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ON THE HOMOLOGY SEQUENCE IN A P-SEMI-ABELIAN CATEGORY

YA. A. KOPYLOV

Abstract. We obtain sufficient conditions for the existence and exactness of the long homology sequence in a category semi-abelian in the sense of Palamodov.

Keywords: strict morphism, semi-stable kernel, semi-stable cokernel, P-semi-abelian category, (co)homology.

Introduction

As is well known, the definition of homology in an abelian category can be formulated in two ways dual to one another (see, for example, [5]). These two definitions make it possible to construct a connecting morphism for the long exact homology sequence corresponding to a short exact sequence of complexes in an abelian category. It is now known that the same holds in quasi-abelian categories [7, 13] and even in the nonadditive setting of homological categories in the sense of Grandis [8] but in these cases the so-obtained long homology sequence is in general not exact (see [7, 8, 13] for details). For another approach to the Snake Lemma and homology in nonadditive categories, see [2].

In [11], we introduced the left and right homology objects in a preabelian category and gave a sufficient condition for these objects to coincide in a P-semi-abelian category. We also discussed the possibility of constructing the long (co)homology sequence for a short strictly exact sequence of cochain complexes in a P-semi-abelian category. In this note, we, basing on the exactness properties of the Ker-Coker-sequence established in [12], prove sufficient conditions for the exactness of a fragment of the long (co)homology sequence in terms of the properties of the complexes and morphisms of the initial sequence.

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The article is organized as follows.

In Section 1, we give the necessary definitions and recall or prove some basic facts. In Section 2, we review the exactness properties of the Ker-Coker-sequence in a P-semi-abelian category as exposed in [12]. In Section 3, we recall the definitions of the left and right (co)homology objects and prove that their canonical isomorphism in a P-semi-abelian category implies that the category is quasi-abelian. Finally, Section 4 contains the main assertions of the article, Theorems 5 and 6, which give some sufficient conditions for the exactness of the long (co)homology sequence.

1. P-semi-abelian categories

We consider *preabelian* categories, i.e., additive categories satisfying the following axiom.

Axiom 1. Each morphism has a kernel and a cokernel.

We denote by $\ker\alpha$ ($\operatorname{coker}\alpha$) an arbitrary kernel (cokernel) of α and by $\operatorname{Ker}\alpha$ ($\operatorname{Coker}\alpha$) the corresponding object; the equality $a = \ker b$ ($a = \operatorname{coker} b$) means that a is a kernel of b (a is a cokernel of b).

In a preabelian category, every morphism α admits a canonical decomposition $\alpha = (\operatorname{im}\alpha)\overline{\alpha}(\operatorname{coim}\alpha)$, where $\operatorname{im}\alpha = \ker\operatorname{coker}\alpha$, $\operatorname{coim}\alpha = \operatorname{coker}\ker\alpha$. A morphism α is called *strict* if $\overline{\alpha}$ is an isomorphism.

We write $\alpha\,|\,\beta$ if $\alpha = \ker\beta$ and $\beta = \operatorname{coker}\alpha$.

Lemma 1. [3, 4, 17, 26] *The following assertions hold in a preabelian category :*

(i) *Strict monomorphisms = kernels, strict epimorphisms = cokernels.*

(ii) *α is a kernel $\Longleftrightarrow \alpha = \operatorname{im}\alpha$, α is a cokernel $\Longleftrightarrow \alpha = \operatorname{coim}\alpha$.*

(iii) *A morphism α is strict if and only if it is representable in the form $\alpha = \alpha_1\alpha_0$ with α_0 a cokernel and α_1 a kernel; in every such representation, $\alpha_0 = \operatorname{coim}\alpha$ and $\alpha_1 = \operatorname{im}\alpha$.*

(iv) *Suppose that a commutative square*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & D \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow{\beta} & B \end{array} \tag{1}$$

is a pullback. Then $\ker f = \alpha\ker g$. If $f = \ker h$ for some h then $g = \ker(h\beta)$. If f is a monomorphism then g is a monomorphism; if f is a kernel then g is a kernel.

In the dual manner, assume that (1) *is a pushout. Then $\operatorname{coker} g = (\operatorname{coker} f)\beta$. If $g = \operatorname{coker} e$ for some e then $f = \operatorname{coker}(\alpha e)$. If g is an epimorphism then f is an epimorphism; if g is a cokernel then f is a cokernel.*

(v) *The relations $\ker\alpha = \ker\operatorname{coim}\alpha$ and $\operatorname{coker}\alpha = \operatorname{coker}\operatorname{im}\alpha$ hold for every morphism α.*

We call a sequence $\dots \xrightarrow{a} B \xrightarrow{b} \dots$ in an additive category *semi-exact at the term B* if $ba = 0$. A sequence $\dots \xrightarrow{a} B \xrightarrow{b} \dots$ is said to be *exact at the term B* if $\operatorname{im} a = \ker b$. Lemma 1(v), which is Lemma 1 of [26], implies that the sequence is exact at the term B if and only if $\operatorname{coker} a = \operatorname{coim} b$.

A preabelian category is abelian if and only if $\overline{\alpha}$ is an isomorphism for every α, that is, if and only if every morphism is strict.

A preabelian category is called *P-semi-abelian* or *semi-abelian* (*in the sense of Palamodov*) [19, 21] if it satisfies

Axiom 2. For every morphism α, $\overline{\alpha}$ is a bimorphism, that is, a monomorphism and an epimorphism.

If the morphism $\overline{\alpha}$ is a monomorphism (an epimorphism) for every α then, following Rump [21, p. 167], we call the preabelian category *left semi-abelian* (*right semi-abelian*).

Theorem 1. [15] *A preabelian category $\mathcal{A}$ is P-semi-abelian if and only if it satisfies one of the equivalent conditions* (i)–(vii) *and one of the equivalent conditions* (i′)–(vii′) *listed below.*

(i) *$\mathcal{A}$ is right semi-abelian.*
(ii) *If $h \circ l$ is a kernel then so is l.*
(iii) *If* (1) *is a pushout and g is a kernel then* (1) *is a pullback.*
(iv) *If* (1) *is a pushout and g is a kernel then f is a monomorphism.*
(v) *If* (1) *is a pushout, g is a kernel and β is a cokernel then f is a monomorphism.*
(vi) *If l and h are kernels and $h \circ l$ is defined then $h \circ l$ is a kernel.*
(vii) *If* (1) *is a pushout and g is strict then the canonical morphism $\hat{\alpha} : \operatorname{Ker} g \to \operatorname{Ker} f$ is an epimorphism.*

(i′) *$\mathcal{A}$ is left semi-abelian.*
(ii′) *If $h \circ l$ is a cokernel then so is h.*
(iii′) *If* (1) *is a pullback and f is a cokernel then* (1) *is a pushout.*
(iv′) *If* (1) *is a pullback and f is a cokernel then g is an epimorphism.*
(v′) *If* (1) *is a pullback, f is a cokernel and α is a kernel then g is an epimorphism.*
(vi′) *If l and h are cokernels and $h \circ l$ is defined then $h \circ l$ is a cokernel.*
(vii′) *If* (1) *is a pullback and f is strict then the canonical morphism $\hat{\beta} : \operatorname{Coker} g \to \operatorname{Coker} f$ is a monomorphism.*

A preabelian category $\mathcal{A}$ is called *left quasi-abelian* (or *left almost abelian* [21]) if it satisfies

Axiom 3. If square (1) is a pullback then f is a cokernel $\Longrightarrow$ g is a cokernel.

Dually, a preabelian category $\mathcal{A}$ is called *right quasi-abelian* (or *right almost abelian* [21]) if it satisfies

Axiom 3*. If (1) is a pushout then g is a kernel $\Longrightarrow$ f is a kernel.

A left and right quasi-abelian category is referred to as *quasi-abelian* [24] (*semi-abelian in the sense of Raĭkov* [20], or *almost abelian* [21]).

As is well-known [17, 20, 21, 24], every quasi-abelian category is P-semi-abelian. Raĭkov thought that every semi-abelian category is quasi-abelian (which was mentioned in [17]). Kuz′minov and Cherevikin [17, Theorem 2] and later Rump [21, Proposition 3] noticed that a P-semi-abelian category is quasi-abelian if and only if it is left or right quasi-abelian. Several years ago Bonet and Dierolf [1], answering a question of W. Rump, constructed an example of a pullback violating Axiom 3 in the category **Bor** of bornological locally convex spaces, thus proving that it is not quasi-abelian. Later Rump [22] gave an algebraic example of a P-semi-abelian but not quasi-abelian category. In [23], he carried out a thorough study of P-semi-abelian subcategories of quasi-abelian categories and proved that **Bor** and the category **Bar** of barreled locally convex spaces are P-semi-abelian but both categories are not quasi-abelian. Recently Wengenroth [25] explained that the non-stability of some cokernels under pullbacks in **Bor** is not unusual. The point is the existence of non-α-regular (or non-β-regular) inductive limits of locally convex spaces (see [18] and [25, Section 3]).

If, for a cokernel f in a preabelian category, in every pullback (1) g is a cokernel (for a kernel g in a preabelian category, in every pushout (1), f is a kernel) then f is called a *semi-stable cokernel* (g is called a *semi-stable kernel*).

We recall some basic properties of semi-stable kernels and cokernels (following from [10, Propositions 5.11 and 5.12]).

Lemma 2. *The following hold in a preabelian category:*

(i) *if gf is a semi-stable kernel then so is f, if gf is a semi-stable cokernel then so is g;*

(ii) *if f and g are semi-stable kernels and gf is defined then gf is a semi-stable kernel; if f and g are semi-stable cokernels and gf is defined then gf is a semi-stable cokernel.*

(iii) *a pushout of a semi-stable kernel is a semi-stable kernel; a pullback of a semi-stable cokernel is a semi-stable cokernel.*

2. The Ker-Coker-Sequence

Consider a commutative diagram of the form

$$\begin{array}{ccccccccc} & & A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\psi_0} & C_0 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\psi_1} & C_1 & & \end{array} \tag{2}$$

in which $\psi_0 = \operatorname{coker}\varphi_0$, $\varphi_1 = \ker\psi_1$, in a preabelian category.

Diagram (2) naturally extends to the diagram

$$\begin{array}{ccccccccc} & & \operatorname{Ker}\alpha & \xrightarrow{\varepsilon} & \operatorname{Ker}\beta & \xrightarrow{\zeta} & \operatorname{Ker}\gamma & & \\ & & {\scriptstyle\ker\alpha}\downarrow & & {\scriptstyle\ker\beta}\downarrow & & {\scriptstyle\ker\gamma}\downarrow & & \\ & & A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\psi_0} & C_0 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\psi_1} & C_1 & & \\ & & {\scriptstyle\operatorname{coker}\alpha}\downarrow & & {\scriptstyle\operatorname{coker}\beta}\downarrow & & {\scriptstyle\operatorname{coker}\gamma}\downarrow & & \\ & & \operatorname{Coker}\alpha & \xrightarrow{\tau} & \operatorname{Coker}\beta & \xrightarrow{\theta} & \operatorname{Coker}\gamma\ , & & \end{array} \tag{3}$$

where $\zeta\varepsilon = 0$ and $\theta\tau = 0$.

Suppose now that ψ_0 is a semi-stable cokernel and φ_1 is a semi-stable kernel (or that the category is P-semi-abelian and one of these is semi-stable). Then, as follows from [6, 14], there is a natural connecting morphism $\delta : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ uniting the first and last rows in (3) into the Ker-Coker-sequence

$$\operatorname{Ker}\alpha \xrightarrow{\varepsilon} \operatorname{Ker}\beta \xrightarrow{\zeta} \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \xrightarrow{\tau} \operatorname{Coker}\beta \xrightarrow{\theta} \operatorname{Coker}\gamma, \tag{4}$$

in which the composition of two consecutive morphisms is zero [6, 14].

The proof of the following assertion may be found in [14].

Theorem 2. *The following hold for a diagram of the form* (2) *in a P-semi-abelian category:*

(i) *If in* (2) ψ_0 *and* $\operatorname{coker}\alpha$ *are semi-stable cokernels and* β *is strict then* (4) *is exact at* $\operatorname{Coker}\alpha$.

If in (2) φ_1 *and* $\ker\gamma$ *are semi-stable kernels and* β *is strict then* (4) *is exact at* $\operatorname{Ker}\gamma$.

(ii) *If in* (2) α *is strict then the upper row in* (3) *is exact.*

If in (2) γ *is strict then the lower row in* (3) *is exact.*

(iii) *If in* (2) α *is strict and* $\ker\gamma$ *and* φ_1 *are semi-stable kernels then* (4) *is exact at* $\operatorname{Ker}\beta$ *and* $\operatorname{Ker}\gamma$. *If* γ *is strict and* $\operatorname{coker}\alpha$ *and* ψ_0 *are semi-stable cokernels then* (4) *is exact at* $\operatorname{Coker}\beta$ *and* $\operatorname{Coker}\alpha$.

3. Left and Right Homology Objects

Suppose first that the ambient category is preabelian.

Given a sequence of the form

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \tag{5}$$

such that $\psi\varphi = 0$, there are a natural morphism $\sigma : A \to \operatorname{Ker}\psi$ such that $\varphi = (\ker\psi)\sigma$ and a natural morphism $\tau : \operatorname{Coker}\varphi \to C$ such that $\psi = \tau \operatorname{coker}\varphi$.

Definition 1. Call $H_-(B) = H_-(B,\varphi,\psi) = \operatorname{Coker}\sigma$ and $H_+(B) = H_+(B,\varphi,\psi) = \operatorname{Ker}\tau$ the *left* and *right homology objects of* (5) *at the term* B.

It is classical that these two notions coincide for abelian categories (see, for example, [5]). This remains valid for quasi-abelian categories [13]. As was shown in [13], in a preabelian category, there is a unique morphism $m : H_-(B) \to H_+(B)$ such that

$$(\ker\tau)m \operatorname{coker}\sigma = (\operatorname{coker}\varphi)(\ker\psi). \tag{6}$$

In [11], we established the following P-semi-abelian version of Lemma 4 of [13] (see [11, Lemma 7]).

Lemma 3. *Let the ambient category be P-semi-abelian. The morphism* $m : H_-(B) \to H_+(B)$ *is a bimorphism. If* $\ker\psi$ *is a semi-stable kernel or* $\operatorname{coker}\varphi$ *is a semi-stable cokernel then* m *is an isomorphism.*

Using the technique used in the proof of Lemma 7 in [11], we may also obtain the following:

Proposition 1. *Let the ambient category be preabelian. If* $\ker\psi$ *is a semi-stable kernel then* m *is a semi-stable kernel and if* $\operatorname{coker}\varphi$ *is a semi-stable cokernel then* m *is a semi-stable cokernel. Thus, if both conditions are fulfilled then* m *is an isomorphism.*

Proof. Indeed, as in the proof of Lemma 7 in in [11], we (easily) conclude that

$$\begin{array}{ccc} \operatorname{Ker}\psi & \xrightarrow{\operatorname{coker}\sigma} & H_-(B) \\ {\scriptstyle \ker\psi}\downarrow & & \downarrow{\scriptstyle (\ker\tau)m} \\ B & \xrightarrow{\operatorname{coker}\varphi} & \operatorname{Coker}\varphi \end{array}$$

is a pushout.

Now, Lemma 2(iii) implies that $(\ker\tau)m$ is a semi-stable kernel, from which by Lemma 2(i), so is m.

By duality, we see that if $\operatorname{coker}\varphi$ is a semi-stable cokernel then m is a semi-stable cokernel.

The proposition follows. □

If the category is quasi-abelian then all kernels and cokernels are stable, and thus the left and right homology objects are canonically isomorphic. The referee conjectured that the coincidence of the left and right homology objects in a preabelian category in turn implies that this category is quasi-abelian. Using the results of Kuz′minov and Cherevikin, we are now able to provethe following weakened version of this conjecture:

Theorem 3. *Suppose that, in a P-semi-abelian category $\mathcal{A}$, for every sequence*

$$A \overset{\varphi}{\to} B \overset{\psi}{\to} C$$

with $\psi\varphi = 0$, the canonical morphism $m : H_-(B, \varphi, \psi) \to H_+(B, \varphi, \psi)$ is an isomorphism. Then the category is quasi-abelian.

Proof. Since $\mathcal{A}$ is P-semi-abelian, from [17, Theorem 2] (see also [21, Proposition 3]) it follows that it suffices to prove that the category $\mathcal{A}$ is left quasi-abelian, that is, that cokernels in $\mathcal{A}$ are stable under pullbacks. As follows from [17, Theorem 2], it suffices to establish this only for pullbacks of cokernels along kernels. (We warn the reader that the terminology of [17] differs from ours: therein, "preabelian" means "P-semi-abelian" and "semiabelian" stands for "quasi-abelian".)

So, consider a pullback

$$\begin{array}{ccc} P & \xrightarrow{u'} & F \\ {\scriptstyle v'}\downarrow & & {\scriptstyle v}\downarrow \\ E & \xrightarrow{u} & G \end{array} \tag{7}$$

in which u is a cokernel and v is a kernel (and thus v' is a kernel) and prove that u' is also a cokernel.

Form the sequence

$$K \overset{\ker u}{\longrightarrow} E \overset{\operatorname{coker} v'}{\longrightarrow} L. \tag{8}$$

In (8), $(\operatorname{coker} v') \ker u = (\operatorname{coker} v')v' \ker u' = 0$.

Note that $v' = \ker(\operatorname{coker} v')$ and $u = \operatorname{coker}(\ker u)$. As above, the relation $(\operatorname{coker} v') \ker u = 0$ gives rise to two morphisms $\sigma : K \to P$ and $\tau : G \to L$ characterized uniquely by $\ker u = v'\sigma$ and $\operatorname{coker} v' = \tau u$. We have $\sigma = \ker u'$. Since u is a cokernel, the pullback (7) is a pushout (by Theorem 1(iii) or the assertion dual to [17, Lemma 5]). Hence, by Lemma 1(iv), $\tau = \operatorname{coker} v$. Then

$$H_-(E, \ker u, \operatorname{coker} v') = \operatorname{Coker} \sigma = \operatorname{Coim} u'$$

and

$$H_+(E, \ker u, \operatorname{coker} v') = \operatorname{Ker} \tau = F.$$

If $m : \operatorname{Coim} u' \to F$ is the canonical isomorphism between the left and right homology objects then

$$vm(\operatorname{coim} u') = uv' = vu'.$$

Since v is a monomorphism, this implies that $u' = m \operatorname{coim} u'$ with m an isomorphism. Thus, u' is a cokernel.

The theorem is proved. □

The proof of Theorem 3 in fact implies the following assertion.

Lemma 4. *Let*

$$\begin{array}{ccc} P & \xrightarrow{u'} & F \\ {\scriptstyle v'}\downarrow & & {\scriptstyle v}\downarrow \\ E & \xrightarrow{u} & G \end{array}$$

be a pullback in a P-semi-abelian category such that v is a kernel, u is a cokernel, and u' is not a cokernel. Let $H_-(E)$ and $H_+(E)$ be the left and right homology objects of the sequence

$$K \overset{\ker u}{\longrightarrow} E \overset{\operatorname{coker} v'}{\longrightarrow} L$$

at the term E. Then the canonical morphism $m : H_-(E) \to H_+(E)$ is not an isomorphism.

Consider the following

Example (Wengenroth [25])**.** Let **Bor** be the category of bornological locally convex spaces. For details on bornological spaces, the reader is referred, for example, to [9, Chapter 13] or [16, Chapter 6, Section 28]. The category **Bor** is P-semi-abelian [23].

In [18] Makarov introduced the following terminology: the inductive limit $X = \varinjlim X_n$ of an increasing sequence of locally convex spaces $X_1 \subset X_2 \subset \dots \subset X_n \subset \dots$ is called *α-regular* if any set bounded in X is contained in some X_n; it is called β-regular if any set that is bounded in X and contained in X_n is bounded in some X_m. The first examples of non-α-regular and non-β-regular inductive limits were given in [18]; other examples (connected with partial differential operators) may be found in [25, Section 3].

The following theorem is proved in [25]:

Theorem 4. *Let $X = \varinjlim X_n$ be a non-α-regular inductive limit of an increasing sequence of locally convex spaces $X_1 \subset X_2 \subset \dots \subset X_n \subset \dots$. Then there is a normed space Y included continuously in X such that, in the pullback*

$$\begin{array}{ccc} P & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle j} \\ \bigoplus X_n & \xrightarrow{s} & X \end{array} \tag{9}$$

where $i : Y \to X$ is the inclusion mapping, the mapping π_Y is not open in the associated bornological topology (*and, hence, not a cokernel in* **Bor**)*.*

By the definition of a pullback, the space P of this theorem is as follows:

$$P = \{((x_n)_{n\in\mathbb{N}}, y) \in \bigoplus X_n \times Y \,:\, \sum_{n\in\mathbb{N}} x_n \in Y\}.$$

and the mappings $\pi_0 : P \to X$ and $\pi_Y : P \to Y$ are the corresponding restricted projections. Note that π_Y, being an open surjection in the (quasi-abelian) category **LCS** of locally convex spaces, is a cokernel in this category.

Since a pullback of a cokernel in a P-semi-abelian category is a pushout (by Theorem 1(iii) or the dual of [17, Lemma 5]), Lemma 1(4) yields $\operatorname{coker}\pi_0 = (\operatorname{coker} i)s$.

As above, consider the sequence

$$\operatorname{Ker} s \xrightarrow{\ker s} \bigoplus X_n \xrightarrow{(\operatorname{coker} i)s} \operatorname{Coker} i,$$

corresponding to (9). If $H_-(\bigoplus X_n)$ is the left homology at $\bigoplus X_n$ and $H_+(\bigoplus X_n)$ is the right homology at $\bigoplus X_n$ then we have the isomorhism of vector spaces $H_-(\bigoplus X_n) \cong H_+(\bigoplus X_n) \cong Y$. However, by Lemma 4, the canonical morphism $m : H_-(X) \to H_+(X)$ (which is a bimorphism by Lemma 3) cannot be an isomorpism in **Bor**.

4. The Long (Co)homology Sequence in a P-Semi-Abelian Category

In [11], we started discussing the possibility of constructing the long exact cohomology sequence for a short strictly exact sequence of complexes in a P-semi-abelian category. Here we improve the main theorem by giving more natural sufficient conditions for the exactness of the (co)homology sequence.

By a (*cochain*) *complex* $\mathfrak{A} = (A^n, d_A^n)_{n\in\mathbb{Z}}$ in an additive category we understand a sequence

$$\dots \longrightarrow A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \xrightarrow{d_A^{n+1}} \dots$$

which is semi-exact at each term, that is, $d_A^{n+1}d_A^n = 0$ for all n. (Here and below, for all objects and morphisms corresponding to a cochain complex $\mathfrak{A}$, we use the subscript A instead of $\mathfrak{A}$.)

Let $\mathfrak{A} = (A^n, d_A^n)_{n\in\mathbb{Z}}$ be a cochain complex in a preabelian category. As was observed in [7], for each $n \in \mathbb{Z}$ the relations $d_A^{n+1}d_A^n = 0$ and $d_A^n d_A^{n-1} = 0$ imply the existence of a unique morphism $a_A^n : \operatorname{Coker} d_A^{n-1} \to \operatorname{Ker} d_A^{n+1}$ satisfying the condition

$$(\ker d_A^{n+1})a_A^n(\operatorname{coker} d_A^{n-1}) = d_A^n.$$

Put $H_-^n(\mathfrak{A}) = H_-(A^n, d_A^{n-1}, d_A^n)$ and $H_+^n(\mathfrak{A}) = H_+(A^n, d_A^{n-1}, d_A^n)$. As follows from the previous section,

$$H_-^n(\mathfrak{A}) = \operatorname{Coker}(a_A^{n-1} \operatorname{coker} d_A^{n-2}) = \operatorname{Coker} a_A^{n-1}$$

and

$$H_+^n(\mathfrak{A}) = \operatorname{Ker}((\ker d_A^{n+1})a_A^n) = \operatorname{Ker} a_A^n.$$

We call the homology objects $H_-^n(\mathfrak{A})$ and $H_+^n(\mathfrak{A})$ the *left* and *right nth (co)homology objects* of the cochain complex $\mathfrak{A}$.

Denote by $m_A^n : H_-^n(\mathfrak{A}) \to H_+^n(\mathfrak{A})$ the morphism of the previous section, defined uniquely by the relation

$$(\ker a_A^n)m_A^n(\operatorname{coker} a_A^{n-1}) = (\operatorname{coker} d_A^{n-1})(\ker d_A^n) \tag{10}$$

From now on, let our category be P-semi-abelian.

Lemma 5. *The following hold:*
(i) *if the cokernel* $\operatorname{coker} d_A^n$ *is semi-stable then so is* $\operatorname{coker} a_A^n$*;*
(ii) *if the kernel* $\ker d_A^n$ *is semi-stable then so is* $\ker a_A^n$*.*

Proof. (i) Writing down relations (10) for $n := n+1$, we obtain the commutative square

$$\begin{array}{ccc} \operatorname{Ker} d_A^{n+1} & \xrightarrow{\ker d_A^{n+1}} & A^{n+1} \\ {\scriptstyle \operatorname{coker} a_A^n}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{coker} d_A^n} \\ H_A^n(\mathfrak{A}) & \xrightarrow{\ker d_A^{n+1} m_A^{n+1}} & \operatorname{Coker} d_A^n, \end{array} \tag{11}$$

which is a pushout (cf. the proof of Lemma 7 in [11]). Since $\ker d_A^{n+1}$ is a kernel, by Theorem 1(iii) (or [17, Lemma 5]), the square (11) is also a pullback. Now, by Lemma 2(iii), the semi-stability of $\operatorname{coker} d_A^n$ implies that of $\operatorname{coker} a_A^n$.

Assertion (ii) is dual to (i).

The lemma is proved. □

Remark 1. *It is easy to observe also that, for a cochain complex* $\mathfrak{A} = (A^n, d_A^n)$*,* a_A^n *is strict if and only if* d_A^n *is strict.*

By a *morphism* of two complexes $\mathfrak{A} = (A^n, d_A^n)_{n\in\mathbb{Z}}$ and $\mathfrak{B} = (B^n, d_B^n)_{n\in\mathbb{Z}}$ we mean a family of morphisms $(\varphi^n : A^n \to B^n)_{n\in\mathbb{Z}}$ such that $\varphi^{n+1}d_A^n = d_B^n\varphi^n$ for all n. For three complexes $\mathfrak{A} = (A^n, d_A^n)_{n\in\mathbb{Z}}$, $\mathfrak{B} = (B^n, d_B^n)_{n\in\mathbb{Z}}$, and $\mathfrak{C} = (C^n, d_C^n)_{n\in\mathbb{Z}}$ and morphisms $\varphi : \mathfrak{A} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathfrak{C}$, we call the sequence

$$0 \to \mathfrak{A} \xrightarrow{\varphi} \mathfrak{B} \xrightarrow{\psi} \mathfrak{C} \to 0$$

short strictly exact if $\varphi^n|\psi^n$ for all n.

A morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ of complexes induces morphisms $\hat{\varphi}^n : \operatorname{Ker} d_A^n \to \operatorname{Ker} d_B^n$ and $\tilde{\varphi}^n : \operatorname{Coker} d_A^{n-1} \to \operatorname{Coker} d_B^{n-1}$. These morphisms are characterized uniquely by the equalities

$$(\ker d_B^n)\hat{\varphi}^n = \varphi^n \ker d_A^n; \quad \tilde{\varphi}^n \operatorname{coker} d_A^{n-1} = (\operatorname{coker} d_B^{n-1}\varphi^n). \tag{12}$$

Like in the quasi-abelian case [7], to a short strictly exact sequence of complexes

$$0 \to \mathfrak{A} \xrightarrow{\varphi} \mathfrak{B} \xrightarrow{\psi} \mathfrak{C} \to 0 \tag{13}$$

in a P-semi-abelian category, there corresponds a commutative diagram

$$\begin{array}{ccccccccc} & & \operatorname{Coker} d_A^{n-1} & \xrightarrow{\tilde{\varphi}^n} & \operatorname{Coker} d_B^{n-1} & \xrightarrow{\tilde{\psi}^n} & \operatorname{Coker} d_C^{n-1} & \longrightarrow & 0 \\ & & {\scriptstyle a_A^n}\downarrow & & {\scriptstyle a_B^n}\downarrow & & {\scriptstyle a_C^n}\downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker} d_A^{n+1} & \xrightarrow{\hat{\varphi}^{n+1}} & \operatorname{Ker} d_B^{n+1} & \xrightarrow{\hat{\psi}^{n+1}} & \operatorname{Ker} d_C^{n+1} & & \end{array} \tag{14}$$

Here $\tilde{\varphi}^n$, $\tilde{\psi}^n$, $\hat{\varphi}^{n+1}$ and $\hat{\psi}^{n+1}$ are defined in accordance with (12), $\tilde{\psi}^n = \operatorname{coker} \tilde{\varphi}^n$, and $\hat{\varphi}^{n+1} = \ker \hat{\psi}^{n+1}$. Diagram (14) yields two semi-exact sequences: the Ker-sequence

$$H_+^n(\mathfrak{A}) \xrightarrow{H_+^n(\varphi)} H_+^n(\mathfrak{B}) \xrightarrow{H_+^n(\psi)} H_+^n(\mathfrak{C}) \tag{15}$$

and the Coker-sequence

$$H_-^{n+1}(\mathfrak{A}) \xrightarrow{H_-^{n+1}(\varphi)} H_-^{n+1}(\mathfrak{B}) \xrightarrow{H_-^{n+1}(\psi)} H_-^{n+1}(\mathfrak{C}). \tag{16}$$

According to Theorem 1 of [11], if ψ^n, $\operatorname{coker} d_B^{n-1}$, $\operatorname{coker} d_C^{n-1}$ are semi-stable cokernels or φ^{n+1}, $\ker d_A^{n+1}$, $\ker d_B^{n+1}$ are semi-stable kernels, the morphism m_A^{n+1} diagram (14) yields a semi-exact Ker-Coker-sequence

$$\begin{aligned} &H_+^n(\mathfrak{A}) \xrightarrow{H_+^n(\varphi)} H_+^n(\mathfrak{B}) \xrightarrow{H_+^n(\psi)} H_+^n(\mathfrak{C}) \\ &\qquad \xrightarrow{\Delta^n} H_-^{n+1}(\mathfrak{A}) \xrightarrow{H_-^{n+1}(\varphi)} H_-^{n+1}(\mathfrak{B}) \xrightarrow{H_-^{n+1}(\psi)} H_-^{n+1}(\mathfrak{C}). \end{aligned} \tag{17}$$

Theorem 5. *Given a short strictly exact sequence* (13) *in a P-semi-abelian category, the corresponding sequences* (15) *and* (16) *have the following exactness properties:*
(i) *if d_A^n is strict then* (15) *is exact*;
(ii) *if d_C^n is strict then* (16) *is exact.*

Proof. The theorem follows from Theorem 2(ii) with account taken of Remark 1. □

Theorem 6. *Given a short strictly exact sequence* (13) *in a P-semi-abelian category, the corresponding sequence* (17) *has the following exactness properties*:
(i) *Suppose that d_B^n is strict.*

If φ^{n+1}, $\ker d_A^{n+1}$, $\ker d_B^{n+1}$, $\ker d_C^n$ are semi-stable kernels then (17) *is exact at $H_+^n(C)$.*

If ψ^n, $\operatorname{coker} d_B^{n-1}$, $\operatorname{coker} d_C^{n-1}$, $\operatorname{coker} d_A^n$ are semi-stable cokernels then (17) *is exact at $H_-^{n+1}(A)$.*

(ii) *Suppose that d_A^n is strict and φ^{n+1}, $\ker d_A^{n+1}$, $\ker d_B^{n+1}$, $\ker d_C^n$ are semi-stable kernels. Then* (17) *is exact at $H_+^n(B)$ and $H_+^n(C)$.*

Suppose that d_C^n is strict and ψ^n, $\operatorname{coker} d_B^{n-1}$, $\operatorname{coker} d_C^{n-1}$, $\operatorname{coker} d_A^n$ are semi-stable cokernels. Then (17) *is exact at $H_-^{n+1}(A)$ and $H_-^{n+1}(B)$.*

Proof. (i) Prove the first assertion. Suppose that d_B^n is strict and φ^{n+1}, $\ker d_A^{n+1}$, $\ker d_B^{n+1}$, $\ker d_C^n$ are semi-stable kernels. Then Lemma 5(i) implies that $\ker a_C^n$ is a semi-stable kernel. Moreover, the equality $(\ker d_B^{n+1})\hat{\varphi}^{n+1} = \varphi^{n+1} \ker d_A^{n+1}$ together with Lemma 2(i,ii) (as in the proof of Theorem 1 in [11]) easily implies that $\hat{\varphi}^{n+1}$ is a semi-stable kernel, and we may apply Theorem 2(i) to obtain the desired exactness of (17) at $H_+^n(C)$. The second assertion of (i) is dual to the first.

(ii) Suppose that d_A^n is strict and φ^{n+1}, $\ker d_A^{n+1}$, $\ker d_B^{n+1}$, $\ker d_C^n$ are semi-stable kernels. Then a_A^n is also strict (Remark 1). Using the same argument as in the proof of (i), we infer that $\hat{\varphi}^{n+1}$ is a semi-stable kernel. Applying Theorem 2(iii), we see that sequence (17) is exact at $H^n_+(B)$ and $H^n_+(C)$. The second assertion follows by duality.

The theorem is proved. □

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Yaroslav A. Kopylov
Sobolev Institute of Mathematics,
Prospekt Akad. Koptyuga 4,
630090, Novosibirsk, Russia
and
Novosibirsk State University
ul. Pirogova 2
630090, Novosibirsk, Russia
E-mail address: yakop@math.nsc.ru