})\in\mathrm{Pres}_{p,q}(\bar{x},\bar{y})\cap\mathrm{Pres}_{q,p}(\bar{y},\bar{x})$ ($\varphi(\bar{x},\bar{y})\in\mathrm{Pres}_{p,q}(\bar{x},\bar{y})\div\mathrm{Pres}_{q,p}(\bar{y},\bar{x})$, where $\div$ is the operation of symmetric difference for sets).

By Lemma 1.3, the symmetry with respect to (p,q)-preservation for the formula $\varphi(\bar{x},\bar{y})$ means that there exist realizations $\bar{a}$ and $\bar{b}$ of types p and q respectively connected by $\varphi(\bar{x},\bar{y})$ such that $\varphi(\bar{x},\bar{y})$ simultaneously witnesses that $\bar{a}$ semi-isolates $\bar{b}$ over A and that $\bar{b}$ semi-isolates $\bar{a}$ over A.

Again, by Lemma 1.3, the non-symmetry with respect to (p,q)-preservation for $\varphi(\bar{x},\bar{y})$ means that $\varphi(\bar{x},\bar{y})$ witnesses that $\bar{a}$ semi-isolates $\bar{b}$ over A, but it cannot witness that $\bar{b}$ semi-isolates $\bar{a}$ over A or, conversely, $\varphi(\bar{x},\bar{y})$ witnesses that $\bar{b}$ semi-isolates $\bar{a}$ over A, but it cannot witness that $\bar{a}$ semi-isolates $\bar{b}$ over A.

The following remark is obvious.

Remark 1.5. *Let $p(\bar{x})$ and $q(\bar{y})$ be complete types over a set A, $\mathfrak{M}\models p(\bar{a})$, $\mathfrak{M}\models q(\bar{b})$. The tuple $\bar{a}$ semi-isolates $\bar{b}$ over A, and $\bar{b}$ does not semi-isolate $\bar{a}$ over A if and only if there exists a formula $\varphi(\bar{x},\bar{y})\in\mathrm{Pres}_{p,q}(\bar{x},\bar{y})$ such that $\mathfrak{M}\models\varphi(\bar{a},\bar{b})$ and there is no formula $\psi(\bar{x},\bar{y})\in\mathrm{Pres}_{q,p}(\bar{x},\bar{y})$ with the properties $\mathfrak{M}\models\psi(\bar{a},\bar{b})$ and $\vdash\psi(\bar{x},\bar{y})\to\varphi(\bar{x},\bar{y})$.*

If $p,q\in S(A)$ then we denote by $\mathrm{SI}_{p,q}$ (in a model $\mathfrak{M}$) the relation of semi-isolation (over A) connecting realizations of types p and q:

$$\mathrm{SI}_{p,q}=\{(\bar{a},\bar{b})\mid\mathfrak{M}\models p(\bar{a})\wedge q(\bar{b})\text{ and }\bar{a}\text{ semi-isolates }\bar{b}\}\cup$$
$$\cup\{(\bar{b},\bar{a})\mid\mathfrak{M}\models p(\bar{a})\wedge q(\bar{b})\text{ and }\bar{b}\text{ semi-isolates }\bar{a}\}.$$

In accordance with Lemma 1.1, the non-symmetry of $\mathrm{SI}_{p,q}$ means that some $(p\to q)$-formula (or $(q\to p)$-formula) φ connects some realizations $\bar{a}$ and $\bar{b}$ of types p and q respectively and there is no $(q\to p)$-formula (accordingly a $(p\to q)$-formula) ψ connecting these tuples and such that $\vdash\psi\to\varphi$.

The following two propositions show that, for the characterization on non-symmetry of $\mathrm{SI}_{p,q}$, it suffices to consider a case when the types p and q are non-principal.

Proposition 1.6. *If $p(\bar{x}),q(\bar{y})\in S(A)$ are principal types isolated by formulas $\theta_p(\bar{x})$ and $\theta_q(\bar{y})$ respectively, then $\theta_p(\bar{x})\wedge\theta_q(\bar{y})$ is a $(p\leftrightarrow q)$-formula. Moreover, for any realizations $\bar{a}$ and $\bar{b}$ in a model $\mathfrak{M}$ of types p and q respectively, the formula $\theta_p(\bar{x})\wedge\theta_q(\bar{y})$ witnesses that $\bar{a}$ semi-isolates $\bar{b}$ over A and that $\bar{b}$ semi-isolates $\bar{a}$ over A.*

Proof is obvious. □

Proposition 1.7. *If T is a countable theory, $p(\bar{x})\in S(A)$ is a non-principal type, $q(\bar{y})\in S(A)$ is a principal type isolated by a formula $\theta_q(\bar{y})$ then the formula $\varphi(\bar{x},\bar{y})\rightleftharpoons\theta_q(\bar{y})$ is non-symmetric with respect to (p,q)-preservation. Moreover, for any realizations $\bar{a}$ and $\bar{b}$ in a model $\mathfrak{M}$ of types p and q respectively, the formula $\varphi(\bar{x},\bar{y})$ witnesses that $\bar{a}$ semi-isolates $\bar{b}$ over A and there is no formula $\psi(\bar{x},\bar{y})$, with parameters in A, witnessing that $\bar{b}$ semi-isolates $\bar{a}$ over A.*

Proof. Since the formula $\theta_q(\bar{y})$ isolates $q(\bar{y})$, we have $\mathfrak{M}\models\varphi(\bar{a},\bar{b})$ $(=\theta_q(\bar{b}))$ and $\varphi(\bar{x},\bar{y})\,(=\theta_q(\bar{y}))\vdash q(\bar{y})$. On the other hand, there is no formula $\psi(\bar{x},\bar{y})$, with parameters in A, witnessing that $\bar{b}$ semi-isolates $\bar{a}$ over A since otherwise, for a model $\mathfrak{N}$ omitting the type p and realizing the principal type $q(\bar{y})$ by some tuple $\bar{b}'$, $\psi(\bar{x},\bar{b}') \vdash p(\bar{x})$ holds, which is impossible in view of $\mathfrak{N} \models \exists x\, \psi(\bar{x},\bar{b}')$. □

We say that a type $q(\bar{x})$ (not necessarily complete) over a set A is *isolated* (or *defined*) by a set $\Phi(\bar{x},A)$ of formulas in q if $\Phi(\bar{x},A) \vdash q(\bar{x})$.

Consider non-principal types $p(\bar{x}), q(\bar{y}) \in S(A)$ over at most countable set A and realized in a countable model $\mathfrak{M}$ of a countable theory T. Let $\Theta(\bar{x}) \subset p(\bar{x})$ and $\Theta'(\bar{y}) \subset q(\bar{y})$ be isolating sets for these types consisting of formulas $\theta_n(\bar{x})$ and $\theta'_n(\bar{y})$, $n \in \omega$, respectively, such that the following conditions are satisfied:

1) $\vdash \forall \bar{x}\, \theta_0(\bar{x}) \wedge \forall \bar{y}\, \theta'_0(\bar{y})$;

2) $\vdash \theta_{n+1}(\bar{x}) \to \theta_n(\bar{x})$, $\models \exists \bar{x}(\theta_n(\bar{x}) \wedge \neg\theta_{n+1}(\bar{x}))$;

3) $\vdash \theta'_{n+1}(\bar{y}) \to \theta'_n(\bar{y})$, $\models \exists \bar{y}(\theta'_n(\bar{y}) \wedge \neg\theta'_{n+1}(\bar{y}))$.[1]

If $p = q$, we assume that $\theta_n = \theta'_n$, $n \in \omega$.

The formula θ_n is called an *n-neighborhood* of type p, and the formula θ'_n is called an *n-neighborhood of type q*. We say that a tuple $\bar{a}$ (accordingly $\bar{b}$) *has the color n* if $\mathfrak{M} \models \theta_n(\bar{a}) \wedge \neg\theta_{n+1}(\bar{a})$ ($\mathfrak{M} \models \theta'_n(\bar{b}) \wedge \neg\theta'_{n+1}(\bar{b})$). The realizations of types p and q are said to have the *infinite color* ∞. Here we assume that $n < \infty$ for any $n \in \omega$.

Proposition 1.8. *For any non-principal types $p(\bar{x})$ and $q(\bar{y})$ in $S(A)$ realized in a countable model $\mathfrak{M}$ of a countable theory T by tuples $\bar{a}$ and $\bar{b}$ accordingly and for any formula $\varphi(\bar{x},\bar{y})$ with parameters in A satisfying the condition $\mathfrak{M} \models \varphi(\bar{a},\bar{b})$, the formula $\varphi(\bar{x},\bar{y})$ witnesses that $\bar{b}$ is semi-isolated over $\bar{a}$ with respect to A if and only if, for any $n' \in \omega$, there exists $n \in \omega$ such that, for any tuple $\bar{a}_n$ of color $\geq n$ with $\mathfrak{M} \models \theta_n(\bar{a}_n)$, any realization of the formula $\varphi(\bar{a}_n,\bar{y})$ in $\mathfrak{M}$ has color $\geq n'$.*

Proof. Suppose that the formula $\varphi(\bar{x},\bar{y})$ witnesses that $\bar{b}$ is semi-isolated over $\bar{a}$ with respect to A. By Lemmas 1.1 and 1.3, it is equivalent to the statement that, for any formula $\theta'_{n'}(\bar{y})$, there exists a formula $\theta_n(\bar{x})$ such that

$$\mathfrak{M} \models \forall \bar{x},\bar{y}\left((\theta_n(\bar{x}) \wedge \varphi(\bar{x},\bar{y})) \to \theta'_{n'}(\bar{y})\right).$$

This means that, for any tuple $\bar{a}_n$ of color $\geq n$ with $\mathfrak{M} \models \theta_n(\bar{a}_n)$, any realization of $\varphi(\bar{a}_n,\bar{y})$ in $\mathfrak{M}$ has a color $\geq n'$. □

Corollary 1.9. *For any non-principal types $p(\bar{x})$ and $q(\bar{y})$ in $S(A)$, realized in a countable model $\mathfrak{M}$ of a countable theory T by tuples $\bar{a}$ and $\bar{b}$ respectively, as well as for any formula $\varphi(\bar{x},\bar{y})$ with parameters in A satisfying the condition $\mathfrak{M} \models \varphi(\bar{a},\bar{b})$, the following conditions are equivalent:*

1) *the formula $\varphi(\bar{x},\bar{y})$ witnesses that $\bar{b}$ is semi-isolated over $\bar{a}$ with respect to A but cannot witness that $\bar{a}$ is semi-isolated over $\bar{b}$ with respect to A;*

2) *the following conditions are satisfied:*

(a) *for any $n' \in \omega$, there exists an $n \in \omega$ such that for any tuple $\bar{a}_n$ of color $\geq n$ with $\mathfrak{M} \models \theta_n(\bar{a}_n)$, any realization of $\varphi(\bar{a}_n,\bar{y})$ in $\mathfrak{M}$ has a color $\geq n'$;*

(b) *there exists $n \in \omega$ such that for any $n' \in \omega$ there are tuples $\bar{a}_n$ and $\bar{b}_{n'}$ of finite colors $< n$ and $\geq n'$ respectively such that $\mathfrak{M} \models \varphi(\bar{a}_n,\bar{b}_{n'})$.*

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[1] The existence of these isolating sets of formulas follows from the fact that the set of formulas with parameters in A is countable. Indeed, we enumerate all formulas belonging, for instance, to the type $p(\bar{x})$: φ_n, $n \in \omega$. For any $n \in \omega$, we denote by ψ_n the formula $\bigwedge\limits_{i<n} \varphi_i$ assuming $\psi_0 = (\bar{x} \approx \bar{x})$. Now we remove from the sequence of formulas ψ_n all formulas equivalent to some their predecessors and obtain the sequence $(\theta_n(\bar{x}))_{n\in\omega}$.

Proof. By Proposition 1.8, the condition that the formula $\varphi(\bar{x},\bar{y})$ witnesses that $\bar{b}$ is semi-isolated over $\bar{a}$ with respect to A is equivalent to the property (a).

Now we assume that the formula $\varphi(\bar{x},\bar{y})$ does not witness that $\bar{a}$ is semi-isolated over $\bar{b}$ with respect to A. Then, by Lemmas 1.1 and 1.3, this means that there exists a formula $\theta_n(\bar{x})$ such that for any formula $\theta'_{n'}(\bar{y})$ the following holds:

$$\mathfrak{M} \models \exists \bar{x},\bar{y}\,(\theta'_{n'}(\bar{y}) \wedge \varphi(\bar{x},\bar{y}) \wedge \neg\theta_n(\bar{x}))\,.$$

This means that there exists an $n \in \omega$ such that for any $n' \in \omega$ there are tuples $\bar{a}_n$ and $\bar{b}_{n'}$ of finite colors $< n$ and $\geq n'$ respectively such that $\mathfrak{M} \models \varphi(\bar{a}_n, \bar{b}_{n'})$, i. e., the property (b) holds. □

2. Ehrenfeucht theories, powerful types, relations of semi-isolation, and quasi-neighborhoods

For a theory T, we denote by $I(T,\lambda)$ the number of pairwise non-isomorphic models of T in a power λ. A theory T is called *Ehrenfeucht* if $1 < I(T,\omega) < \omega$.

Definition (M. Benda [12]). A type $p(\bar{x}) \in S(T)$ is said to be *powerful* in a theory T if every model $\mathfrak{M}$ of T realizing p also realizes every type $q \in S(T)$, that is, $\mathfrak{M} \models S(T)$.

Since for any type $p \in S(T)$ there exists a countable model $\mathfrak{M}$ of T, realizing p, and the model $\mathfrak{M}$ realizes exactly countably many types, the availability of a powerful type implies that T is *small*, that is, the set $S(T)$ is countable. Hence for any type $q \in S(T)$ and its realization $\bar{a}$, there exists a model $\mathfrak{M}(\bar{a})$ prime over $\bar{a}$. Since all prime models over realizations of q are isomorphic, we often denote these models by $\mathfrak{M}_q$.

The condition that $p(\bar{x})$ is a powerful type means that every type in $S(T)$ is realized in $\mathfrak{M}_p$, that is, $\mathfrak{M}_p \models S(T)$. Every type in $S(T)$ of ω-categorical theory T is powerful.

Proposition 2.1 (M. Benda [12]). *Every Ehrenfeucht theory T has a powerful type.*

Proof. Assume on the contrary that, for any type $q(\bar{x}) \in S(\varnothing)$ there exists a type $r_q(\bar{y}) \in S(\varnothing)$ such that a model $\mathfrak{M}_q$ (of T) omits $r_q(\bar{y})$. Denote by $p_0(\bar{x})$ an arbitrary type in $S(\varnothing)$ and by $p_1(\bar{x},\bar{y})$ a type in $S(\varnothing)$ containing the type $p_0(\bar{x}) \cup r_{p_0}(\bar{y})$. Now we construct by induction a sequence $p_n \in S(\varnothing)$, $n \in \omega$, such that $p_n \subset p_{n+1}$ and $\mathfrak{M}_{p_n}$ omits p_{n+1}. By the construction we have that $p_n \subset p_m$ for $n < m$ and hence $\mathfrak{M}_{p_n}$ omits p_m if and only if $n < m$. Then $\mathfrak{M}_{p_n} \not\cong \mathfrak{M}_{p_m}$ for $n \neq m$. Thus $I(T,\omega) \geq \omega$ and we get a contradiction. □

As illustrations, we consider the following *Ehrenfeucht examples* [13] of theories T_n, $n \in \omega$, with $I(T_n,\omega) = n \geq 3$.

Example 2.2. Let T_n be the theory of a structure $\mathfrak{M}^n$ obtained from the structure $\langle \mathbb{Q}; < \rangle$ by adding constants c_k, $k \in \omega$, such that $\lim\limits_{k\to\infty} c_k = \infty$ and unary predicates $P_0,\ldots,P_{n-3}$ which form a partition of the set $\mathbb{Q}$ of rationals, with

$$\models \forall x,y\,((x<y) \to \exists z\,((x<z) \wedge (z<y) \wedge P_i(z))),\ i = 0,\ldots,n-3.$$

The theory T_n has exactly n pairwise non-isomorphic models:

(a) a prime model $\mathfrak{M}^n$ ($\lim\limits_{k\to\infty} c_k = \infty$);

(b) prime models $\mathfrak{M}^n_i$ over realizations of types $p_i(x) \in S^1(\varnothing)$, isolated by sets of formulas $\{c_k < x \mid k \in \omega\} \cup \{P_i(x)\}$, $i = 0,\ldots,n-3$ $\big(\lim\limits_{k\to\infty} c_k \in P_i\big)$;

(c) a saturated model $\overline{\mathfrak{M}}^n$ (the limit $\lim\limits_{k\to\infty} c_k$ is irrational). □

If $p \in S(T)$ then SI_p (in the model $\mathfrak{M}$) denotes the relation of semi-isolation (over $\varnothing$) on a set of realizations of p:

$$\mathrm{SI}_p \rightleftharpoons \{(\bar{a}, \bar{b}) \mid \mathfrak{M} \models p(\bar{a}) \wedge p(\bar{b}) \text{ and } \bar{a} \text{ semi-isolates } \bar{b}\}.$$

Analogously, we denote by I_p (in the model $\mathfrak{M}$) the isolation relation (over $\varnothing$) on the set of realizations of p:

$$I_p \rightleftharpoons \{(\bar{a}, \bar{b}) \mid \mathfrak{M} \models p(\bar{a}) \wedge p(\bar{b}) \text{ and } \bar{a} \text{ isolates } \bar{b}\}.$$

Repeating the arguments on the preorder of semi-isolation for the set of realizations of type p, we obtain that the relation SI_p is also a preorder. The preorder SI_p is called the *preorder of semi-isolation* on the set of realizations of type p.

At the same time, contrasting to the semi-isolation, it is easy to construct an example of a theory with a non-transitive relation I_p, i. e., generally speaking, the isolation can be not preserved under two-step transitions by the relation of isolation.

Proposition 2.3 (A. Pillay [7]). *If $p \in S(T)$ is a non-principal powerful type having a realization in a model $\mathfrak{M}$ of T, then the relation SI_p on the set of realization of p in $\mathfrak{M}$ is non-symmetric. Moreover, there exist realizations $\bar{a}$ and $\bar{b}$ of p in $\mathfrak{M}$ such that the type $\mathrm{tp}(\bar{b}/\bar{a})$ is principal and $\bar{b}$ does not semi-isolate $\bar{a}$.*

Proof. At first we consider the set

$$q(\bar{x}, \bar{y}) \rightleftharpoons p(\bar{x}) \cup p(\bar{y}) \cup \{\neg\varphi(\bar{x}, \bar{y}) \mid \varphi(\bar{x}, \bar{y}) \text{ is a } p\text{-preserving formula}\}$$

and show that it is consistent. Since any disjunction of p-preserving formulas is p-preserving as well, by Compactness Theorem it suffices to prove that any formula $\theta(\bar{y}) \wedge \neg\varphi(\bar{a}, \bar{y})$ is consistent, where $\theta(\bar{y}) \in p(\bar{y})$, $\varphi(\bar{x}, \bar{y})$ is a p-preserving formula, and $\mathfrak{M} \models p(\bar{a})$. The consistency of this formula follows from $\varphi(\bar{a}, \bar{y}) \vdash p(\bar{y})$ and the fact that the non-principality of p implies the existence of a tuple $\bar{b} \in M$ such that $\mathfrak{M} \not\models p(\bar{b})$ and $\mathfrak{M} \models \theta(\bar{b})$. This tuple $\bar{b}$ realizes the formula $\theta(\bar{y}) \wedge \neg\varphi(\bar{a}, \bar{y})$ in $\mathfrak{M}$.

Since the set $q(\bar{x}, \bar{y})$ is consistent, it could be extended to a complete type $r(\bar{x}, \bar{y}) \in S(T)$. As p is powerful, the type r is realized in any model realizing p. So there exists a consistent formula $\psi(\bar{x}, \bar{y}, \bar{z})$ such that $\psi(\bar{c}, \bar{y}, \bar{z}) \vdash r(\bar{y}, \bar{z})$ for any (some) realization $\bar{c}$ of p. Assuming that $\mathfrak{M} \models \psi(\bar{c}, \bar{a}, \bar{b})$ for realizations $\bar{a}, \bar{b}, \bar{c}$ of p, we get $(\bar{a}, \bar{b}) \notin \mathrm{SI}_p$.

Now assuming on the contrary, that SI_p is symmetric, we get that SI_p is an equivalence relation. Since $(\bar{c}, \bar{a}\,\hat{}\,\bar{b}) \in \mathrm{SI}_{p,r}$ then, because $\bar{c}$ semi-isolates $\bar{a}$ and $\bar{b}$, we have $(\bar{c}, \bar{a}) \in \mathrm{SI}_p$ and $(\bar{c}, \bar{b}) \in \mathrm{SI}_p$. Thus, $\bar{a}$ and $\bar{b}$ belong to the same SI_p-class, which contradicts to $(\bar{a}, \bar{b}) \notin \mathrm{SI}_p$.

Since the type p is powerful, the type $q(\bar{x}, \bar{y})$ is realized by some pair $(\bar{b}, \bar{c})$ in the model $\mathfrak{M}_p$, which is $\mathfrak{M}(\bar{a})$ for some realization $\bar{a}$ of p. Then $(\bar{a}, \bar{b}) \in I_p$ and $\bar{b}$ cannot semi-isolate $\bar{a}$, since otherwise, by transitivity of semi-isolation, $\bar{b}$ semi-isolates $\bar{c}$ in spite of definition of q. □

Thus the availability of a non-principal powerful type $p(\bar{x})$ presumes the existence of a formula $\varphi(\bar{x}, \bar{y})$, $l(\bar{x}) = l(\bar{y})$, such that, for any (some) realization $\bar{a}$ of p, the following conditions hold:

(1) $\varphi(\bar{a}, \bar{y}) \vdash p(\bar{y})$;

(2) $\varphi(\bar{x}, \bar{a}) \not\vdash p(\bar{x})$, and moreover, there exists a tuple $\bar{b}$ which realizes type p and is such that $\models \varphi(\bar{b}, \bar{a})$ and $\bar{a}$ does not semi-isolate $\bar{b}$.

Every formula $\varphi(\bar{x},\bar{y})$, satisfying the conditions 1 and 2, is called a *formula, witnessing that the relation* SI_p *is non-symmetric*.

Definition (B. S. Baizhanov [14]). Let $p(\bar{x})$ be some (may be incomplete) n-type over a set $A\subseteq M$ in a model $\mathfrak{M}$ of a theory T, B be a set in the model $\mathfrak{M}$. A *quasi-neighborhood of* B *in* p is the set $\mathrm{QV}_{p,\mathfrak{M}}(B)$ of all tuples $\bar{c}\in M$ such that there exist a tuple $\bar{b}\in B$ and a $(\mathrm{tp}(\bar{b}/A),p)$-preserving formula $\varphi(\bar{x},\bar{y})$ with $\mathfrak{M}\models\varphi(\bar{b},\bar{c})$.

A *quasi-neighborhood of* B *in* $S^n(A)$ is a set

$$\mathrm{QV}^n_{A,\mathfrak{M}}(B)\rightleftharpoons\bigcup_{p\in S^n(A)}\mathrm{QV}_{p,\mathfrak{M}}(B).$$

A *quasi-neighborhood of* B *in* $S(A)$ is a set

$$\mathrm{QV}_{A,\mathfrak{M}}(B)\rightleftharpoons\bigcup_{n\in\omega}\mathrm{QV}^n_{A,\mathfrak{M}}(B).$$

For a tuple $\bar{a}=\langle a_1,\ldots,a_n\rangle$, we write $\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$ (accordingly $\mathrm{QV}^n_{A,\mathfrak{M}}(\bar{a})$, $\mathrm{QV}_{A,\mathfrak{M}}(\bar{a})$) instead of $\mathrm{QV}_{p,\mathfrak{M}}(\{a_1,\ldots,a_n\})$ ($\mathrm{QV}^n_{A,\mathfrak{M}}(\{a_1,\ldots,a_n\})$, $\mathrm{QV}_{A,\mathfrak{M}}(\{a_1,\ldots,a_n\})$).

Obviously, any quasi-neighborhood of form $\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$, where p is a n-type and $\mathfrak{M}\models p(\bar{a})$, is nonempty: $\bar{a}\in\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$. Thus, $\bar{a}\in\mathrm{QV}^n_{A,\mathfrak{M}}(\bar{a})$. At the same time, the set $\mathrm{QV}_{p,\mathfrak{M}}(B)$ can be empty (for instance, one can take the empty set for B and a non-principal type for p).

Notice that for any tuples $\bar{a}$ and $\bar{b}$ in $\mathfrak{M}$, the tuple $\bar{a}$ semi-isolates $\bar{b}$ if and only if $\bar{b}\in\mathrm{QV}_{\mathrm{tp}(\bar{b}),\mathfrak{M}}(\bar{a})$. In particular, the relation SI_p on the set of realizations of type p in the model $\mathfrak{M}$ coincides with the set of pairs $(\bar{a},\bar{b})$ such that $\mathfrak{M}\models p(\bar{a})$ and $\bar{b}\in\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$.

The reflexivity and the transitivity of the semi-isolation correspond to the following properties:

1. Let $\bar{a}\in M$ be a realization of type $p\in S(A)$. Then $\bar{a}\in\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$.

2. Let q, r be types in $S(A)$, $\bar{a}$ be a tuple in $\mathfrak{M}$, $\bar{b}\in\mathrm{QV}_{q,\mathfrak{M}}(\bar{a})$, and $\bar{c}\in\mathrm{QV}_{r,\mathfrak{M}}(\bar{b})$. Then $\bar{c}\in\mathrm{QV}_{r,\mathfrak{M}}(\bar{a})$.

Thus, the relation $\bar{a}\in\mathrm{QV}_{p,\mathfrak{M}}(\bar{b})$ is a preorder on the set of realizations of p in $\mathfrak{M}$. By the same way we get that the relation $\bar{a}\in\mathrm{QV}^n_{A,\mathfrak{M}}(\bar{b})$ is a preorder on M^n and that $\bar{a}\in\mathrm{QV}_{A,\mathfrak{M}}(\bar{b})$ is a preorder on the set of all tuples in M.

The transitivity property above implies that if $\bar{b}\in\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$ then $\mathrm{QV}_{p,\mathfrak{M}}(\bar{b})\subseteq\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$.

Proposition 2.4 *The relation* SI_p *on the set of realizations of a type* p *in a model* $\mathfrak{M}$ *is non-symmetric if and only if, for every (some) realization* $\bar{a}$ *of* p *in* $\mathfrak{M}$*, there exists a tuple* $\bar{b}\in\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$ *such that* $\mathrm{QV}_{p,\mathfrak{M}}(\bar{b})\subset\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$.

Proof follows directly from the definitions. □

In view of Proposition 2.4, Proposition 2.3 admits the following reformulation:

Proposition 2.5 *If* p *is a non-principal powerful type realized in a model* $\mathfrak{M}$ *via some tuple* $\bar{a}$ *then there exists a tuple* $\bar{b}\in\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$ *such that* $\mathrm{QV}_{p,\mathfrak{M}}(\bar{b})\subset\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$.

Definition. Let $p(\bar{x})$ be some (may be incomplete) n-type over a set $A\subseteq M$ in a model $\mathfrak{M}$ of a theory T and let B be a set in $\mathfrak{M}$. The *neighborhood of* B *in the* *type* p is the set $V_{p,\mathfrak{M}}(B)$ consisting of all tuples $\bar{c} \in M$ such that $\mathfrak{M} \models p(\bar{c})$ and there exist a tuple $\bar{b} \in B$ and a $(\mathrm{tp}(\bar{b}/A) \leftrightarrow \mathrm{tp}(\bar{c}/A))$-formula $\varphi(\bar{x}, \bar{y})$ such that $\mathfrak{M} \models \varphi(\bar{b}, \bar{c})$.

The set

$$V^n_{A,\mathfrak{M}}(B) \rightleftharpoons \bigcup_{p \in S^n(A)} V_{p,\mathfrak{M}}(B)$$

is the *neighborhood of set* B *in* $S^n(A)$.

The set

$$V_{A,\mathfrak{M}}(B) \rightleftharpoons \bigcup_{n \in \omega} V^n_{A,\mathfrak{M}}(B)$$

is the *neighborhood of set* B *in* $S(A)$.

Note the following easy properties of neighborhoods.

1. Let p, q be types in $S^n(A)$, $\bar{a}$ be a realization of p in a model $\mathfrak{M}$ of a theory T, $A \subseteq M$. Then $\bar{b} \in V_{q,\mathfrak{M}}(\bar{a})$ if and only if there exists a $(p \leftrightarrow q)$-formula $\varphi(\bar{x}, \bar{y})$ such that $\mathfrak{M} \models \varphi(\bar{a}, \bar{b})$.

2. Let $\bar{a}$ be a realization of a type $p \in S^n(A)$ in a model $\mathfrak{M}$ of a theory T, $A \subseteq M$. Then $\bar{a} \in V_{p,\mathfrak{M}}(\bar{a})$ and $\bar{a} \in V^n_{A,\mathfrak{M}}(\bar{a})$.

3. Let $\bar{a}$ be a realization of a type $p \in S^n(A)$ and $\bar{b}$ be a realization of a type $q \in S^n(A)$, both in a model $\mathfrak{M}$ of a theory T, $A \subseteq M$. If $\bar{a} \in V_{p,\mathfrak{M}}(\bar{b})$ then $\bar{b} \in V_{q,\mathfrak{M}}(\bar{a})$. Besides, if $\bar{a} \in V^n_{A,\mathfrak{M}}(\bar{b})$ then $\bar{b} \in V^n_{A,\mathfrak{M}}(\bar{a})$.

4. Let p and q be types in $S(A)$ and B be a set in a model $\mathfrak{M}$ of theory T, $A \subseteq M$. If $\bar{b} \in V_{p,\mathfrak{M}}(B)$ and $\bar{c} \in V_{q,\mathfrak{M}}(\bar{b})$ then $\bar{c} \in V_{q,\mathfrak{M}}(B)$. A similar property is satisfied for $V^n_{A,\mathfrak{M}}$ and for $V_{A,\mathfrak{M}}$.

Thus, the relation $\bar{a} \in V_{p,\mathfrak{M}}(\bar{b})$ is an equivalence relation on the set of realizations of type p in the model $\mathfrak{M}$, as well as $\bar{a} \in V^n_{A,\mathfrak{M}}(\bar{b})$ is an equivalence relation on the set M^n, and $\bar{a} \in V_{A,\mathfrak{M}}(\bar{b})$ is an equivalence relation on the set of all tuples in the model $\mathfrak{M}$.

Finally we obtain that, for instance, the relation $\bar{a} \in \mathrm{QV}_{A,\mathfrak{M}}(\bar{b})$ modulo the equivalence relation $\bar{a} \in V_{A,\mathfrak{M}}(\bar{b})$ forms a partial order on the set of equivalence classes of tuples in $\mathfrak{M}$.

If a theory has a non-principal powerful type p then this preorder contains an infinite chain in any model realizing p.

3. Colorings of structures and semi-isolation

The notions we consider here enable us to justify some possibilities for colorings of neighborhoods of types described in Corollary 1.9 and guaranteeing the non-symmetry of semi-isolation.

Let $\mathfrak{M}$ be a structure. Any function Col: $M \to \lambda \cup \{\infty\}$, where λ is a power and ∞ is a symbol of infinity, is said to be a *coloring of structure* $\mathfrak{M}$. Here, for any $a \in M$, the value $\mathrm{Col}(a)$ is said to be the *color of element* a. A pair $\langle \mathfrak{M}, \mathrm{Col} \rangle$ is said to be a *colored structure*.

Below, colored structures $\langle \mathfrak{M}, \mathrm{Col} \rangle$ will be identified with expansions of $\mathfrak{M}$ by unary predicates $\mathrm{Col}_\mu = \{a \in M \mid \mathrm{Col}(a) = \mu\}$, $\mu < \lambda$.

Definition (S. V. Sudoplatov [1, 15, 16]). A coloring Col of a structure $\mathfrak{M}$ is n-*inessential*, $n \in \omega \setminus \{0\}$, if for any model $\langle \mathfrak{M}', \mathrm{Col}' \rangle \models \mathrm{Th}(\langle \mathfrak{M}, \mathrm{Col} \rangle)$ the type $\mathrm{tp}(\bar{a})$ of each tuple $\bar{a}$ in $\langle \mathfrak{M}', \mathrm{Col}' \rangle$ of length n is isolated by the type of this tuple in $\mathfrak{M}'$ being united with the set of formulas describing colors for elements in $\bar{a}$.

Let $\mathfrak{M}$ be a model of a theory T and $\varphi(x,y)$ be a formula of T. A coloring Col: $M \to \lambda \cup \{\infty\}$ (where λ is an infinite cardinality) is said to be *φ-ordered* if the following conditions are satisfied:

(a) for any $\mu \leq \nu < \lambda$ there exist elements $a, b \in M$ such that $\models \mathrm{Col}_\mu(a) \wedge \mathrm{Col}_\nu(b) \wedge \varphi(a,b)$;

(b) if $\mu < \nu < \lambda$ then there are no elements $c, d \in M$ such that $\models \mathrm{Col}_\mu(c) \wedge \mathrm{Col}_\nu(d) \wedge \varphi(d,c)$.

Recall, that a theory T is said to be *transitive* if T has unique 1-type over the empty set.

Note that if Col: $M \to \lambda \cup \{\infty\}$ is a surjective 1-inessential coloring of a model $\mathfrak{M}$ of a transitive theory T, then the set of complete 1-types of $\mathrm{Th}(\langle \mathfrak{M}, \mathrm{Col}\rangle)$ over $\varnothing$ consists of types $p_\mu(x)$, $\mu \in \lambda \cup \{\infty\}$, where $p_\mu(x)$ is a type isolated by the formula $\mathrm{Col}_\mu(x)$, $\mu \in \lambda$, and $p_\infty(x)$ is a (unique) non-principal type isolated by set of formulas $\{\neg \mathrm{Col}_\mu(x) \mid \mu < \lambda\}$.

The colorings in Section 1 can be transformed into colorings of structures, and the Ehrenfeucht examples illustrate this transformation.

In the Ehrenfeucht example of theory T_3 with three countable models, the model expansion of a transitive theory $\mathrm{Th}(\langle \mathbb{Q}; <\rangle)$ by constants c_k, $k \in \omega$, can be interpreted as an inessential coloring Col specified by the following conditions:

$$\mathrm{Col}(a) = \begin{cases} 0 & \text{if } a < c_0, \\ 2k+1 & \text{if } a = c_k, \\ 2k+2 & \text{if } c_k < a < c_{k+1}. \end{cases}$$

It is easy to note that Col is φ-ordered, where $\varphi(x,y) \rightleftharpoons x < y$. In addition, the relation SI_{p_∞} on the set of realizations of powerful type p_∞ is non-symmetric, and the formula φ witnesses this. In Ehrenfeucht examples of T_n, $n \geq 4$, constant expansions of the structures $\langle \mathbb{Q}; <, P_0, \ldots, P_{n-3}\rangle$ can also be seen as color models with inessential ordered colorings.

Now we show that if a theory is obtained by means of an 1-inessential ordered coloring of a transitive theory and has a unique non-principal complete 1-type then this coloring has the cardinality $\lambda = \omega$.

Proposition 3.1. *If $\varphi(x,y)$ is a formula of a transitive theory T and* Col: $M \to \lambda \cup \{\infty\}$ *is a surjective* 1*-inessential φ-ordered coloring of a model $\mathfrak{M}$ of a theory T then $\lambda = \omega$.*

Proof. As it was noticed above, the 1-inessentiality of coloring implies that there exists the unique non-principal complete 1-type of theory $\mathrm{Th}(\langle \mathfrak{M}, \mathrm{Col}\rangle)$.

Now assume that $\lambda > \omega$. Consider the following sets of formulas:

$$q_0(x) = \{\exists y, z(\mathrm{Col}_n(y) \wedge \mathrm{Col}_\omega(z) \wedge \varphi(y,x) \wedge \varphi(x,z)) \mid n \in \omega\},$$

$$q_1(x) = \{\exists y(\mathrm{Col}_\mu(y) \wedge \varphi(y,x)) \mid \mu < \lambda\}.$$

By Compactness Theorem, each q_i, $i = 0, 1$, is consistent. Since the coloring Col is φ-ordered, the set $q_0(x) \cup q_1(x)$ is inconsistent. At the same time, since the coloring Col is surjective and φ-ordered, each q_i, $i = 0, 1$, is extensible to a complete non-principal 1-type of theory $\mathrm{Th}(\langle \mathfrak{M}, \mathrm{Col}\rangle)$. Thus we get a contradiction. □

Consider sufficient conditions for a φ-ordered 1-inessential coloring to imply a non-symmetry of relation SI_{p_∞} witnessed by φ.

Proposition 3.2 (S. V. Sudoplatov [1, 16]). *Let $\varphi(x,y)$ be a principal formula of a transitive theory T and* Col *be an* 1*-inessential φ-ordered coloring of a model $\mathfrak{M}$ of T such that $\langle\mathfrak{M}',\mathrm{Col}'\rangle \models \mathrm{Th}(\langle\mathfrak{M},\mathrm{Col}\rangle)$ and $\langle\mathfrak{M}',\mathrm{Col}'\rangle \models \varphi(a,b)$ imply that the type $\mathrm{tp}_{\langle\mathfrak{M}',\mathrm{Col}'\rangle}(ab)$ is isolated both by $\mathrm{tp}_{\mathfrak{M}'}(ab)$ and by formulas for colors of a and b. Then for any (i. e., some) realization a of type $p_\infty(x)$ the following conditions are satisfied:*

(1) *if $\models \varphi(a,b)$ then $\models p_\infty(b)$ and a semi-isolates b;*

(2) *if $\models \varphi(a,b)$ then b does not semi-isolate a.*

Proof. By Proposition 3.1, w.l.o.g. $\lambda = \omega$.

(1) Consider the following set of formulas:

$$q(x) = \{\neg\mathrm{Col}_m(x) \mid m < \omega\} \cup \left\{ \neg\exists y \left(\varphi(x,y) \wedge \bigvee_{n\in w} \mathrm{Col}_n(y)\right) \;\middle|\; w \text{ is a finite set of natural numbers} \right\}.$$

By the item (b) of the definition of φ-ordered coloring, that set is locally consistent (it suffices to consider a formula

$$\neg\exists y \left(\varphi(x,y) \wedge \bigvee_{n\in w} \mathrm{Col}_n(y)\right)$$

with a finite set $w = \{i_1,\dots,i_k\}$, a finite set of formulas $\neg\mathrm{Col}_j(x)$, $j = j_1,\dots,j_m$, and to take, for a realization, an element of a color $k < \omega$, greater than all $i_1,\dots,i_k,j_1,\dots,j_m$). By Compactness Theorem, the set $q(x)$ is consistent. Since the coloring Col is 1-inessential, the theory $\mathrm{Th}(\langle\mathfrak{M},\mathrm{Col}\rangle)$ has the unique non-principal 1-type $p_\infty(x)$, and the consistency of this type with the set $q(x)$ implies the validity of the inclusion $q(x) \subset p_\infty(x)$. Finally we notice that, by the definition of $q(x)$, the formula $\varphi(a,y)$, where $\models p_\infty(a)$, cannot have realizations b with some condition $\mathrm{Col}(b) = n$, $n < \omega$. So $\models \varphi(a,b)$ implies $\models p_\infty(b)$, $\varphi(a,y) \vdash p_\infty(y)$, and thus a semi-isolates b.

(2) At first we show that the formula $\varphi(x,y)$ is not p_∞-preserving with respect to the first coordinate. To do so, we consider, for an arbitrary $m < \omega$, the following set of formulas in the language of the structure $\langle\mathfrak{M},\mathrm{Col}\rangle$:

$$r_m(y) \rightleftharpoons \{\neg\mathrm{Col}_n(y) \mid n < \omega\} \cup \{\exists x(\mathrm{Col}_m(x) \wedge \varphi(x,y))\}.$$

Since the set $r_m(y)$ is consistent with the type $p_\infty(y)$ by item (a) of the definition of φ-ordering, and the φ-ordered coloring Col is 1-inessential, we obtain an inclusion $r_m(y) \subset p_\infty(y)$. This means that, for any realization b of $p_\infty(y)$ in a model $\mathfrak{N}$ of $\mathrm{Th}(\langle\mathfrak{M},\mathrm{Col}\rangle)$ and for any $m < \omega$, $\mathfrak{N} \models \varphi(a_m,b) \wedge \mathrm{Col}_m(a_m)$ holds for some element a_m in N. Hence, $\varphi(x,y)$ is not p_∞-preserving with respect to the first coordinate.

Assume now to the contrary, that $\models p_\infty(a)$, $\models \varphi(a,b)$, and b semi-isolates a. The condition that the formula $\varphi(x,y)$ is not p_∞-preserving with respect to the first coordinate implies that $\varphi(x,b)$ cannot witness that b semi-isolates a. On the other hand, by the assumption, there is a formula $\psi(x,y)$ such that $\models \psi(a,b)$ and $\psi(x,b) \vdash p_\infty(x)$. In this case the set $p_\infty(x) \cup p_\infty(y) \cup \{\varphi(x,y) \wedge \psi(x,y)\}$ is consistent. By Compactness Theorem and since $p_\infty(x)$ is a non-principal type, $p_\infty(x) \cup p_\infty(y) \cup \{\varphi(x,y) \wedge \neg\psi(x,y)\}$ is consistent too. Hence the set

$$\{\neg\mathrm{Col}_m(x) \wedge \neg\mathrm{Col}_m(y) \mid m < \omega\} \cup \{\varphi(x,y)\}$$

does not semi-isolate a complete type. The latter conflicts with the fact that $\varphi(x, y)$ is a principal formula in T, and with the property that, for any (a, b) with $\models \varphi(a, b)$, the type of (a, b) is isolated by the type of this tuple in $\mathfrak{M}'$ being united with the set of formulas describing colors of a and b. Thus, $\models \varphi(a, b)$ and $\models p_\infty(a)$ imply that b does not semi-isolate a. □

4. $(p_1, \ldots, p_n)$-TYPES, THE STRICT ORDER PROPERTY, RELATIONS OF SEMI-ISOLATION, AND POWERFUL TYPES

S. V. Sudoplatov [1, 17] introduced the notion of (n, p)-type. Before it was used implicitly in R. Woordow [18, 19] and A. Tsuboi [3]. The following notion generalizes this definition.

Definition (K. Ikeda, A. Pillay, A. Tsuboi [20]). Let $p_1(\bar{x}_1), \ldots, p_n(\bar{x}_n)$ be types in $S(T)$ with disjoint free variables. A type $q(\bar{x}_1, \ldots, \bar{x}_n) \in S(T)$ is said to be a $(p_1, \ldots, p_n)$*-type* if $q(\bar{x}_1, \ldots, \bar{x}_n) \supseteq \bigcup\limits_{i=1}^{n} p_i(\bar{x}_i)$. The set of all $(p_1, \ldots, p_n)$-types of T is denoted by $S_{p_1,\ldots,p_n}(T)$. A theory T is *almost ω-categorical* if for any types $p_1(x_1), \ldots, p_n(x_n) \in S(T)$ there are only finitely many types $q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T)$.

It is shown in [20] that if T is an almost ω-categorical theory with $I(T, \omega) = 3$, then a dense linear ordering is interpretable in T.

If $p_1(\bar{x}) = \ldots = p_n(\bar{x}) = p(\bar{x})$, a $(p_1, \ldots, p_n)$-type $q(\bar{x}_1, \ldots, \bar{x}_n)$ is said to be a (n, p)*-type*. The set of all (n, p)-types of T is denoted by $S_{n,p}(T)$ and elements of $S_p(T) \rightleftharpoons \bigcup\limits_{n\in\omega\setminus\{0\}} S_{n,p}(T)$ are called *p-types*.

A type $q(\bar{y})$ in $S_{p_1,\ldots,p_n}(T)$, where $\bar{y}$ is a concatenation of tuples $\bar{y}_i$, $p_i = p_i(\bar{y}_i)$, $i = 1, \ldots, n$, is said to be $(p_1, \ldots, p_n)$*-principal* if there is a formula $\varphi(\bar{y}) \in q(\bar{y})$ such that $\cup\{p_i(\bar{y}_i) \mid i = 1, \ldots, n\} \cup \{\varphi(\bar{y})\} \vdash q(\bar{y})$. If $q(\bar{y})$ is a $(p, \ldots, p)$-principal p-type then this type is said to be *p-principal*.

The following proposition is obvious.

Proposition 4.1. *For any types $p_1(\bar{x}_1), \ldots, p_n(\bar{x}_n) \in S(\varnothing)$ the following conditions are equivalent:*

(1) *the set of $(p_1, \ldots, p_n)$-types with free variables in $(\bar{x}_1, \ldots, \bar{x}_n)$ is finite;*

(2) *any $(p_1, \ldots, p_n)$-type is $(p_1, \ldots, p_n)$-principal.*

By Proposition 4.1, a theory T is almost ω-categorical if and only if for any types $p_1(x_1), \ldots, p_n(x_n) \in S(T)$ any $(p_1, \ldots, p_n)$-type is $(p_1, \ldots, p_n)$-principal. Notice also that Proposition 4.1 admits a natural generalization for uncomplete types $p_1(\bar{x}_1), \ldots, p_n(\bar{x}_n)$.

Recall, that a theory T has the *strict order property* if there exists a formula $\varphi(\bar{x}, \bar{y})$ of T and tuples $\bar{a}_i$, $i \in \omega$ such that the following equivalence holds:

$$\vdash \varphi(\bar{a}_i, \bar{y}) \to \varphi(\bar{a}_j, \bar{y}) \Leftrightarrow i \leq j. \tag{2}$$

Proposition 4.2. *If $p(\bar{x})$ is a non-principal powerful type of a theory T without strict order property then, for any tuple $\bar{a}$ realizing p in a model $\mathfrak{M} \models T$, the set $\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$ is not $\bar{a}$-definable (i. e., a set of solutions of a formula $\varphi(\bar{a}, \bar{y})$, in $\mathfrak{M}$).*

Proof. Assume that $\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$ is $\bar{a}$-definable by some formula $\varphi(\bar{a}, \bar{y})$. Using Proposition 2.5 we obtain that some $\mathrm{QV}_{p,\mathfrak{M}}(\bar{b}) \subset \mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$ is $\bar{b}$-definable by the formula

$\varphi(\bar{b}, \bar{y})$. Since tuples $\bar{a}$ and $\bar{b}$ realize the same type p, there exists an automorphism f in some homogeneous elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ taking $\bar{b}$ to $\bar{a}$. Denoting $f^i(\bar{b})$ by $\bar{a}_i$, $f^0 = \mathrm{id}_N$, $i \in \omega$, we obtain the condition (2), which contradicts the assumption that T does not have the strict order property. □

Remark 4.3. The proof implies that in the theory T without the strict order property each formula $\varphi(\bar{x}, \bar{y})$ witnessing that SI_p is non-symmetric has the following property: $\varphi(\bar{b}, \mathfrak{M}) \not\subset \varphi(\bar{a}, \mathfrak{M})$ for any realizations $\bar{a}$ and $\bar{b}$ of p in a model $\mathfrak{M} \models T$ such that $\bar{a}$ semi-isolates $\bar{b}$ by $\varphi(\bar{x}, \bar{y})$, and $\bar{b}$ does not semi-isolate $\bar{a}$. Consider a p-preserving formula $H_\varphi(\bar{x}, \bar{y}) \rightleftharpoons \exists \bar{z}(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{y}, \bar{z}) \wedge \neg\varphi(\bar{x}, \bar{z}))$ for which we obviously have $\models H_\varphi(\bar{a}, \bar{b})$. Since $\bar{a}$ and $\bar{b}$ realize the same type p, there exists a sequence $\bar{b}_0 = \bar{b}, \bar{b}_1, \ldots, \bar{b}_n, \ldots$ of realizations of p that forms on the set of realizations of p with the relation defined by $H_\varphi(\bar{x}, \bar{y})$ the graph with distances $\rho(\bar{b}_i, \bar{b}_j) = |i - j|$. The diameter of this graph equals ∞. □

By the definition, theories with infinite definable linear orders have the strict order property. In particular, the Ehrenfeucht examples (see Example 2.2) have this property because they are almost ω-categorical.

The following proposition, which is implicitly contained in R. E. Woodrow [19], clarifies that the described situation is impossible for the theories without strict order property.

Proposition 4.4. *If $p(\bar{x})$ is a non-principal powerful type of theory T and T does not have strict order property, then $|S_{2,p}(T)| = \omega$. Moreover, for any model $\mathfrak{M}$ of T realizing the type p, there are infinitely many p-preserving formulas which are pairwise non-equivalent on the set of realizations of p in $\mathfrak{M}$.*

Proof. Notice that the set $\mathrm{QV}_{p,\mathfrak{M}}(\bar{a})$ is $\bar{a}$-definable if and only if there exists the greatest by inclusion set $\varphi(\bar{a}, \mathfrak{M})$, where $\varphi(\bar{x}, \bar{y})$ is a p-preserving formula. Then, by Proposition 4.2, there are infinitely many p-preserving formulas which are pairwise non-equivalent on the set of realizations of p. Indeed, if there were only finitely many such formulas, we could take their disjunction to obtain a p-preserving formula producing the greatest set. □

Propositions 2.1 and 4.4. imply

Corollary 4.5. *Any almost ω-categorical Ehrenfeucht theory has the strict order property.*

Proposition 4.6 (S. V. Sudoplatov [1, 17]). *If a non-p-principal p-type q is realized in a model $\mathfrak{M}(a)$, where a is a realization of p, then, for every element b_i of a realization $\bar{b}$ of q in $\mathfrak{M}(a)$, the pair (a, b_i) belongs to I_p and (b_i, a) does not belong to SI_p.*

Proof. Let a be a realization of type p and $\varphi(a, \bar{y})$ be a formula isolating a non-p-principal p-type $q(\bar{y})$. Assume, that some element b_i of a realization $\bar{b}$ of $q(\bar{y})$ in $\mathfrak{M}(a)$ semi-isolates the element a. Consider a formula $\psi(b_i, x)$ witnessing that b_i semi-isolates a. Then the type $q(\bar{y})$ is isolated by set

$$\cup\{p(y_i) \mid y_i \in \bar{y}\} \cup \{\exists x\, (\varphi(x, \bar{y}) \wedge \psi(y_i, x))\}.$$

This is impossible since the p-type $q(\bar{y})$ is not p-principal. □

5. Semi-isolation and limit models

Definition (S. V. Sudoplatov [1, 16, 21]). A countable model $\mathfrak{M}$ of a theory T is *limit* (accordingly *limit over a type* $p \in S(T)$) if $\mathfrak{M}$ is not prime over a tuple and $\mathfrak{M} = \bigcup\limits_{n\in\omega} \mathfrak{M}(\bar{a}_n)$, where $(\mathfrak{M}(\bar{a}_n))_{n\in\omega}$ is an elementary chain of prime models over tuples $\bar{a}_n$ (and $\mathfrak{M} \models p(\bar{a}_n)$), $n \in \omega$.

Theorem 5.1 (S. V. Sudoplatov [1, 16, 21]). *Any countable model of a small theory T is either prime over a tuple or limit.*

Proof. Let $\mathfrak{M}$ be an arbitrary countable model of T. It suffices to construct an elementary chain $\mathfrak{C}$ of prime models $\mathfrak{M}(\bar{a}_i)$ over tuples $\bar{a}_i$, $i \in \omega$, such that $\mathfrak{M} = \bigcup\limits_{i\in\omega} \mathfrak{M}(\bar{a}_i)$. For this purpose, we enumerate all elements of $\mathfrak{M}$: $M = \{b_k \mid k \in \omega\}$, and also all formulas of the form $\varphi(x, \bar{c})$, $\bar{c} \in M$: $\Phi = \{\varphi_m(x, \bar{c}_m) \mid m \in \omega\}$. We shall construct $\mathfrak{C}$ inductively and, at any step k, some finite sequence of tuples $\bar{a}_0, \ldots, \bar{a}_n$ will be defined, and each such a tuple will be connected to a finite set X_i^k, $0 \le i \le n$, such that unions of these sets by all k with respect to fixed i will define universes of models $\mathfrak{M}(\bar{a}_i)$. If a tuple $\bar{a}_i$ is not defined before the step k then the sets X_i^l are supposed to be empty for any $l < k$.

At the initial step, we fix the tuple $\bar{a}_0 \rightleftharpoons \langle b_0 \rangle$ and for the formula $\varphi_m(x, b_0)$ from Φ having the minimal number and satisfying $\mathfrak{M} \models \exists x\, \varphi_m(x, b_0)$ we find a realization d_m of a principal complete type $p(x, b_0)$ containing $\varphi_m(x, b_0)$. Now we let $X_0^0 \rightleftharpoons \{b_0, d_m\}$.

Suppose that at step k we have already found tuples $\bar{a}_0, \ldots, \bar{a}_n$ and have formed finite sets $X_0^k, \ldots, X_n^k$ satisfying the following conditions:

(1) all elements of $\bar{a}_i$ are contained in the set of elements of $\bar{a}_{i+1}$, $i < n$, and belong to X_i^k;

(2) $\{b_0, \ldots, b_k\} \subseteq X_n^k$;

(3) $X_i^k \subset X_{i+1}^k$, $i < n - 1$;

(4) for the formula $\varphi_m(x, \bar{c}_m)$ we chose at step k, which is minimal with respect to m and is not considered before, contains only elements of the maximal nonempty set X_j^{k-1}, and satisfies $\mathfrak{M} \models \exists x\, \varphi_m(x, \bar{c}_m)$, we have found a realization $d_m \in M$ of a principal complete type $p(x, X_j^{k-1} \cup \{b_k\})$ containing $\varphi_m(x, \bar{c}_m)$ so that for any tuple $\bar{a}_i$ with $\bar{c}_m \in X_i^{k-1}$ and for any tuple $\bar{d} \in X_i^{k-1} \cup \{d_m\}$ the type $\mathrm{tp}(\bar{d}/\bar{a}_i)$ is principal; this realization is added to the minimal set X_i^k with respect to i such that $\bar{c}_m \in X_i^{k-1}$.

At step $k+1$, we consider the element b_{k+1}. If it belongs to X_n^k then the sequence $\bar{a}_0, \ldots, \bar{a}_n$ remains the same and we construct sets X_i^{k+1} by adding to X_i^k some element d_m satisfying the conditions (3) and (4) for $k + 1$ instead of k.

If $b_{k+1} \notin X_n^k$ and, starting from some $i_0 \le n$, all types $\mathrm{tp}(\bar{b}/\bar{a}_i)$, $\bar{b} \in X_i^k \cup \{b_{k+1}\}$, are principal, we do not extend the sequence $\bar{a}_0, \ldots, \bar{a}_n$ and add the element b_{k+1} to the set $X_{i_0}^k$ as well as to all consequent sets X_i^k, $i_0 \le i \le n$. Then we obtain sets X_i^{k+1} by adding an element d_m satisfying the conditions (3) and (4) for $k + 1$ instead of k.

If some type $\mathrm{tp}(\bar{b}/\bar{a}_n)$, $\bar{b} \in X_n^k \cup \{b_{k+1}\}$, is not principal, we add to the sequence $\bar{a}_0, \ldots, \bar{a}_n$ a tuple $\bar{a}_{n+1}$ consisting of all elements of the set $X_n^k \cup \{b_{k+1}\}$. Then we add this set to the (initially empty) set X_{n+1}^k and form the sets X_i^{k+1}, $0 \le i \le n+1$, by adding a realization d_m of a principal complete type $p(x, X_n^k \cup \{b_{k+1}\})$ containing the minimal (with respect to m) formula $\varphi_m(x, \bar{c}_m)$ which was not considered before and contains only elements of X_n^k and satisfying $\mathfrak{M} \models \exists x\, \varphi_m(x, \bar{c}_m)$, such that for any tuple $\bar{a}_i$ with $\bar{c}_m \in X_i^k$ and for any tuple $\bar{d} \in X_i^k \cup \{d_m\}$, the type $\mathrm{tp}(\bar{d}/\bar{a}_i)$ is principal. We add the element d_m to the minimal (with respect to i) set X_i^k, and also to the consequent sets such that $\bar{c}_m \in X_j^k$, $i \leq j \leq n$. Now we let $X_{n+1}^{k+1} \rightleftharpoons X_n^k \cup \{b_{k+1}, d_m\}$.

By construction, the sets $X_i \rightleftharpoons \bigcup\limits_{k\in\omega} X_i^k$ are the universes of prime models $\mathfrak{M}(\bar{a}_i)$ over tuples $\bar{a}_i$. Moreover, we have $\mathfrak{M}(\bar{a}_i) \preccurlyeq \mathfrak{M}(\bar{a}_{i+1})$ and $\mathfrak{M} = \bigcup\limits_i \mathfrak{M}(\bar{a}_i)$. If the number of indices i is finite, the model $\mathfrak{M}$ is prime over the greatest tuple $\bar{a}_i$ and we add the elementary chain of the models $\mathfrak{M}(\bar{a}_i)$ to the countable chain taking the model $\mathfrak{M}$ countably many times. □

If $I(T, \omega) < \omega$, for any elementary chain $(\mathfrak{M}_i)_{i\in\omega}$ of prime models over tuples, we obtain that there is an infinite subsequence of models $(\mathfrak{M}_{i_j})_{j\in\omega}$ such that all its elements are isomorphic to a model $\mathfrak{M}_p$. Thus the following corollary holds.

Corollary 5.2 (S. V. Sudoplatov [1, 16]). *Any countable model of an Ehrenfeucht theory T is either prime over a tuple or limit over a type.*

The following proposition gives a syntactical characterization of the existence of limit model over a type.

Proposition 5.3 (S. V. Sudoplatov [1, 16]). *A small theory T has a limit model over a type $p \in S(T)$ if and only if for any (some) realization $\bar{a}$ of type p there are a realization $\bar{b}$ of p in $\mathfrak{M}(\bar{a})$ and a tuple $\bar{c} \in M(\bar{a})$ such that* $\mathrm{tp}(\bar{c}/\bar{b})$ *is a non-principal type.*

Proof. Suppose that there exists a limit model $\mathfrak{M} = \bigcup\limits_{n\in\omega} \mathfrak{M}_n$ of T over p, where $\mathfrak{M}_n \simeq \mathfrak{M}_p$, $\mathfrak{M}_0 = \mathfrak{M}(\bar{a})$, $\models p(\bar{a})$, and for any $\bar{b} \in p(\mathfrak{M}_0)$, $\bar{c} \in M_0$ the type $\mathrm{tp}(\bar{c}/\bar{b})$ is principal. Then models $\mathfrak{M}_n$ (and hence also $\mathfrak{M}$) realize just principal types over any realizations of type p lying in $\mathfrak{M}_n$ (in $\mathfrak{M}$). Hence the model $\mathfrak{M}$ is prime over a realization of p, which contradicts the assumption that $\mathfrak{M}$ is limit.

Conversely, assume that for some tuple $\bar{a}$ realizing p there are tuples $\bar{b} \in p(\mathfrak{M}_0)$ and $\bar{c} \in M_0$ such that $q(\bar{x}, \bar{b}) = \mathrm{tp}(\bar{c}/\bar{b})$ is a non-principal type. Our goal is to construct an elementary chain $(\mathfrak{M}(\bar{a}_n))_{n\in\omega}$ over p satisfying the following conditions: $\bar{a}_0 = \bar{b}$, $\bar{a}_1 = \bar{a}$, and $\mathrm{tp}(\bar{a}_{n+1}\bar{a}_n) = \mathrm{tp}(\bar{a}\bar{b})$. We argue to show that $\mathfrak{M} = \bigcup\limits_{n\in\omega} \mathfrak{M}(\bar{a}_n)$ and $\mathfrak{M}_p$ are non-isomorphic. By way of contradiction, find a tuple $\bar{d} \in p(\mathfrak{M}(\bar{a}_n))$ such that $\mathfrak{M} = \mathfrak{M}(\bar{d})$. By the construction of $\mathfrak{M}$, however, the type $q(\bar{x}, \bar{a}_n)$ is omitted in the model $\mathfrak{M}(\bar{d})$ but is realized in the model $\mathfrak{M}$, a contradiction. □

Lemma 5.4 (B. Kim [4], P. Tanović [22, 23]). (1) *If a tuple $\bar{a}$ isolates a tuple $\bar{b}$, whereas $\bar{b}$ does not isolate $\bar{a}$, then $\bar{b}$ does not semi-isolate $\bar{a}$.*

(2) *If $(\bar{a}, \bar{b}) \in I_p$ and $(\bar{b}, \bar{a}) \in \mathrm{SI}_p$ then $(\bar{b}, \bar{a}) \in I_p$.*

Proof. (1) Suppose that $\varphi(\bar{a}, \bar{y})$ isolates $\mathrm{tp}(\bar{b}/\bar{a})$. Assume the contrary (i. e. $\bar{b}$ semi-isolates $\bar{a}$) and take a formula $\psi(\bar{x}, \bar{b})$ witnessing that $\bar{b}$ semi-isolates $\bar{a}$. Now as $\mathrm{tp}(\bar{a}/\bar{b})$ is non-isolated, there exists a formula $\chi(\bar{x}, \bar{y})$ such that $\varphi(\bar{x}, \bar{b}) \wedge \psi(\bar{x}, \bar{b}) \wedge \chi(\bar{x}, \bar{b})$ and $\varphi(\bar{x}, \bar{b}) \wedge \psi(\bar{x}, \bar{b}) \wedge \neg\chi(\bar{x}, \bar{b})$ are both consistent. Moreover both formulas imply $\mathrm{tp}(\bar{a})$. Hence $\varphi(\bar{a},\bar{y})\wedge\chi(\bar{a},\bar{y})$ and $\varphi(\bar{a},\bar{y})\wedge\neg\chi(\bar{a},\bar{y})$ are both consistent. This contradicts the fact that $\varphi(\bar{a},\bar{y})$ is a principal formula.

(2) follows immediately from (1). □

Definition (A. Pillay [24]). A type $p(\bar{x})\in S(T)$ is *good*, if for any realizations $\bar{a}$ and $\bar{b}$ of p, $(\bar{a},\bar{b})\in I_p$ implies $(\bar{b},\bar{a})\in I_p$, i. e., the relation I_p is symmetric.

Lemma 5.4 immediately implies

Corollary 5.5 (P. Tanović [22, 23]). *If $p(\bar{x})$ is a complete type of theory T and for any model of T the relation SI_p is symmetric, then the type p is good.*

P. Tanović [22, 23] noticed that there exist good types p with non-symmetric SI_p:

Example 5.6. In $\mathrm{Th}(\langle\omega;<\rangle)$, the unique non-algebraic 1-type $p\in S^1(\varnothing)$ is good and SI_p is non-symmetric.

Indeed, if $\models p(a)$ then any principal formula $\varphi(a,y)$ describes a finite number of steps for taking finitely many successors or predecessors to come to the (unique) realization b of this formula. The existence of an reverse way from b to a means that there exists a principal formula $\psi(x,b)$ for which $\models\psi(a,b)$.

At the same time, if a and b are realizations of p, $a<b$, they having infinitely many intermediate elements, then a semi-isolates b by the formula $a<y$, but b does not semi-isolate a. □

Theorem 5.7. *Let $p(\bar{x})$ be a complete type of a small theory T. The following conditions are equivalent:*

(1) *there exists a limit model over p;*

(2) *the relation I_p of isolation on a set of realizations of p in a (any) model $\mathfrak{M}\models T$ realizing p is non-symmetric;*

(3) *in some (any) model $\mathfrak{M}\models T$ realizing p, there exist realizations $\bar{a}$ and $\bar{b}$ of p such that the type $\mathrm{tp}(\bar{b}/\bar{a})$ is principal and $\bar{b}$ does not semi-isolate $\bar{a}$ and, in particular, SI_p is non-symmetric on $\mathfrak{M}$.*

Proof. At first we consider the conditions (2) and (3) for *some* model $\mathfrak{M}$.

(1) ⇒ (2). Assume that the theory T has a limit model over p and the non-empty relation I_p is symmetric on the set of realizations of p in $\mathfrak{M}$.

Consider realizations $\bar{a}$ and $\bar{b}$ of p in the model $\mathfrak{M}$ and a tuple $\bar{c}\in M$, which exist by Proposition 5.3, such that $\bar{b},\bar{c}\in M(\bar{a})$, $\mathfrak{M}(\bar{a})\preccurlyeq\mathfrak{M}$, and $\mathrm{tp}(\bar{c}/\bar{b})$ is non-principal. Choose a principal formula $\varphi(\bar{a},\bar{y},\bar{z})$ for which $\mathfrak{M}(\bar{a})\models\varphi(\bar{a},\bar{b},\bar{c})$ holds. Since $\mathrm{tp}(\bar{b}/\bar{a})$ is principal, by the hypothesis, the type $\mathrm{tp}(\bar{a}/\bar{b})$ is also principal. Take a principal formula $\psi(\bar{x},\bar{b})$ for which $\mathfrak{M}(\bar{a})\models\psi(\bar{a},\bar{b})$ holds. Now we consider the formula

$$\chi(\bar{b},\bar{z})\rightleftharpoons\exists\bar{x}(\varphi(\bar{x},\bar{b},\bar{z})\wedge\psi(\bar{x},\bar{b})).$$

Clearly, $\chi(\bar{b},\bar{z})\in\mathrm{tp}(\bar{c}/\bar{b})$. On the other hand, any two solutions of the formula $\chi(\bar{b},\bar{z})$ are connected by an automorphism (in an elementary extension of $\mathfrak{M}$) fixing $\bar{b}$.

Indeed, let $\bar{c}'$ and $\bar{c}''$ be tuples for which $\models\chi(\bar{b},\bar{c}')\wedge\chi(\bar{b},\bar{c}'')$ holds. Take tuples $\bar{a}'$ and $\bar{a}''$ with

$$\models\varphi(\bar{a}',\bar{b},\bar{c}')\wedge\psi(\bar{a}',\bar{b})\wedge\varphi(\bar{a}'',\bar{b},\bar{c}'')\wedge\psi(\bar{a}'',\bar{b}).$$

Since the formula $\psi(\bar{x},\bar{b})$ is principal, there exists an automorphism g fixing $\bar{b}$ and taking $\bar{a}'$ to $\bar{a}''$. As the formula $\varphi(\bar{a}'',\bar{y},\bar{z})$ is principal with

$$\models \varphi(\bar{a}'',\bar{b},g(\bar{c}')) \wedge \varphi(\bar{a}'',\bar{b},\bar{c}''),$$

there exists an automorphism h fixing the tuples $\bar{a}''$, $\bar{b}$ and taking $g(\bar{c}')$ to $\bar{c}''$. Thus, the $\bar{b}$-automorphism $g \circ h$ maps $\bar{c}'$ to $\bar{c}''$ and the type $\mathrm{tp}(\bar{c}/\bar{b})$ is principal in spite of the assumption.

The obtained contradiction shows that, having the symmetric non-empty relation I_p on the set of realizations of p in $\mathfrak{M}$, we have no limit models over p.

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) follow immediately from Lemma 5.4 and Proposition 5.3 respectively.

The equivalence of the existence and of the generality pointed out in (2) and (3) for models realizing the type p is true since the considered properties are reduced to the model $\mathfrak{M}(\bar{a})$, where $\models p(\bar{a})$, and $\mathfrak{M}(\bar{a})$ is isomorphic to an elementary submodel of any model which realizes the type p. □

Note that for the proof of Theorem 5.7 we can also use the following facts:

(1) (A. N. Gavryushkin [25]) there is no limit structure over a type p if and only if the structure $\mathfrak{M}_p$ is homogeneous;

(2) if $\mathfrak{M}_p$ is homogeneous then I_p is symmetric.

Proposition 2.3 and Theorem 5.7 imply the following

Corollary 5.8 (S. V. Sudoplatov [1, 16]). *If $p \in S(T)$ is a non-principal powerful type then there exists a limit model over p.*

Another argument for Corollary 5.8 is that any countable saturated model of T is limit over a (any) non-principal powerful type, if T has these types.

6. The Tsuboi and Kim theorems

Recall that a theory T is *ω-categorical* if $I(T,\omega) = 1$.

By a well-known Ryll-Nardzewski Theorem, a theory T is ω-categorical if and only if for any tuple $\bar{x}$ there are only finitely many types $p(\bar{x}) \in S(T)$, i. e., there are only finitely many pairwise non-equivalent formulas $\varphi(\bar{x})$.

Recall [26, 27] that a (*strict*) *dense order* is an (ir)reflexive, transitive, anti-symmetric relation such that any two different comparable elements a, b have an intermediate element, i. e. an element greater than a and less than b, or less than a and greater than b. An order is *identical*, if, relative to that order, only coincident elements are comparable.

Obviously, a non-identical dense order, on a set containing at least two comparable elements, has an infinite *chain*, i. e., infinitely many pairwise comparable elements. There are also infinitely many pairwise comparable elements for any strict dense order, connecting at least two elements.

An order $\leq$ (accordingly a strict order $<$) on a set A of tuples in a structure $\mathfrak{M}$ is (formula) *definable* if there exists a formula $\varphi(\bar{x},\bar{y})$, $l(\bar{x}) = l(\bar{y})$, such that for any tuples $\bar{a},\bar{b}$ in A,

$$\bar{a} \leq \bar{b} \Leftrightarrow \mathfrak{M} \models \varphi(\bar{a},\bar{b}) \quad \left(\bar{a} < \bar{b} \Leftrightarrow \mathfrak{M} \models \varphi(\bar{a},\bar{b})\right).$$

Clearly if a theory T (i. e. some model of T) has a definable order with an infinite chain on a definable set, then T has the strict order property. A theory T has also the strict order property if T has a definable set with a definable strict order having infinitely many pairwise comparable elements.

Now we present the Tsuboi theorem on non-representability of Ehrenfeucht theories without definable non-identical dense orders and, in particular, without the strict order property, as unions of ω-categorical theories.

Theorem 6.1 (A. Tsuboi [3]). *If T is an Ehrenfeucht theory and is a union of ω-categorical theories T_m, $m \in \omega$, such that $T_m \subseteq T_{m+1}$, $m \in \omega$, then there exists a formula $\psi(\bar{x}, \bar{y})$ of T which defines a non-identical dense order on a definable set.*

Proof. By Propositions 2.1 and 2.3, the Ehrenfeucht theory T has a non-principal powerful type $p(\bar{x})$ and a formula $\varphi(\bar{x}, \bar{y})$ satisfying the following conditions:

1) the formula $\varphi(\bar{x}, \bar{y})$ witnesses that the relation SI_p is non-symmetric on the set of realizations of p in a model $\mathfrak{M} \models T$;

2) for any realization $\bar{a}$ of p, the formula $\varphi(\bar{a}, \bar{y})$ is principal.

Consider a theory T_n such that $\varphi(\bar{x}, \bar{y})$ is a formula of T_n. We set $\varphi^0(\bar{x}, \bar{y}) \rightleftharpoons (\bar{x} \approx \bar{y})$, $\varphi^1(\bar{x}, \bar{y}) \rightleftharpoons \varphi(\bar{x}, \bar{y})$, $\varphi^{k+1}(\bar{x}, \bar{y}) \rightleftharpoons \exists \bar{z}(\varphi^k(\bar{x}, \bar{z}) \wedge \varphi(\bar{z}, \bar{y}))$, $k \in \omega \setminus \{0\}$. Since all $\varphi^k(\bar{x}, \bar{y})$ are formulas of T_n, and T_n is ω-categorical, there are at most finitely many pairwise non-equivalent, in T_n, formulas $\varphi^0(\bar{x}, \bar{y}), \varphi^1(\bar{x}, \bar{y}), \ldots, \varphi^{m-1}(\bar{x}, \bar{y})$ among all formulas $\varphi^k(\bar{x}, \bar{y})$, $k \in \omega$. For each $i, j \in \omega$ let $F(i, j)$ be the set

$$\{k < m \mid\vdash \exists \bar{z}(\varphi^i(\bar{x}, \bar{z}) \wedge \varphi^j(\bar{z}, \bar{y})) \leftrightarrow \varphi^k(\bar{x}, \bar{y})\}.$$

Using that F, define D_l, $l \in \omega$, by the following induction:

$$D_0 \rightleftharpoons m; \quad D_{l+1} \rightleftharpoons \bigcup \{F(i, j) \mid i, j \in D_l\}.$$

It is clear that, for each $l \in \omega$, $D_{l+1} \subseteq D_l$ and $D_l \neq \varnothing$. Since m is finite, $D \rightleftharpoons \bigcap\limits_{l \in \omega} D_l$ is a non-empty subset of cardinality m which contains 0 and some non-zero cardinal. For this D, we put

$$\psi(\bar{x}, \bar{y}) \rightleftharpoons \bigvee_{i \in D} \varphi^i(\bar{x}, \bar{y}).$$

By Compactness Theorem is suffices to show that $\psi(\bar{x}, \bar{y})$ defines a non-identical dense order on the set of realizations of p in $\mathfrak{M}$, i. e. the set

$$P \rightleftharpoons \{(\bar{a}, \bar{b}) \mid \mathfrak{M} \models \psi(\bar{a}, \bar{b}), \mathfrak{M} \models p(\bar{a}), \mathfrak{M} \models p(\bar{a})\}$$

is that order.

Since $D \setminus \{0\} \neq \varnothing$, P is non-identical. As $0 \in D$, P is reflexive. The relation P is antisymmetric; indeed, if $(\bar{a}, \bar{b}) \in P$ then $\bar{a}$ semi-isolates $\bar{b}$, and $\bar{b}$ can semi-isolate $\bar{a}$ only for $\bar{a} = \bar{b}$.

By the definitions of F and D we have

$$p(\bar{x}) \vdash (\psi(\bar{x}, \bar{y}) \wedge \neg \bar{x} \approx \bar{y}) \leftrightarrow \exists \bar{z}(\psi(\bar{x}, \bar{z}) \wedge \psi(\bar{z}, \bar{y}) \wedge \neg \bar{x} \approx \bar{z} \wedge \neg \bar{z} \approx \bar{y}).$$

Thus P is transitive and dense. □

Notice that, in the conditions of previous theorem, to prove the strict order property for the theory T (it is asserted in a weak version of Tsuboi theorem [2]) it suffices to consider the set $\Phi(\bar{a}, \mathfrak{M}) \rightleftharpoons \bigcup\limits_{k \in \omega} \varphi^k(\bar{a}, \mathfrak{M})$ which is an $\bar{a}$-*formula*, i. e. equals $\chi(\bar{a}, \mathfrak{M})$ for some formula $\chi(\bar{x}, \bar{y})$, and to prove that the strict order property holds for the formula $\chi(\bar{x}, \bar{y})$ on a set of realizations of $p(\bar{x})$.

Indeed, since $\varphi(\bar{x}, \bar{y})$ witnesses that SI_p is non-symmetric, there exists a realization $\bar{b}$ of p in $\mathfrak{M}$ such that $\bar{b} \in \chi(\bar{a}, \mathfrak{M})$ and $\chi(\bar{b}, \mathfrak{M}) \subset \chi(\bar{a}, \mathfrak{M})$. Since $\bar{a}$

and $\bar{b}$ realize the same type p, there are realizations $\bar{a}_i$, $i \in \omega$, of p in $\mathfrak{M}$ such that the following holds:

$$\mathfrak{M} \models \forall \bar{y}(\chi(\bar{a}_i, \bar{y}) \to \chi(\bar{a}_j, \bar{y})) \Leftrightarrow i \leq j.$$

It means that T has the strict order property.

Since the strict order property for a theory T implies that T is unstable, the following Corollary holds.

Corollary 6.2 (A. Tsuboi [2, 3]). *If T is a countable theory without non-identical definable dense orders on definable sets* (*in particular, if T is stable*) *and T is obtained from an ω-categorical theory by addition of axioms for new constants, then $I(T, \omega) = 1$ or $I(T, \omega) \geq \omega$.*

Recall several notions of Stability Theory related to the class of simple theories [28, 29, 30].

Let $k \in \omega$. A formula $\varphi(\bar{x}, \bar{a})$ in a theory T *k-divides* over a set A if there are tuples $\bar{a}^n$, $n \in \omega$, of type $\mathrm{tp}(\bar{a}/A)$ such that the set $\{\varphi(\bar{x}, \bar{a}^n) \mid n \in \omega\}$ of formulas is *k-inconsistent*, i. e., for every $w \subset \omega$ of cardinality k the formula $\bigwedge\limits_{n \in w} \varphi(\bar{x}, \bar{a}^n)$ is inconsistent in T.

A partial type $\pi(\bar{x})$ *k-divides* over A if there is a formula $\varphi(\bar{x})$ implied by $\pi(\bar{x})$ which is k-divides over A. A formula or a partial type *divides* over A if they k-divide for some $k \in \omega$.

A partial type $\pi(\bar{x})$ *forks* over A if there are $n \in \omega$ and formulas $\varphi_0(\bar{x}), \ldots, \varphi_n(\bar{x})$ such that $\pi(\bar{x}) \vdash \bigvee\limits_{i \leq n} \varphi_i(\bar{x})$, and each $\varphi_i(\bar{x})$ divides over A.

If $p \in S(A)$, $q \supset p$, and q forks (does not fork) over A then q is called to be a (*non-*)*forking extension* of p and it is denoted by $q \supset_{\mathrm{f}} p$ ($q \supset_{\mathrm{nf}} p$).

A theory T is called (*super*)*simple* if, for any type $p \in S(B)$, p does not fork over a subset A of B with $|A| \leq |T|$ ($|A| < \omega$).

Remark 6.3. By the definition, a supersimple theory is simple. Moreover, T is supersimple if and only if there are no $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_i \ldots$ and $p_i \in S(A_i)$, $i \in \omega$, such that $p_{i+1} \supset_{\mathrm{f}} p_i$ for each $i \in \omega$.

Remark 6.4. Every (super)stable theory is (super)simple.

The following properties of non-forking in simple theories are shown by B. Kim [31].

1. (*Extension*) For any $p \in S(A)$ and $A \subseteq B$, p has a $q \supset_{\mathrm{nf}} p$ in $S(B)$.

2. (*Symmetry*) A type $\mathrm{tp}(\bar{b}/A\bar{c})$ does not fork over A if and only if $\mathrm{tp}(\bar{c}/A\bar{b})$ does not fork over A.

3. (*Transitivity*) If $A \subseteq B \subseteq C$ and $p \in S(C)$, then p does not fork over A if and only if p does not fork over B and $p \upharpoonright B$, the restriction of p to B, does not fork over A.

A tuple $\bar{a}$ is *dependent* of $\bar{b}$ over A if $\mathrm{tp}(\bar{a}/A\bar{b})$ divides over A. If $\bar{a}$ is not dependent of $\bar{b}$ over A, one say that $\bar{a}$ is *independent* of $\bar{b}$ over A.

Remark 6.5. In view of forking symmetry, the (in)dependence in simple theories is symmetric too.

By Remark 6.5, if $\bar{a}$ is (in)dependent of $\bar{b}$ over A we may say that $\bar{a}$ and $\bar{b}$ are (*in*)*dependent* over A. Tuples being (in)dependent over $\varnothing$ are called simply *(in)dependent.* A sequence (set) of tuples is said to be *independent* (over A) if each tuple of this sequence (set) is independent (over A) with every tuple formed by coordinates of other elements of the sequence (set).

A theory T *has the infinite weight* if there exist a tuple $\bar{a}$, a set A, and an infinite independent sequence $(\bar{a}_n)_{n\in\omega}$ over A such that the tuples $\bar{a}$ and $\bar{a}_n$ are dependent over A for each $n \in \omega$.

Definition (A. Tsuboi [3]). A stable theory T is *pseudo-superstable* if T fails to have the infinite weight.

Similarly to the previous definition, we say that a simple theory T is *pseudo-supersimple* if T fails to have infinite weight.

Remark 6.6. Every supersimple theory is pseudo-supersimple. Actually the proof of the Kim theorem [4] implies that any pseudo-supersimple theory is not an Ehrenfeucht theory.

Example 6.7 (A. Tsuboi [3]). Let T be the theory of refining equivalence relations $E_n(x, y)$, $n \in \omega$, such that each E_n-class is divided into infinitely many E_{n+1}-classes. The theory T is stable since $|S(M)| = |M|$ for any model $\mathfrak{M} \models T$ of cardinality 2^ω. The theory T is pseudo-supersimple since the dependence of elements a and b means that a and b belong to some common E_n-class. And T is non-supersimple by Remark 6.3 since each type $p \in S(\bar{a})$ that is realized by elements b being non-E_n-equivalent to the elements of $\bar{a}$, has a forking extension $q \in S(\bar{a}b)$, that is realized by elements c being E_n-equivalent and non-E_{n+1}-equivalent to the elements of $\bar{a}b$.

Recall an example of a supersimple unstable theory [4]: this is the theory of countable *bipartite random graph* $\mathfrak{M}$, consisting of disjoint infinite sets U, V with the relation R between U, V such that for any finite disjoint subsets A, B of U there is $c \in V$ such that $(a, c) \in R$ for $a \in A$ and $(b, c) \notin R$ for $b \in B$, and vice versa.

Taking a disjoint union of a union of pseudo-superstable theories (like Example 6.7) and of a theory of random graph we get a theory T which is the union of pseudo-supersimple theories T_n with $T_n \subseteq T_{n+1}$ such that T is not a union of pseudo-superstable theories T'_n with $T'_n \subseteq T'_{n+1}$, $n \in \omega$.

Definition (A. Tsuboi [3]). Let S be a subset of $S(A)$. A nonempty set $R \subseteq S(A)$, consisting of some types $\mathrm{tp}(\bar{a}\bar{b}/A)$ with $\mathrm{tp}(\bar{a}/A) \in S$ and $\mathrm{tp}(\bar{b}/A) \in S$ is said to be a *transitive forking A-class* (on S) if the following conditions hold:

(a) if $\mathrm{tp}(\bar{a}\bar{b}/A) \in R$ then $\mathrm{tp}(\bar{a}/A\bar{b}) \supset_{\mathrm{f}} \mathrm{tp}(\bar{a}/A)$;

(b) if $\mathrm{tp}(\bar{a}\bar{b}/A) \in R$ and $\mathrm{tp}(\bar{b}\bar{c}/A) \in R$ then $\mathrm{tp}(\bar{a}\bar{c}/A) \in R$.

$\varnothing$-classes are called simply *classes.*

Let T be a simple theory, R be a transitive forking A-class, and p be a type in $S(A)$. The *R-weight* $w_R(p)$ of p in R is the maximal cardinal $\varkappa$ such that, for every $\lambda < \varkappa$, there is a realization $\bar{a}$ of p and tuples $\bar{b}_i$, $i \leq \lambda$, such that $(\bar{b}_i)_{i\leq\lambda}$ is independent over A and $\mathrm{tp}(\bar{a}\bar{b}_i/A) \in R$ for each $i \leq \lambda$.

Remark 6.8. If T is pseudo-supersimple then $w_R(p) \leq \omega$.

Proposition 6.9 (B. Kim [31]). *Let T be a simple theory, $\bar{a}$, $\bar{b}$ be realizations of a type $p \in S(T)$. If $(\bar{a}, \bar{b}) \in \mathrm{SI}_p$ and $(\bar{b}, \bar{a}) \notin \mathrm{SI}_p$ then $\mathrm{tp}(\bar{a}/\bar{b})$ forks over $\varnothing$.*

Proof. Take a formula $\varphi(\bar{x}, \bar{y})$ witnessing that $(\bar{a}, \bar{b}) \in \mathrm{SI}_p$. Let $\bar{c}$ be *any* tuple such that $\mathrm{tp}(\bar{a}\bar{b}) = \mathrm{tp}(\bar{b}\bar{c})$.

We claim that $\psi(\bar{x},\bar{a},\bar{c}) \rightleftharpoons \varphi(\bar{a},\bar{x}) \wedge \varphi(\bar{x},\bar{c})$ forks over $\varnothing$, and so $\mathrm{tp}(\bar{b}/\bar{a}\bar{c})$ forks over $\varnothing$.

Indeed, let $\bar{a}_0 = \bar{a}$, $\bar{b}_0 = \bar{b}$, $\bar{c}_0 = \bar{c}$. There exists a sequence $(\bar{a}_i\bar{b}_i\bar{c}_i)_{i\in\omega}$ such that for all $i \in \omega$, $\mathrm{tp}(\bar{a}_i\bar{b}_i\bar{c}_i) = \mathrm{tp}(\bar{a}\bar{b}\bar{c})$ and $\mathrm{tp}(\bar{a}\bar{b}) = \mathrm{tp}(\bar{c}_i\bar{a}_{i+1})$. By transitivity of semi-isolation, $(\bar{a}_i,\bar{a}_j) \in \mathrm{SI}_p$ for every $i \le j$. It suffices to show that $\{\psi(\bar{x},\bar{a}_i,\bar{c}_i) \mid i \in \omega\}$ is 2-inconsistent. If not, then there is $\bar{d}$ such that $\models \varphi(\bar{a}_j,\bar{d}) \wedge \varphi(\bar{d},\bar{c}_i)$ for some $j > i$. Then $(\bar{a}_j,\bar{d}),(\bar{d},\bar{c}_i) \in \mathrm{SI}_p$, so $(\bar{a}_j,\bar{c}_i) \in \mathrm{SI}_p$, and $(\bar{a}_{i+1},\bar{a}_j) \in \mathrm{SI}_p$ implies $(\bar{a}_{i+1},\bar{c}_i) \in \mathrm{SI}_p$. But since $\mathrm{tp}(\bar{a}\bar{b}) = \mathrm{tp}(\bar{c}_i\bar{a}_{i+1})$, it contradicts to $(\bar{b},\bar{a}) \notin \mathrm{SI}_p$.

Now if $\bar{a}$ and $\bar{b}$ are independent (over $\varnothing$), then by properties of non-forking we can find a tuple $\bar{c}'$ such that $\mathrm{tp}(\bar{a}\bar{b}) = \mathrm{tp}(\bar{b}\bar{c}')$ and $\{\bar{a},\bar{b},\bar{c}'\}$ is independent. This contradicts the claim above. Thus $\mathrm{tp}(\bar{a}/\bar{b})$ forks over $\varnothing$. □

Lemma 5.4 and Proposition 6.9 imply

Corollary 6.10 (B. Kim [31]). *Let T be a simple theory, $\bar{a}$, $\bar{b}$ be realizations of a type $p \in S(T)$. If $(\bar{a},\bar{b}) \in I_p$ and $(\bar{b},\bar{a}) \notin I_p$ then $\mathrm{tp}(\bar{a}/\bar{b})$ forks over $\varnothing$.*

Corollary 6.11. *Let T be a simple theory, $p(\bar{x}) \in S(T)$,*

$$R \rightleftharpoons \{\mathrm{tp}(\bar{a}\bar{b}) \mid (\bar{a},\bar{b}) \in \mathrm{SI}_p \text{ and } (\bar{b},\bar{a}) \notin \mathrm{SI}_p\} \neq \varnothing.$$

Then R is a transitive forking class on $\{p\}$.

Repeating the proof of [3, Proposition 3.3] we obtain

Proposition 6.12. *Let T be a pseudo-supersimple theory and R be a transitive forking A-class on S. Then there is a type $p \in S$ with $w_R(p) = 1$.*

Proof. By way of a contradiction, assume that, for any $p \in S$, $w_R(p) \geq 2$. We shall construct by induction a sequence $(\bar{a}_i)_{i\in\omega}$ of realizations of types in S such that

(1) both $\mathrm{tp}(\bar{a}_{2i}\bar{a}_{2i+1}/A)$ and $\mathrm{tp}(\bar{a}_{2i}\bar{a}_{2i+2}/A)$ belong to R;

(2) $\{\bar{a}_{2j+1} \mid j \le i\} \cup \{\bar{a}_{2i+2}\}$ are independent over A.

Let $(\bar{a}_j)_{j\le 2i}$ be already defined. We have to define $\bar{a}_{2i+1}$ and $\bar{a}_{2i+2}$. Since $\bar{a}_{2i}$ realizes a type in S, by the assumption, there are two realizations $\bar{b}$ and $\bar{c}$ of types in S such that

(1)′ both $\mathrm{tp}(\bar{a}_{2i}\bar{b}/A)$ and $\mathrm{tp}(\bar{a}_{2i}\bar{c}/A)$ belong to R;

(2)′ $\bar{b}$ and $\bar{c}$ are independent over A.

Now we choose tuples $\bar{a}_{2i+1}$ and $\bar{a}_{2i+2}$ so that

(3) $\mathrm{tp}(\bar{a}_{2i+1}\bar{a}_{2i+2}/A \cup \{\bar{a}_{2j+1} \mid j < i\} \cup \{\bar{a}_{2i}\}) \supset_{\mathrm{nf}} \mathrm{tp}(\bar{b}\bar{c}/A \cup \{\bar{a}_{2i}\})$.

We prove that these $\bar{a}_{2i+1}$ and $\bar{a}_{2i+2}$ satisfy the conditions (1) and (2) above. By (3) and non-forking symmetry, we have

$$\mathrm{tp}((\bar{a}_{2j+1})_{j<i}/A\bar{a}_{2i}\bar{a}_{2i+1}\bar{a}_{2i+2}) \supset_{\mathrm{nf}} \mathrm{tp}((\bar{a}_{2j+1})_{j<i}/A\bar{a}_{2i}).$$

By the induction hypothesis, $\mathrm{tp}((\bar{a}_{2j+1})_{j<i}/A\bar{a}_{2i})$ does not fork over A. Thus, by non-forking transitivity, we have

(4) $\mathrm{tp}((\bar{a}_{2j+1})_{j<i}/A\bar{a}_{2i+1}\bar{a}_{2i+2}) \supset_{\mathrm{nf}} \mathrm{tp}((\bar{a}_{2i+1})_{j<i}/A)$.

Since $(\bar{a}_{2j+1})_{j<i}$ is independent over A, (4) shows that $(\bar{a}_{2j+1})_{j\le i}$ is also independent over A. Again by (4),

$$\mathrm{tp}(\bar{a}_{2i+1}\bar{a}_{2i+2}/A \cup \{\bar{a}_{2j+1} \mid j < i\}) \supset_{\mathrm{nf}} \mathrm{tp}(\bar{a}_{2i+1}\bar{a}_{2i+2}/A).$$

Thus we have

$$\mathrm{tp}(\bar{a}_{2i+2}/A \cup \{\bar{a}_{2j+1} \mid j \leq i\}) \supset_{\mathrm{nf}} \mathrm{tp}(\bar{a}_{2i+2}/A\bar{a}_{2i+1}).$$

Since $\bar{b}$ and $\bar{c}$ are independent over A, $\bar{a}_{2i+1}$ and $\bar{a}_{2i+2}$ are also independent over A. Hence we have

$$\mathrm{tp}(\bar{a}_{2i+2}/A \cup \{\bar{a}_{2j+1} \mid j \leq i\}) \supset_{\mathrm{nf}} \mathrm{tp}(\bar{a}_{2i+2}/A).$$

Thus the sequence $\{\bar{a}_{2j+1} \mid j \leq i\} \cup \{\bar{a}_{2i+2}\}$ is independent over A.

Now we obtain the independent set $\{\bar{a}_{2i+1} \mid i \in \omega\}$ over A and, by the transitivity of R, each $\mathrm{tp}(\bar{a}_0\bar{a}_{2i+1})$ belongs to R. Hence $\bar{a}_0$ and $\bar{a}_{2i+1}$ are dependent over A, $i \in \omega$. This is a contradiction to the assumption that T is pseudo-supersimple. □

Corollary 6.13. *Let T_n be pseudo-supersimple theories, $T_n \subseteq T_{n+1}$, $n \in \omega$, T be the union of all T_n, and p be a type in $S(T)$. If $R \rightleftharpoons \{\mathrm{tp}(\bar{a}\bar{b}) \mid (\bar{a}, \bar{b}) \in \mathrm{SI}_p$ and $(\bar{b}, \bar{a}) \notin \mathrm{SI}_p\} \neq \varnothing$ then $w_R(p) = 1$.*

Proof. If we assume $w_R(p) \geq 2$, as in the proof of Proposition 6.12, we have a sequence $(\bar{a}_i)_{i\in\omega}$ of realizations of p satisfying the following conditions:

(1) both $\mathrm{tp}(\bar{a}_{2i}\bar{a}_{2i+1})$ and $\mathrm{tp}(\bar{a}_{2i}\bar{a}_{2i+2})$ belong to R;

(2) $\{\bar{a}_{2j+1} \mid j \leq i\} \cup \{\bar{a}_{2i+2}\}$ are independent.

Moreover, there exist formulas $\varphi_j(\bar{x}, \bar{y})$, witnessing that SI_p is non-symmetric and such that

(3) $\models \varphi_j(\bar{a}_{2i}, \bar{a}_{2i+j})$, $j = 1, 2$;

(4) every type $q(\bar{x}, \bar{y}) \in S(T)$ containing $p(\bar{x}) \cup \{\varphi_j(\bar{x}, \bar{y})\}$ belongs to R, $j = 1, 2$.

Notice that for $\psi(\bar{x}, \bar{y}) \rightleftharpoons \varphi_1(\bar{x}, \bar{y}) \vee \varphi_2(\bar{x}, \bar{y})$, all $\psi^k(\bar{x}, \bar{y})$, $k > 1$, witness that SI_p is non-symmetric. By Proposition 6.9, this implies that each element of the sequence $(\bar{a}_{2i+1})_{i\in\omega}$ is dependent with $\bar{a}_0$ and this also true for the theory T_n, where $\psi(\bar{x}, \bar{y})$ is a formula of T_n. But this is a contradiction, since we are assuming that all T_n are pseudo-supersimple. □

Now we are ready to prove a generalization of both the Tsuboi theorem for unions of pseudo-superstable theories [3] and the Kim theorem for supersimple theories [4].

Theorem 6.14. *Let T be a union of pseudo-supersimple theories T_n, where $T_n \subseteq T_{n+1}$, $n \in \omega$. Then $I(T, \omega) = 1$ or $I(T, \omega) \geq \omega$.*

Proof. Suppose that T is an Ehrenfeucht theory. By Proposition 2.1, there exists a powerful type $p(\bar{x}) \in S(T)$. In view of Proposition 2.3 any non-empty relation SI_p is non-symmetric. By Corollary 6.11, the set

$$R \rightleftharpoons \{\mathrm{tp}(\bar{a}\bar{b}) \mid (\bar{a}, \bar{b}) \in \mathrm{SI}_p \text{ and } (\bar{b}, \bar{a}) \notin \mathrm{SI}_p\}$$

is a transitive forking class on $\{p\}$. Corollary 6.13 implies $w_R(p) = 1$.

At the same time, there are independent realizations $\bar{b}$ and $\bar{c}$ of p. Since each type $q \in S(T)$ is realized in the model $\mathfrak{M}_p$ and, by Proposition 2.3, there exist realizations $\bar{a}$, $\bar{d}$ of p such that $(\bar{a}, \bar{d}) \in I_p$ and $(\bar{d}, \bar{a}) \notin \mathrm{SI}_p$, we can find such independent $\bar{b}$ and $\bar{c}$ in $\mathfrak{M}(\bar{d})$ and then consider an elementary extension $\mathfrak{M}(\bar{a})$ of $\mathfrak{M}(\bar{d})$. Thus $(\bar{a}, \bar{b}) \in I_p$, $(\bar{b}, \bar{a}) \notin \mathrm{SI}_p$, $(\bar{a}, \bar{c}) \in I_p$, $(\bar{c}, \bar{a}) \notin \mathrm{SI}_p$. Proposition 6.9 implies that $\bar{a}$ and $\bar{b}$ are dependent, and $\bar{a}$ and $\bar{c}$ are dependent. Hence we have $w_R(p) \geq 2$, which leads to a contradiction. □

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