S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 9, стр. 261–265 (2012) УДК 517.98
MSC 47B99

CONDITIONS FOR FACTORABLE MATRICES TO BE HYPONORMAL AND DOMINANT

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Abstract. Sufficient conditions are given for a lower triangular factorable matrix M, acting as a bounded linear operator on ℓ^2, to be hyponormal. Necessary conditions are given for M to be a dominant operator on ℓ^2. The results are then applied to several examples, including the H-J Cesàro operators, the q-Cesàro operators and other weighted mean matrices, and some Toeplitz matrices.

Keywords: hyponormal operator, dominant operator, factorable matrix, weighted mean matrix

1. Introduction

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space H, then $A \in B(H)$ is said to be *hyponormal* if $A^*A - AA^* \geq 0$. The operator A is *dominant* if $Ran(A-\lambda) \subset Ran(A-\lambda)^*$ for all λ in the spectrum of A. Hyponormal operators are necessarily dominant.

A lower triangular infinite matrix $M = M(\{a_i\},\{c_j\})$, acting through multiplication to give a bounded linear operator on $H = \ell^2$, is *factorable* if its nonzero entries m_{ij} satisfy $m_{ij} = a_i c_j$ where a_i depends only on i and c_j depends only on j. A factorable matrix is *terraced* (see **[6]**, **[8]**) if $c_j = 1$ for all j. A *weighted mean matrix* is a lower triangular matrix with entries p_j/P_i, where $\{p_j\}$ is a nonnegative sequence with $p_0 > 0$, and $P_i = \sum_{j=0}^{i} p_j$. A weighted mean matrix is factorable, with $a_i = 1/P_i$ and $c_j = p_j$ for all i,j.

In earlier papers **[9]** , **[10]**, sufficient conditions were found for terraced matrices to be hyponormal. Here we extend one of those results to factorable matrices,

Rhaly, H. C., Jr., and Rhoades, B. E., Conditions for Factorable Matrices to be Hyponormal and Dominant.

Received March 20, 2012; published May 14, 2012.

and we also obtain necessary conditions for a factorable matrix to be dominant. Throughout this paper we will restrict our attention to those factorable matrices that are lower triangular and give bounded linear operators on ℓ^2.

2. Sufficient Conditions for Hyponormality of a Factorable Matrix

Our goal here is to obtain restrictions on the sequences $\{a_i\}, \{c_j\}$ that are sufficient to guarantee that the factorable matrix $M :\equiv M(\{a_i\}, \{c_j\}) \in B(\ell^2)$ is hyponormal. We follow the procedure used for terraced matrices in **[9]**, involving the use of a diagonal matrix D which satisfies $M^*M \geq M^*DM \geq MM^*$.

Theorem 1. *Assume that $M :\equiv M(\{a_i\}, \{c_j\})$ is a bounded linear operator on ℓ^2 with $a_i, c_j > 0$ for all i, j, and $\{a_k/c_k\}$ is strictly decreasing to 0. If $0 < c_0a_0 \leq 1$ and*

$$(a_k/c_k)(1 - c_ka_k) \leq a_{k+1}/c_{k+1} \leq (a_k/c_k)/(1 + c_{k+1}^2[a_k/c_k])$$

for each nonnegative integer k, then M is hyponormal.

Proof. Assume that $\{a_k/c_k\}$ is strictly decreasing to 0, where $a_k, c_k > 0$ for all k, and $sup\{d_k : k = 0, 1, 2, ...\} < \infty$, where $d_k \equiv (c_{k+1}a_k - c_ka_{k+1})/(c_kc_{k+1}a_k^2)$ for all k. Let D denote $diag\{d_k : k = 0, 1, 2, ...\}$. Then it is straightforward to verify that

$$M^*DM = \begin{pmatrix} c_0a_0 & c_0a_1 & c_0a_2 & c_0a_3 & \cdots \\ c_0a_1 & c_1a_1 & c_1a_2 & c_1a_3 & \cdots \\ c_0a_2 & c_1a_2 & c_2a_2 & c_2a_3 & \cdots \\ c_0a_3 & c_1a_3 & c_2a_3 & c_3a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We consider the equation

$$M^*M - MM^* = (M^*M - M^*DM) + (M^*DM - MM^*)$$

and investigate the summands separately. Since

$$\langle (M^*M - M^*DM)f, f\rangle = \langle (I - D)Mf, Mf\rangle,$$

the first summand is a positive operator if $d_k \leq 1$ for all k; or, equivalently, $a_{k+1}/c_{k+1} \geq (a_k/c_k)(1-c_ka_k)$ for all k. The second summand has the representation $M^*DM - MM^* =$

$$\begin{pmatrix} c_0a_0(1-s_0a_0) & c_0a_1(1-s_0a_0) & c_0a_2(1-s_0a_0) & c_0a_3(1-s_0a_0) & \cdots \\ c_0a_1(1-s_0a_0) & c_1a_1(1-s_1a_1) & c_1a_2(1-s_1a_1) & c_1a_3(1-s_1a_1) & \cdots \\ c_0a_2(1-s_0a_0) & c_1a_2(1-s_1a_1) & c_2a_2(1-s_2a_2) & c_2a_3(1-s_2a_2) & \cdots \\ c_0a_3(1-s_0a_0) & c_1a_3(1-s_1a_1) & c_2a_3(1-s_2a_2) & c_3a_3(1-s_3a_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $s_n = (\sum_{k=0}^n c_k^2)/c_n$. To show that $M^*DM - MM^*$ is a positive operator, it suffices to show that, for each positive integer N, the N^{th} finite section has nonnegative determinant. For $k = N-1, N-2, ..., 2, 1, 0$ (in that order), multiply row k by a_{k+1}/a_k and subtract from row $k+1$. This results in an upper triangular matrix whose diagonal elements are $c_0a_0(1 - c_0a_0)$ and

$$c_{k+1}a_{k+1}(1 - c_{k+1}a_{k+1} - c_ka_{k+1}/[c_{k+1}a_k])$$

for $0 \leq k \leq N-1$. It then follows that this determinant is nonnegative if $0 < c_0a_0 \leq 1$ and $a_{k+1}/c_{k+1} \leq (a_k/c_k)/(1 + c_{k+1}^2[a_k/c_k])$ for $0 \leq k \leq N-1$. □

Example 1. *(Toeplitz Matrix) Suppose that M is the factorable matrix with entries $m_{ij} = a_i c_j$ where $a_i = (1-r^2)r^i$, $c_j = 1/r^j$ for all i, j for $0 < r < 1$. One easily verifies that Theorem 1 is satisfied, so M is hyponormal. It follows that the Toeplitz matrix $T :\equiv [1/(1-r^2)]M$ is also hyponormal when $0 < r < 1$.*

Corollary 1. *If $p_0 > 0$, $p_k \le p_{k+1}$ for all $k \ge 0$, and*

$$p_{k+1}(\sum_{j=0}^{k-1} p_j)(\sum_{j=0}^{k+1} p_j) \le p_k(\sum_{j=0}^{k} p_j)^2 \text{ for all } k \ge 1,$$

then the associated weighted mean matrix (whenever bounded) is hyponormal.

Example 2. *(Weighted mean matrix) The hypothesis of Corollary 1 is satisfied by the sequence $p_n = 2 - 1/2^n$ for all $n \ge 0$, so the associated weighted mean matrix is hyponormal.*

Example 3. *For each of the following examples, the factorable matrix M is hyponormal since Theorem 1 holds.*
(a) Define M by taking $c_j = 1$ and $a_i = 1/\sqrt{(i+1)(i+2)}$ for all i, j.
(b) For fixed $k > 0$, M is defined by $a_i = 1/\sqrt{(i+k)(i+k+1)}$,
$c_j = \sqrt{j+k}/\sqrt{j+k+1}$ for all i, j.
(c) Define M by taking $c_j = \beta^j$ and $a_i = \beta^i/(\sum_{k=o}^{i} \beta^{2k})$, where $\beta > 1$, for all i,j.

Example 4. *The following matrices were studied by Hausdorff and Jakimovski (see [**3**], [**4**],[**5**],[**14**]), so we refer to them as the H-J Cesàro matrices. Suppose that the sequence $\{\lambda_n\}$ satisfies the following conditions:*

$$0 \le \lambda_0 < \lambda_1 < ... < \lambda_n < ...$$

with $\lim \lambda_n = \infty$, but so slowly that $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$. If we take $a_0 = 1/(\lambda_0 + 1)$, $c_0 = 1$, $a_i = (\lambda_1 ... \lambda_i)/\prod_{m=0}^{i}(\lambda_m + 1)$ for $i \ge 1$, and $c_j = \prod_{m=0}^{j-1}(\lambda_m + 1)/(\lambda_1 ... \lambda_j)$ for $j \ge 1$, then Theorem 1 tells us that the associated matrix M will be hyponormal if

$$(\lambda_n + \sqrt{\lambda_n^2 + 4\lambda_n})/2 \le \lambda_{n+1} \le \lambda_n + 1 \text{ for all } n.$$

*We note that the special case $\lambda_n = n - 1 + \alpha$ for fixed $\alpha \ge 1$ and all $n \ge 0$ yields the hyponormal generalized Cesàro operators of order one; see [**11**], [**13**]. The growth rate of acceptable λ_n in the inequality above goes from n^α for $0 < \alpha < 1$ to $n(logn)^\alpha$, for any $\alpha > 0$.*

Remark 1. *We observe that the conditions presented in Theorem 1 are not necessary for the hyponormality of a factorable matrix. For consider the case when $c_j = 1$ for each j and $a_i = (i+3)/(i+2)^2$ for each i. This example is known to be hyponormal since it satisfies the hypothesis of [**10**, Theorem 2.2], but it does not satisfy the inequality in Theorem 1.*

3. Necessary Conditions for Dominance of a Factorable Matrix

Next we find necessary conditions for the factorable matrix M to be dominant. Let $\{e_n\}$ denote the standard orthonormal basis for ℓ^2.

Theorem 2. *For the factorable matrix M with $a_0 c_0 \ne 0$ to be a dominant operator on ℓ^2, it is necessary that $\{c_n a_1 \prod_{j=2}^{n-1}(c_0 a_0 - c_j a_j)/(c_0^n a_0^n)\}_{n \ge 3} \in \ell^2$.*

Proof. First we observe that $e_1 \in Ran(M - c_0a_0I)$ since

$$(M - c_0a_0I)(\{c_1/[c_0^2a_0]\}e_0 - \{1/[c_0a_0]\}e_1) = e_1.$$

If M is to be dominant, then we must also have $e_1 \in Ran(M - c_0a_0I)^*$. If $(M - c_0a_0I)^*x = e_1$ for some $x =< x_0, x_1, x_2, x_3, ... >^T$, then it is necessary that
(1) $c_0a_0x_1 = (c_1/c_0)(c_0\sum_{j=1}^{\infty} a_jx_j) - \{(c_1a_1 - c_0a_0)x_1 + c_1\sum_{j=2}^{\infty} a_jx_j\} =$
$(c_1/c_0)e_1(0) - e_1(1) = -1$, so $x_1 = -1/(c_0a_0)$;
(2) $(c_2/c_1)(c_1a_1 - c_0a_0)x_1 + c_0a_0x_2 =$
$(c_2/c_1)\{(c_1a_1 - c_0a_0)x_1 + c_1\sum_{j=2}^{\infty} a_jx_j\} - \{(c_2a_2 - c_0a_0)x_2 + c_2\sum_{j=3}^{\infty} a_jx_j\}$
$= (c_2/c_1)e_1(1) - e_1(2) = (c_2/c_1)$, so $x_2 = c_2a_1/(c_0^2a_0^2)$;
(3) $(c_3/c_2)(c_2a_2 - c_0a_0)x_2 + c_0a_0x_3 =$
$(c_3/c_2)\{(c_2a_2 - c_0a_0)x_2 + c_2\sum_{j=3}^{\infty} a_jx_j\} - \{(c_3a_3 - c_0a_0)x_3 + c_3\sum_{j=4}^{\infty} a_jx_j\}$
$= (c_3/c_2)e_1(2) - e_1(3) = 0$, so $x_3 = c_3a_1(c_0a_0 - c_2a_2)/(c_0^3a_0^3)$;
and similarly, by induction,
(4) $(c_n/c_{n-1})(c_{n-1}a_{n-1} - c_0a_0)x_{n-1} + c_0a_0x_n =$
$(c_n/c_{n-1})\{(c_{n-1}a_{n-1} - c_0a_0)x_{n-1} + c_{n-1}\sum_{j=n}^{\infty} a_jx_j\}$
$-\{(c_na_n - c_0a_0)x_n + c_n\sum_{j=n+1}^{\infty} a_jx_j\} =$
$(c_n/c_{n-1})e_1(n-1) - e_1(n) = 0$, so $x_n = c_na_1\prod_{j=2}^{n-1}(c_0a_0 - c_ja_j)/(c_0^na_0^n)$ for $n \geq 3$.
Since we need $x \in \ell^2$, the proof is complete. □

Example 5. *(Generalized Cesàro operators of order one for $0 < k \leq \frac{1}{2}$) We return to the operators mentioned at the conclusion of Example 4. In simplified terms, the generalized Cesàro operators of order one are the terraced matrices C_k associated with $a_i = 1/(i + k)$ for all i, where $k > 0$ is fixed. It was noted in Example 4 that these operators are hyponormal (and hence dominant) for $k \geq 1$.*

*Now we are concerned with $k < 1$. For C_k to be dominant, it is necessary that, in the notation of the proof of Theorem 2, $x_1 = -k$ and $x_n = (n - 1)!k^2/\prod_{j=1}^{n-1}(k + j)$ for all $n \geq 2$. By Raabe's test, $x \in \ell^2$ for $k > 1/2$ and $x \notin \ell^2$ for $0 < k < 1/2$; it follows from a refinement (see [**7**, Theorem III, p. 396]) of Raabe's test that $x \notin \ell^2$ for $k = 1/2$. In summary, we see that C_k is not dominant (and hence also not hyponormal) when $0 < k \leq 1/2$. This result is a slight improvement on that in [**11**]; see also [**12**]. We note that the question of dominance for the case $1/2 < k < 1$ is unresolved, although it is known that those operators are not hyponormal.*

Example 6. *(q-Cesàro matrix for $q > 1$; see [**1**], [**15**].) Suppose M is the factorable matrix with nonzero entries $m_{ij} = a_ic_j$ where $a_i = (q - 1)/(q^{i+1} - 1)$ and $c_j = q^j$ for all i, j. Using the notation from the proof of Theorem 2, we compute $x_1 = -1$ and $x_n = q^n(q-1)/(q^n-1)$ for all $n \geq 2$. This results in $x_n \to q-1 > 0$ as $n \to +\infty$, so $x \notin \ell^2$. Therefore M cannot be dominant, so M is also not hyponormal.*

Example 7. *(q-Cesàro matrix for $0 < q < 1$; see [**2**], [**15**].) Suppose M is the factorable matrix with nonzero entries $m_{ij} = a_ic_j$ where $a_i = (1 - q)q^i/(1 - q^{i+1})$ and $c_j = 1/q^j$ for all i, j. Again using the notation from the proof of Theorem 2, we compute $x_1 = -1$ and $x_n = (1 - q)/[q(1 - q^n)]$ for all $n \geq 2$. This results in $x_n \to (1 - q)/q > 0$ as $n \to +\infty$, so $x \notin \ell^2$. It follows once again that M is not dominant, so M is also not hyponormal.*

In anticipation of the next corollary, we observe that the q-Cesàro matrices qualify as weighted mean matrices: for $0 < q < 1$, take $p_n = 1/q^n$ for all n; and for $q > 1$, take $p_n = q^n$ for all n.

Corollary 2. *If the weighted mean matrix associated with the sequence* $\{p_n\}$ *is to be dominant, it is necessary that* $\{p_n/\sum_{k=0}^{n-1} p_k\}_{n\geq 1} \in \ell^2$.

References

[1] G. Bennett, *An inequality for Hausdorff means*, Houston J. Math., **25**:4 (1999), 709–744. MR1829131

[2] J. Bustoz and L. F. Gordillo, *q-Hausdorff summability*, J. Comput. Anal. Appl, **7**:1 (2005), 35–48. MR2223776

[3] F. Hausdorff, *Summationsmethoden und momentfolgen. I.* Math Z., **9**:1-2 (1921), 74–109. MR1544453

[4] F. Hausdorff, *Summationsmethoden und momentfolgen. II.* Math Z., **9**:3-4 (1921), 280–299. MR1544467

[5] A. Jakimovski, *The product of summability methods: New classes of transformations and their properties*, Tech. Note (4), Contract No. AF **61**(052)-187 (1959).

[6] G. Leibowitz, *Rhaly matrices*, J. Math. Anal. Appl., **128**:1 (1987), 272–286. MR0915983

[7] J. M. H. Olmsted, *Advanced Calculus*, Appleton-Century-Crofts, New York (1961).

[8] H. C. Rhaly Jr., *Terraced matrices*, Bull. Lond. Math. Soc., **21**:4 (1989), 399–406. MR0998639

[9] H. C. Rhaly Jr., *Hyponormal terraced matrices*, Far East J. Math. Sci., **5**:3 (1997), 425–428. MR1619791

[10] H. C. Rhaly Jr., *Posinormal terraced matrices*, Bull. Korean Math. Soc., **46**:1 (2009), 117–123. MR2488507

[11] H. C. Rhaly Jr., *Remarks concerning some generalized Cesàro operators on* ℓ^2, J. Chungcheong Math. Soc., **23**:3 (2010), 425–433.

[12] H. C. Rhaly Jr., *A pathological example of a dominant terraced matrix*, Bull. Math. Anal. Appl., **3**:3 (2011), 9–12.

[13] B. E. Rhoades, *Generalized Hausdorff matrices bounded on* ℓ^p *and* c, Acta Sci. Math. (Szeged), **43**:3-4 (1981), 333–345. MR0640310

[14] B. E. Rhoades, *The point spectra of generalized Hausdorff matrices*, J. Adv. Math. Stud., **2**:1 (2009), 91–98. MR2542710

[15] T. Selmanogullari, E. Savaş and B. E. Rhoades, *On q-Hausdorff matrices*, Taiwanese J. Math., **15**:6 (2011), 2429–2437.

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