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BRAHMAGUPTA FORMULA FOR CYCLIC QUADRILATERALS IN THE HYPERBOLIC PLANE

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Abstract. The Heron formula relates the area of an Euclidean triangle to its side lengths. Indian mathematician and astronomer Brahmagupta, in the seventh century, gave the analogous formulas for a convex cyclic quadrilateral. Several non-Euclidean versions of the Heron theorem have been known for a long time.

In this paper we consider a convex hyperbolic quadrilateral inscribed in a circle, horocycle or one branch of an equidistant curve. This is a natural hyperbolic analog of the cyclic quadrilateral in the Euclidean plane. We find a few versions of the Brahmahupta formula for such quadrilaterals.

Keywords: Heron formula, Brahmagupta formula, cyclic polygon, hyperbolic quadrilateral.

1. Introduction

Heron of Alexandria (c. 60 BC) is credited with the formula that relates the area $\mathcal{A}$ of a triangle to its side lengths a, b, and c :

$$\mathcal{A}^2 = (s-a)(s-b)(s-c)s,$$

where $s = (a+b+c)/2$ is the semiperimeter. For polygons with more than three sides, the side lengths do not in general determine the area, but they do if the polygon is convex and cyclic (inscribed in a circle). Brahmagupta, in the seventh

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century, gave the analogous formula for a convex cyclic quadrilateral with side lengths a, b, c, and d :

$$\mathcal{A}^2 = (s-a)(s-b)(s-c)(s-d),$$

where $s = (a+b+c+d)/2$. See [18] for the elementary proof. An interesting consideration of the problem can be found in the Möbius paper [21]. Independently, D. P. Robbins [22] and V. V. Varfolomev [23] found a way to generalize these formulas. The main idea of both papers was to determine the squared area $\mathcal{A}^2$ as a root of an algebraic equation whose coefficients are integer polynomials in the squares of the side lengths. See also the papers [20], [19] and [3] for more detailed consideration.

The paper [23] was inspired by the result of I. Kh. Sabitov [2] who obtained a similar result for the squared volume of a three dimensional Euclidean polyhedron. Recently, A. Gaifullin [24] obtained a four dimensional version of this result.

In the present paper we deal with the hyperbolic plane instead of the Euclidean one. The hyperbolic plane under consideration is equipped by a Riemannian metric of constant curvature $k = -1$. All necessary definitions from hyperbolic geometry can be found in the books [6] and [7].

By definition, a *cyclic polygon* in the hyperbolic plane is a convex polygon inscribed in a circle, horocycle or one branch of equidistant curve. Useful information about cyclic polygons can be found in the papers [9] and [10]. In particular, it is shown in [9] that any cyclic polygon in the hyperbolic plane is uniquely determined (up to isometry) by the ordered sequence of its side lengths. Moreover, among all polygons in the hyperbolic plane with fixed positive side lengths there exist polygons of maximal area. Each such maximal polygon is cyclic [10].

The following four non-Euclidean versions of the Heron formula in the hyperbolic plane have been known for a long time.

Theorem 1.1. The area $\mathcal{A}$ of a hyperbolic triangle with side lengths $a, b,$ and c is given by each of the following formulas

(i) Sine of 1/2 Area Formula

$$\sin^2 \frac{\mathcal{A}}{2} = \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s)}{4\cosh^2(\frac{a}{2})\cosh^2(\frac{b}{2})\cosh^2(\frac{c}{2})};$$

(ii) Tangent of 1/4 Area Formula

$$\tan^2 \frac{\mathcal{A}}{4} = \tanh(\frac{s-a}{2})\tanh(\frac{s-b}{2})\tanh(\frac{s-c}{2})\tanh(\frac{s}{2});$$

(iii) Sine of 1/4 Area Formula

$$\sin^2 \frac{\mathcal{A}}{4} = \frac{\sinh(\frac{s-a}{2})\sinh(\frac{s-b}{2})\sinh(\frac{s-c}{2})\sinh(\frac{s}{2})}{\cosh(\frac{a}{2})\cosh(\frac{b}{2})\cosh(\frac{c}{2})};$$

(iv) Bilinski Formula

$$\cos \frac{\mathcal{A}}{2} = \frac{\cosh a + \cosh b + \cosh c + 1}{4\cosh(\frac{a}{2})\cosh(\frac{b}{2})\cosh(\frac{c}{2})}.$$

The first two formulas are contained in the books ([7], p. 66) and ([6], p. 36). See also equations 10 and 14 in Chapter 14 of [4].

The third formula can be obtained by the squaring of the product of the first two. The forth one was derived by Stanko Bilinski [8] (see also [10]). It should be noted that the analogous formula in spherical geometry are also known. For example, the spherical version of (i) is called the Cagnoli's Theorem in ([5], sec. 100), (ii) is called the Lhuilier's Theorem ([5], sec. 102), (iii) is proven in ([5], sec. 103), (iv) is proven in ([5], sec. 103) and appears as an exercise in ([14], p. 167).

2. Preliminary results on cyclic quadrilaterals

We recall a few well known facts about cyclic quadrilaterals. A convex Euclidean quadrilateral whose interior angles are A, B, C, D is cyclic if and only if $A + C = B + D = \pi$. This is an elementary fact which can be found, for example, in ([1], p. 52). A similar result for hyperbolic quadrilateral was obtained by V. F. Petrov [12] and W. Lienhard [13].

Proposition 2.1. *A convex hyperbolic quadrilateral whose interior angles are A, B, C, D is cyclic if and only if $A + C = B + D$.*

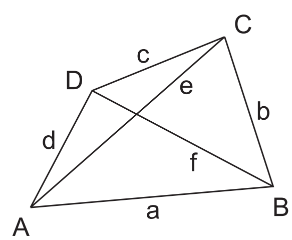

Fig. 1.

We remark that the sum of angles of a hyperbolic quadrilateral is less then 2π. Hence, for any cyclic hyperbolic quadrilateral we have $A + C = B + D < \pi$.

Now, let us suppose that side and diagonal lengths of a quadrilateral are as indicated on Fig. 1. Then, in the Euclidean case, a quadrilateral is cyclic if and only if $e\,f = a\,c + b\,d$. This is the Ptolemy's theorem (see, for example [1], p. 61). A similar result for hyperbolic quadrilaterals is contained in the paper by J. E. Valentine [15].

Proposition 2.2. *A convex hyperbolic quadrilateral with side lengths a, b, c, d and diagonal lengths e, f is cyclic if and only if*

$$\sinh\frac{e}{2}\sinh\frac{f}{2} = \sinh\frac{a}{2}\sinh\frac{c}{2} + \sinh\frac{b}{2}\sinh\frac{d}{2}.$$

The analogous formula in spherical geometry is where sinh is replaced with sin and the spherical version of the formula is found in proposition 4 on page 180 of [14].

An important supplement to the Ptolemy's theorem is the following property of a cyclic quadrilateral in the Euclidean plane. Its side and diagonal lengths are related by the equation

$$\frac{e}{f} = \frac{a\,d + b\,c}{a\,b + c\,d}. \tag{1}$$

(See [1], p. 62). Together with the Ptolemy's theorem this equation allows to find the diagonal lengths of a cyclic quadrilateral through its side lengths.

It was noted in the paper [16] that the above mentioned relationships between sides and diagonals of a cyclic quadrilateral are valid also in the hyperbolic geometry. To make them true ones can change the length side a by the quantity $s(a) = \sinh \frac{a}{2}$. In particular, formula (1) can be rewritten in the following way.

Proposition 2.3. *The side lengths a, b, c, d and diagonal lengths e, f of a cyclic hyperbolic quadrilateral are related by the following equation*

$$\frac{s(e)}{s(f)} = \frac{s(a)s(d) + s(b)s(c)}{s(a)s(b) + s(c)s(d)}.$$

By making use of Propositions 2.2 and 2.3 we derive the following formulas for the diagonal lengths e, f of a cyclic hyperbolic quadrilateral:

$$s^2(e) = \frac{s(a)s(d) + s(b)s(c)}{s(a)s(b) + s(c)s(d)}(s(a)s(c) + s(b)s(d)), \tag{2}$$

$$s^2(f) = \frac{s(a)s(b) + s(c)s(d)}{s(a)s(d) + s(b)s(c)}(s(a)s(c) + s(b)s(d)). \tag{3}$$

It is important to note that formulas (2) and (3) take a place also in the Euclidean and spherical geometries. In these cases, instead of function $s(a)$ one should take the functions $s(a) = a$ and $s(a) = \sin \frac{a}{2}$ respectively. See ([14], p. 180), [16] and [17] for the arguments in the spherical case.

All the above propositions will be used in the next section to obtain a few versions of the Brahmahupta formula for a cyclic hyperbolic quadrilateral.

3. Brahmahupta formula for a cyclic hyperbolic quadrilateral

In this section we consider the four versions of the Brahmahupta formula for a cyclic hyperbolic quadrilateral. They are generalizations of the respective statements (i) - (iv) of Theorem 1.1.

In particular, the first statement (i) has the following analog.

Theorem 3.1 (Sine of 1/2 Area Formula). *The area $\mathcal{A}$ of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d is given by the formula*

$$\sin^2 \frac{\mathcal{A}}{2} = \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s-d)}{4\cosh^2\frac{a}{2}\cosh^2\frac{b}{2}\cosh^2\frac{c}{2}\cosh^2\frac{d}{2}}(1-\varepsilon),$$

where $\varepsilon = \dfrac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\sinh\frac{d}{2}}{\cosh\frac{s-a}{2}\cosh\frac{s-b}{2}\cosh\frac{s-c}{2}\cosh\frac{s-d}{2}}$ *and* $s = \dfrac{a+b+c+d}{2}$.

We note that the number ε vanish if $d = 0$. In this case, we get formula (i) again.

The second statement (ii) for the case of a hyperbolic quadrilateral has the following form.

Theorem 3.2 (Tangent of 1/4 Area Formula). *The area $\mathcal{A}$ of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d is given by the formula*

$$\tan^2 \frac{\mathcal{A}}{4} = \frac{1}{1-\varepsilon}\tanh\frac{s-a}{2}\tanh\frac{s-b}{2}\tanh\frac{s-c}{2}\tanh\frac{s-d}{2},$$

where s and ε are the same as in Theorem 3.1.

It follows from Theorem 3.1 that for any $a, b, c, d \neq 0$ we have $1 - \varepsilon > 0$ and $\varepsilon > 0$. Hence, $0 < \varepsilon < 1$. Taking into account these inequalities as an immediate consequence of theorems 3.1 and 3.2 we obtain the following corollary.

Corollary 3.1. *For any cyclic hyperbolic quadrilateral the following inequalities take a place*

$$\sin^2 \frac{A}{2} < \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s-d)}{4\cosh^2\left(\frac{a}{2}\right)\cosh^2\left(\frac{b}{2}\right)\cosh^2\left(\frac{c}{2}\right)\cosh^2\left(\frac{d}{2}\right)}$$

and

$$\tan^2 \frac{A}{4} > \tanh\left(\frac{s-a}{2}\right)\tanh\left(\frac{s-b}{2}\right)\tanh\left(\frac{s-c}{2}\right)\tanh\left(\frac{s-d}{2}\right).$$

By squaring the product of formulas in Theorems 3.1 and 3.2 we have the following result. It can be considered as a direct generalization of the third statement in Theorem 1.1.

Theorem 3.3 (Sine of $1/4$ Area Formula). *The area $\mathcal{A}$ of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d is given by the formula*

$$\sin^2 \frac{\mathcal{A}}{4} = \frac{\sinh\frac{s-a}{2}\sinh\frac{s-b}{2}\sinh\frac{s-c}{2}\sinh\frac{s-d}{2}}{\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh\frac{d}{2}},$$

where $s = \dfrac{a+b+c+d}{2}$.

The analogous formula in spherical geometry is where sinh is replaced with sin, cosh with cos and the spherical version of the formula is found in proposition 5 on page 182 of [14].

To obtain one more corollary we consider a circumscribed quadrilateral. In this case (see Fig. 2) we have $s-a=c$, $s-b=d$, $s-c=a$, $s-d=b$. As a result we obtain the following assertion.

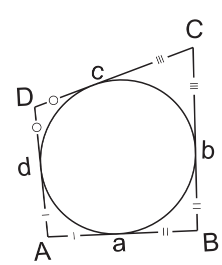

Fig. 2.

Corollary 3.2 (Brahmagupta formula for inscribed and circumscribed quadrilateral). *The area $\mathcal{A}$ of a cyclic (inscribed) and circumscribed hyperbolic quadrilateral with side lengths a, b, c and d can be found by the formula*

$$\sin^2 \frac{\mathcal{A}}{4} = \tanh\left(\frac{a}{2}\right)\tanh\left(\frac{b}{2}\right)\tanh\left(\frac{c}{2}\right)\tanh\left(\frac{d}{2}\right).$$

The analogous formula in spherical geometry is where tanh is replaced with tan and the spherical version of the formula is contained in ([14], p. 46).

An Euclidean version of this result is well known. See, for example [11] and ([1], p. 91). In this case

$$A^2 = a\,b\,c\,d.$$

Finally, we get the following version of the Bilinski Theorem for a cyclic quadrilateral.

Theorem 3.4 (Bilinski Formula). *The area $\mathcal{A}$ of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d is given by the formula*

$$\cos\frac{\mathcal{A}}{2} = \frac{\cosh a + \cosh b + \cosh c + \cosh d - 4\sinh\left(\frac{a}{2}\right)\sinh\left(\frac{b}{2}\right)\sinh\left(\frac{c}{2}\right)\sinh\left(\frac{d}{2}\right)}{4\cosh\left(\frac{a}{2}\right)\cosh\left(\frac{b}{2}\right)\cosh\left(\frac{c}{2}\right)\cosh\left(\frac{d}{2}\right)}.$$

4. The proof of the Brahmagupta theorem

Consider a cyclic hyperbolic quadrilateral with side lengths a, b, c and d and interior angles A, B, C and D shown on Fig. 1. Denote by $\mathcal{A}$ its area. By the Gauss-Bonnet formula we obtain

$$\mathcal{A} = 2\pi - A - B - C - D.$$

To prove Theorem 3.1 let us find the quantities $\sin^2\frac{\mathcal{A}}{4}$ and $\cos^2\frac{\mathcal{A}}{4}$ through a, b, c and d. Since $A + C = B + D$ (see Proposition 2.1) we have

$$2\sin^2\frac{\mathcal{A}}{4} = 1 - \cos\frac{\mathcal{A}}{2} = 1 - \cos(\pi - (A + C)) = 1 + \cos(A + C).$$

Hence,

$$\sin^2\frac{\mathcal{A}}{4} = \frac{1 + \cos A\cos C - \sin A\sin C}{2}. \tag{4}$$

First of all, we show that $\cos A$, $\cos C$ and the product $\sin A\sin C$ can be expressed in terms of elementary functions of a, b, c and d. To find $\cos A$ we have to use the Cosine rule for hyperbolic triangle ABD :

$$\cosh f = \cosh a\cosh d - \sinh a\sinh d\cos A.$$

Hence,

$$\cos A = \frac{\cosh a\cosh d - \cosh f}{\sinh a\sinh d}. \tag{5}$$

We note that $\cosh f = 2s^2(f) + 1$, $\cosh a = 2s^2(a) + 1$ and $\cosh d = 2s^2(d) + 1$. Putting these identities into equations (3) and (5) we find $\cos A$ through a, b, c and d. After straightforward computer calculations we obtain

$$\cos A = \frac{(2s^2(a)+1)(2s^2(d)+1) - (2s^2(f)+1)}{2s(a)\cosh\frac{a}{2}\cdot 2s(d)\cosh\frac{d}{2}} \tag{6}$$

$$= \frac{s^2(a) - s^2(b) - s^2(c) + s^2(d) + 2s(a)s(b)s(c)s(d) + 2s^2(a)s^2(d)}{2(s(a)s(d) + s(b)s(c))\cosh\frac{a}{2}\cosh\frac{d}{2}}.$$

In a similar way we get the formula

$$\cos C = \frac{-s^2(a) + s^2(b) + s^2(c) - s^2(d) + 2s(a)s(b)s(c)s(d) + 2s^2(b)s^2(c)}{2(s(a)s(d) + s(b)s(c))\cosh\frac{b}{2}\cosh\frac{c}{2}}. \tag{7}$$

We note that $\sin A \sin C > 0$ and $\sin^2 A \sin^2 C = (1-\cos^2 A)(1-\cos^2 C)$. Squaring the latter equation, where $\cos A$ and $\cos C$ are given by formulas (6) and (7) we obtain

$$\sin A \sin C =$$
$$4\cosh\frac{a+b-c-d}{4}\cosh\frac{a-b+c-d}{4}\cosh\frac{a-b-c+d}{4}\cosh\frac{a+b+c+d}{4}\times$$
$$\sinh\frac{-a+b+c+d}{4}\sinh\frac{a-b+c+d}{4}\sinh\frac{a+b-c+d}{4}\sinh\frac{a+b+c-d}{4}\Big/ \tag{8}$$
$$((\sinh\frac{a}{2}\sinh\frac{d}{2}+\sinh\frac{b}{2}\sinh\frac{c}{2})^2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh\frac{d}{2}).$$

Substituting (6), (7) and (8) into (4) and simplifying we get

$$\sin^2\frac{\mathcal{A}}{4} = \frac{\sinh\frac{-a+b+c+d}{4}\sinh\frac{a-b+c+d}{4}\sinh\frac{a+b-c+d}{4}\sinh\frac{a+b+c-d}{4}}{\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh\frac{d}{2}}. \tag{9}$$

This proves Theorem 3.3.

In a similar way, from identity $2\cos^2\frac{\mathcal{A}}{4} = 1+\cos\frac{\mathcal{A}}{2} = 1-\cos(A+C)$ we have

$$\cos^2\frac{\mathcal{A}}{4} = \frac{\cosh\frac{a+b-c-d}{4}\cosh\frac{a-b+c-d}{4}\cosh\frac{a-b-c+d}{4}\cosh\frac{a+b+c+d}{4}}{\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh\frac{d}{2}}. \tag{10}$$

The following lemma can be easy proved by straightforward calculations.

Lemma 3.1. *The expression*

$$Q = \frac{\cosh\frac{a+b-c-d}{4}\cosh\frac{a-b+c-d}{4}\cosh\frac{a-b-c+d}{4}\cosh\frac{a+b+c+d}{4}}{\cosh\frac{-a+b+c+d}{4}\cosh\frac{a-b+c+d}{4}\cosh\frac{a+b-c+d}{4}\cosh\frac{a+b+c-d}{4}}$$

can be rewritten in the form $Q = 1-\varepsilon$, *where*

$$\varepsilon = \frac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\sinh\frac{d}{2}}{\cosh\frac{s-a}{2}\cosh\frac{s-b}{2}\cosh\frac{s-c}{2}\cosh\frac{s-d}{2}} \text{ and } s = \frac{a+b+c+d}{2}.$$

Taking four times product of equations (9) and (10) we have

$$\sin^2\frac{\mathcal{A}}{2} = \frac{\sinh\frac{-a+b+c+d}{2}\sinh\frac{a-b+c+d}{2}\sinh\frac{a+b-c+d}{2}\sinh\frac{a+b+c-d}{2}}{4\cosh^2\frac{a}{2}\cosh^2\frac{b}{2}\cosh^2\frac{c}{2}\cosh^2\frac{d}{2}}\cdot Q, \tag{11}$$

where Q is the same as in Lemma 3.1.

Then the statement of Theorem 3.1 follows from equation (11), Lemma 3.1 and the evident identity $s-a = \frac{-a+b+c+d}{2}$.

To prove Theorem 3.2 we divide (9) by (10). As a result we have

$$\tan^2\frac{\mathcal{A}}{4} = \frac{\sinh\frac{-a+b+c+d}{4}\sinh\frac{a-b+c+d}{4}\sinh\frac{a+b-c+d}{4}\sinh\frac{a+b+c-d}{4}}{\cosh\frac{a+b-c-d}{4}\cosh\frac{a-b+c-d}{4}\cosh\frac{a-b-c+d}{4}\cosh\frac{a+b+c+d}{4}}. \tag{12}$$

Hence, applying Lemma 3.1 we obtain the statement of Theorem 3.2.

Finally, the Bilinski formula (Theorem 3.4) follows from the identity $\cos\frac{\mathcal{A}}{2} = \cos^2\frac{\mathcal{A}}{4} - \sin^2\frac{\mathcal{A}}{4}$ and the above mentioned equations (9) and (10).

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