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ON THE SURFACES REPRESENTABLE AS DIFFERENCE OF CONVEX FUNCTIONS

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Abstract. This is a translation into English of the classical paper of A.D. Aleksandrov (1912–1999) about DC surfaces.

Keywords: DC surface, tangent cone, second differential, intrinsic metric.

1. Introduction

Under a surface representable as difference of convex functions, or a "DC surface" for short, we understand the surface that is defined in the Cartesian coordinates by the equation $z = f(x, y)$, where f is the difference of convex functions defined on some plane domain:

$$z = f(x, y) = h_1(x, y) - h_2(x, y), \tag{1}$$

with h_1 and h_2 such that the equations $z = h_1(x, y)$ and $z = h_2(x, y)$ define convex surfaces. We impose no additional regularity conditions on the surfaces under consideration, so that h_1 and h_2 are only subject to the convexity condition.

The convexity condition means that the straight line segment between two arbitrary points of $z = h_1(x, y)$ $\big($or $z = h_2(x, y)\big)$ lies above this surface (every point on it has the value of z at least that of the point on the surface with the same values of x and y).

We can, of course, regard as convex the surfaces lying below the chords. This is utterly immaterial since if $z = h_1(x, y)$ is convex in the first sense then $z = -h_1(x, y)$ is convex in the second sense, and conversely.

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We will understand convexity in the sense of the first definition.

In general, by a DC surface we can understand a surface whose every point has a neighborhood admitting representation in some coordinates as difference of convex functions. But we avoid using this generalized concept, restricting this exposition to the surfaces determined by equations of the form (1).

Assume that the coordinates axes x, y, and z are fixed once and for all. Therefore, the projection along the z-axis and similar operations are assumed defined without specifying the choice of coordinate systems.

The DC surfaces are of interest for the following reasons: Once as I managed to develop the intrinsic geometry of arbitrary convex surfaces (1), the question arose naturally whether it is possible to construct a general theory of the surfaces restricted neither by the convexity requirement nor by regularity requirements in the spirit of classical differential geometry. Together with that, the surfaces to be studied must be regular enough, so that we could count on obtaining for them at least as meaningful a system of concepts and statements as that available now for general convex surfaces. Consequently, we aim at indicating an appropriate sufficiently wide class of surfaces. Of course, in this form the problem is certainly indeterminate in a strict mathematical sense, but we do not mean to pose it as such, rather we only wish to explain the reasons for studying DC surfaces. Namely, it turns out that the class of these surfaces meets our general requirements to a sufficient degree. In the framework of this article we are unable to establish this in a proper volume, but restrict exposition only to some simple propositions that are the starting point for studying DC surfaces deeper. I already presented some of these results, like the general reasons just indicated, in the last section of my *Intrinsic Geometry of Convex Surfaces* [1].

2. On the Volume of a Class of DC Surfaces

In this section we verify that the class of DC surfaces includes all twice continuously differentiable surfaces, as well as all polyhedra. In addition, it trivially includes all convex surfaces. Consequently, it covers all surfaces which have so far been studied in detail in geometry.

Theorem 1. *Every function whose first partial derivatives satisfy the Cauchy–Lipschitz condition is the difference of convex functions.*

Namely, if $f(x, y)$ is such that

$$\begin{aligned} \bigl|f_x(x+\Delta x, y+\Delta y) - f_x(x, y)\bigr| &\le M\bigl(|\Delta x + \Delta y|\bigr), \\ \bigl|f_y(x+\Delta x, y+\Delta y) - f_y(x, y)\bigr| &\le M\bigl(|\Delta x + \Delta y|\bigr), \end{aligned} \tag{2}$$

then the function

$$g(x, y) = M(x^2 + y^2) - f(x, y) \tag{3}$$

is convex, so that $f(x, y)$ is the difference of the convex functions $M(x^2 + y^2)$ and $g(x, y)$.

Proof. Obviously, it suffices to show that on every line

$$x = a + \alpha s, \quad y = b + \beta s \tag{4}$$

the derivative $\frac{dg}{ds}$ is a monotone nondecreasing function of s. Therefore, $g(a+\alpha s, b+\beta s)$ turns out a convex function of s; and since the line (4) is arbitrary, this means precisely that the function $g(x, y)$ itself is convex.

Calculate the derivative

$$g' = 2M(\alpha x + \beta y) - (\alpha f_x + \beta f_y).$$

Calculate the difference $\Delta g'$ for $\Delta x = \alpha \Delta s$ and $\Delta y = \beta \Delta s$:

$$\Delta g' = 2M(\alpha^2 + \beta^2)\Delta s - (\alpha \Delta f_x + \beta \Delta f_y). \tag{5}$$

By (2) we have

$$|\Delta f_x| \leq M\big(|\Delta x| + |\Delta y|\big) = M\big(|\alpha| + |\beta|\big)|\Delta s|, \tag{6}$$

and similarly for Δf_y.

Therefore, (5) implies for $\Delta s > 0$ that

$$\frac{\Delta g'}{\Delta s} \geq 2M(\alpha^2 + \beta^2) - M\big(|\alpha| + |\beta|\big)^2 \geq 0. \tag{7}$$

and this means that g' is a nondecreasing function of s. The proof of the theorem is complete.

Theorem 1 obviously implies the next theorem.

Theorem 1a. *Every twice continuously differentiable surface, and in general every surface with the bounded curvature of normal sections, is a DC surface.*

Thus, the class of DC surfaces includes all surfaces that are studied in the classical differential geometry.

Theorem 2. *Every polyhedron is a DC surface. In other words, every piecewise linear function $f(x, y)$ is the difference of two convex andalsopiecewiselinear functions.*

Proof. Consider a polyhedral surface F with an equation $z = f(x, y)$. Assume that the domain of f (the projection of the polyhedron onto the plane (x, y)) is convex. This is not restriction of generality since we can, if need be, extend this polyhedron F so that the projection becomes convex. The dihedral angle at every edge opens either upwards or downwards: it includes an infinite part of the semiaxis $z > 0$ or the semiaxis $z < 0$.

A dihedral angle opening upwards is determined by a convex function (in the sense that it lies below the chord). Naturally, a function v_i of this type, determining the dihedral angle V_i of the surface F, is defined on the entire plane (x, y), and we assume that its domain is the same as that of the function $f(x, y)$ determining the polyhedron F.

Define now the function $g(x, y)$ as the sum of all functions $v_i(x, y)$ corresponding to all angles of F opening upwards:

$$g(x, y) = \sum_i v_i(x, y). \tag{8}$$

This function is convex and piecewise linear as the sum of convex piecewise linear functions v_i. In other words, the equation $z = g(x, y)$ determines a convex polyhedron; denote the latter by G. This G is in a sense the sum of dihedral angles V_i. Each of its edges lies below the corresponding edge of the polyhedron F: the projection of every edge of G includes the projection of an edge of F with an angle V_i opening upwards. Conversely, each edge of F with an angle opening upwards lies below the corresponding edge of G.

Verify that the difference $h(x, y) = g(x, y) - f(x, y)$ is a convex function as well.

Since g and f are piecewise linear functions, so is $h = g - f$; thus, the equation $z = h(x, y)$ represents a polyhedron which we denote by H.

The projection of each edge of H onto the plane (x, y) coincides necessarily with the projection of an edge of one of the polyhedra F and G since everywhere on F and G away from the edges both functions f and g are linear, and so is their difference. Verify that the dihedral angles at all edges of H open upwards. To this end, consider the three possible cases of the location of the projection of an edge p of H onto the plane (x, y):

(1) this projection coincides with the projection of an edge of F, but not G;

(2) it coincides with the projection of an edge of G, but not F;

(3) it coincides with the projection of some edges of both F and G.

In the first case the dihedral angle at the corresponding edge q of F opens downwards. Indeed, every edge with the angle opening upwards corresponds to an edge of G, but in this case that is assumed excluded.

Thus, in a neighborhood of the projection of an edge p of H the function g determining the polyhedron G is linear, which corresponds to a facet of G; but the function f determining the polyhedron F is “concave” since the angle at q opens downwards. In result, the difference $h = g - f$ is convex, which means that the angle at p opens upwards.

In the second case an edge of G, but not of F, corresponds to p. Therefore, the function f is linear in a neighborhood of the projection of p, while g is everywhere convex. Consequently, in this case $h = g - f$ is convex and the angle at p opens upwards.

In the third case some edges q and r of F and G correspond to p. Then by the main property of the polyhedron G the angle V_k at the edge q of F opens upwards, and by (8) the function $v_k(x, y)$ in the equation $z = v(x, y)$ representing this angle appears as a summand in $g(x, y)$. Obviously, $v_k(x, y)$ near q coincides with $f(x, y)$.

Therefore, in a neighborhood of p we have

$$h(x, y) = g(x, y) - f(x, y) = \sum_i v_i(x, y) - v_k(x, y).$$

The right-hand side amounts to the sum of the convex functions v_i, with the exception of v_k; thus, it is a convex function. Consequently, $h(x, y)$ in a neighborhood of the projection of p is convex, and the angle at the edge opens upwards.

Thus, we have proved that all dihedral angles of the polyhedron H open upwards, so that in a neighborhood of each edge it is convex downwards. It is known that a locally convex polyhedron of this type is globally convex.[1] Therefore, the proof of the theorem is complete.

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[1]This claim holds since we assume that the projection of F, and so of H as well, onto the plane (x, y) is convex. Obviously we can assume that it represents a polygon. To prove the convexity of H, take the plane P parallel to the plane (x, y) so that H lies entirely below it (i.e., in the domain of smaller z). Connecting all points of H to this plane by perpendicular line segments, we obtain a solid polyhedron K bounded below by H, above by its projection onto P, and on the sides by a prismatic surface. Since all angles of H are convex downwards, all dihedral angles of K are less than π.

If all dihedral angles of a polyhedral angle are less than π, then it is convex. Therefore, all vertex figures of K are convex. Hence, it is now easy to conclude that K itself is convex, and so is H.

3. Derivatives of Differences of Convex Functions and the Corresponding Properties of DC Surfaces

Since the differential of the difference of two functions is the difference of their differentials, many properties of the differentials of convex functions carry over easily to differences of convex functions. In this section we indicate the main properties of the derivatives of differences of convex functions, obtained in this way from the properties of the derivatives of convex functions. These properties translate directly into equivalent geometric properties of DC surfaces.

The contingency of M at a point A is the figure formed by the limits of all possible sequences of rays going from A into a variable point X of M as $X \to A$. If the contingency is a surface, then it is obviously a cone with apex A, and then we speak of the tangent cone. Dihedral angles and planes also belong here; if the tangent cone reduces to a plane then it is nothing but the tangent plane.

Lemma. *In order for the surface F with an equation $z = f(x, y)$ to have tangent cone at $A(x_0, y_0)$ also representable by an equation of the form $z = g(x, y)$, it is necessary and sufficient that $f(x, y)$ at (x_0, y_0) have the derivative in every direction depending continuously on the direction and, furthermore, the convergence of the ratios of differences to the derivatives be uniform in all directions. The derivative in a given direction is the angular coefficient of the corresponding generator of the tangent cone.*

This lemma is completely obvious from the relation between the derivative and tangent line. The only part of it possibly not so obvious is that the existence of the tangent cone implies uniform convergence of the ratios of differences to the derivatives, or, which is the same, of the angular coefficients of the secant lines to those of the tangent lines generating the cone. Let us verify this claim of the lemma.

Suppose that at A the surface F has tangent cone K. Suppose that the convergence of the angular coefficients of secant lines to those of the generators of K is not uniform in all directions. Then there is a sequence of secant lines AX_n such that $X_n \to A$, but the angles between AX_n and the generators L_n of K with the same directions of the projections onto the plane (x, y) remain greater than some $\varepsilon > 0$. Choosing a converging sequence of directions, we may assume that the generators L_n converge. Then their limit is a generator L of K. But the limit of the secant lines AX_n is distinct from L, and by definition it is also a generator of K. This is impossible since by assumption to every direction in the plane (x, y) there corresponds only one generator of the tangent cone (obviously, this means precisely that this cone is represented by an equation of the form $z = g(x, y)$). Thus, we have established our claim.

Theorem 3. *Every DC surface has tangent cone at each point which is a DC surface itself.*

Proof. Each convex surface has tangent cone at every point. Using the lemma, we translate this into the language of derivatives of convex functions. For derivatives it is clear that the same result is valid for the differences of convex functions. Again by our lemma, this means that every DC surface has tangent cones. Suppose that $z = g(x, y)$ is an equation of a tangent cone of the surface $z = h_1(x, y) - h_2(x, y)$, where h_1 and h_2 are convex functions. The function g is the difference of functions determining respectively the tangent cones to the convex surfaces $z = h_1(x, y)$ and $z = h_2(x, y)$.

Consequently, each tangent cone to a DC surface is itself a DC surface.

Let us point out a series of properties of tangent cones to a DC surface, which follow directly from the corresponding properties of tangent cones to convex surfaces.

Theorem 4. *The tangent cones to every DC surface enjoy the following properties:*

(A) *The set of conical points, i.e., those where the tangent cone fails to reduce to a dihedral angle or to a plane, is at most countable. The set of points where the tangent cone reduces to a dihedral angle in the projection onto the plane (x, y) has measure zero; moreover, it lies on an at most countable collection of rectifiable curves. Consequently, every DC surface has tangent planes almost everywhere.*

(B) *If a DC surface has tangent plane P at a point A then the tangent cones at $X \to A$ converge to this plane. The angle formed by the secant line XY passing through arbitrary points X and Y of a surface with tangent plane P tends to zero as $X, Y \to A$; we express this by saying that P is the tangent plane in a strong sense.*

(C) *In every closed domain inside a DC surface the slopes of the tangent lines are bounded. In other words, derivatives in all directions are bounded.*

Verify claim (A). Suppose that $z = g(x, y) - h(x, y)$ is an equation for some surface F and, furthermore, the equations $z = g(x, y)$ and $z = h(x, y)$ represent convex surfaces G and H.

Take a conical point A on F. Then, since the tangent cone K_F at it is the "difference" of the tangent cones K_G and K_H at the corresponding points of G and H, it follows that there are open only the two possibilities for K_G and K_H:

(1) at least one of the cones K_G and K_H fails to reduce to a plane or a dihedral angle;

(2) K_G and K_H are dihedral angles with nonparallel edges.

In other cases K_F is inevitably a plane or a dihedral angle.

On a convex surface the set of conical points is at most countable, and so the set of points A for which the first possibility is realized is at most countable.

Suppose that the second possibility is realized. Consider the surface S with equation $z = g(x, y) + h(x, y)$. This S is convex. Its tangent cone K_S at the point corresponding to A is the "sum" of the dihedral angles K_S and K_H. Since their edges are not parallel and both angles open upwards, their "sum" is a tetrahedral angle. Consequently, in the second case the point on S corresponding to A is conical. Since S is convex, it follows that the set of these points is at most countable.

Thus, both possibilities can be realized at at most countably many points. Therefore, the set of conical points of a DC surface is at most countable.

The second part of claim (A) follows obviously from the similar properties of convex surfaces. Besides, we should observe that by Theorem 3 claim (A) turns out a corollary to a general theorem on tangent sets (contingencies) by Šmidov and Verčenko [2].

Claims (B) and (C) are also obvious corollaries to the corresponding properties of convex surfaces. For the latter they are established quite simply.

(The first part of claim (B) for convex surfaces is easily deduced from the fact that the limit of supporting planes is a supporting plane. The second part also follows since if we take the supporting planes at X and Y and verify that when the angle between it and P tends to zero, so does the angle between the chord XY and P.)

Theorem 5. *The difference of convex functions has second differential almost everywhere: the right (or left) partial derivatives f_x and f_y, existing everywhere by Theorem 3, are almost everywhere differentiable; furthermore, wherever this holds, $\frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x}$ and, in addition, the convergence of the ratios of the differences $\frac{\Delta f_x}{\Delta s}$ and $\frac{\Delta f_y}{\Delta s}$ to the derivatives $\frac{df_x}{ds}$ and $\frac{df_y}{ds}$ in every direction is uniform in all directions $\Big($i.e., $\left|\frac{\Delta f_x}{\Delta s} - \frac{df_x}{ds}\right| < \varepsilon$, where ε depends only on the displacement $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}\Big)$.*

This follows from the similar properties of convex functions $\big($see (3) and (4)$\big)$.

For DC surfaces Theorem 5 means that at almost all points they enjoy all local properties of twice differentiable surfaces (the Dupin indicatrix, Meusnier's, Euler's, Rodriguez's theorems, and so on.)

Theorem 6. *If $a(x, y)$ is the difference of convex functions, then $\frac{df(a+\alpha s, b+\beta s)}{ds}$ is always a function of s of bounded variation.*[2] *This means that the section of the surface $z = f(x, y)$ by the plane parallel to the axis z represents a curve with bounded swerve. By the latter we understand the least upper bound of the sums of angles between the (right and left) tangents at the successive points of curve.*

(For a regular curve this is the integral of the curvature, and for a broken line, the sum of complements of its angles to π. All quantities are taken without sign, only in absolute value).

In a plane section of the surface $z = f(x, y)$ we obtain a curve that is the difference of convex curves. Every convex curve or, more exactly, every convex function of one variable, has monotone derivative. Consequently, for the difference of convex functions the derivative has bounded variation.

The first claim of the theorem is verified; the second just translates it into different terms.

The convex function of one variable is characterized by the fact that the derivative is monotone. The difference of monotone functions is nothing but a function of bounded variation.

Hence, we immediately deduce

Theorem 7. *In order for a function of one variable to be the difference of convex functions, it is necessary and sufficient that the function be the integral of a function of bounded variation.*

We are unaware of any similar characterization of the differences of convex functions of two or more variables.

Also, we are unaware of any geometric definition of DC surfaces. In order for a plane curve to be the difference of convex components, it is necessary and sufficient that its swerve (or, if so wished, the variation of the swerve) be finite. We think that something similar should hold for surfaces. Is the property indicated in Theorem 7 characteristic of DC surfaces and, accordingly, differences of convex functions?

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[2]In any case, this holds on a closed interval inside the domain of the function, since as the point approaches the boundary, the derivative can become infinite.

Furthermore, if a function $y = f(x)$ is the difference of convex components, then finding its convex components and, moreover, the minimal ones, reduces to finding the derivative and its variation. Indeed, $\frac{1}{2}\bigl[\mathrm{var} f'(x) + f'(x)\bigr]$ and $\frac{1}{2}\bigl[\mathrm{var} f'(x) - f'(x)\bigr]$ are monotone components of the derivative $f'(x)$, and the integrals of them yield those minimal convex functions whose difference is $f(x)$.

By the minimal convex components of a function f (of arbitrarily many variables) we understand convex functions g and h such that

(1) $f = g - h$,

(2) g, $h \leq 0$,

(3) for every convex g and h satisfying the first two conditions, $g \geq g_1$ and $h \geq h_1$.

Here it is convenient to pass to the other understanding of a convex surface (function): as one lying below its every chord. The functions $g' = -g$ and $h' = -h$ are convex in this sense and the definition reduces to the conditions:

(1) $f = h' - g'$,

(2) g', $h' \geq 0$,

(3) if g'_1 and h'_1 are convex and satisfy (1) and (2) then $g' \leq g'_1$ and $h' \leq h'_1$.

In this case the intuitive meaning of the definitions becomes clearer: the surface $z = g'(x, y)$ $\bigl($and respectively $z = h'(x, y)\bigr)$ lies below the plane (x, y) and, together with that, below every surface $z = g'_1(x, y)$.

The existence of minimal components for every function that is the difference of convex functions follows from the next fact.

If two functions h'_ξ are convex in the second sense and nonnegative then the function h' equal for all x and y to the infimum of the values of h'_ξ is also convex. This is just an analytic expression of the fact that the intersection of convex sets is convex.

Thus, the minimal convex components always exist provided that the function is the difference of convex functions. For a function f of one variable these components are the integrals of the monotone components of its derivative f' $\bigl($the integrals of $\frac{1}{2}[\mathrm{var} f' + f']$ and $\frac{1}{2}[\mathrm{var} f' - f']\bigr)$.

We can give a different method for constructing these "convex components" of a function $f(x)$. We construct the maximal convex function $g_1(x)$ such that $f(x) \geq g_1(x)$. This function is drawn as the curve subtending the curve $y = f(x)$ as the boundary of the convex hull. Furthermore, we construct the function $g_2(x)$ from $g_1(x) - f(x)$ in the same way as $g_1(x)$ is constructed from $f(x)$. Then we construct $g_3(x)$ in the same way from $g_2(x) - g_1(x) + f(x)$, and so on. The sums of $g_i(x)$ with odd and even indices yield convex components of the function $f(x)$:[3]

$$f(x) = \sum_{i=1}^{\infty} g_{2i-1}(x) - \sum_{i=1}^{\infty} g_{2i}(x).$$

The convergence of the series $\sum g_i(x)$ at least at one point is a necessary and sufficient condition for $f(x)$ to be the difference of convex functions.

We have no method for finding "convex components" of a function of two variables, not only minimal, but arbitrary components (if it is known that the function is the difference of some convex functions).

Thus, two problems remain open:

1. Give a characterization of the differences of convex functions and DC surfaces.
2. Give a method for finding convex components of a function which would always converge when the function is the difference of convex functions and diverge otherwise.

4. Convergence and Approximation of DC Surfaces

For differences of convex functions it is natural to introduce the concept of "strong convergence", understanding by this that f_n strongly converges to f if there exist convex functions g_n, h_n, g, and h such that $f_n = g_n - h_n$, $f = g - h$, and $g_n \to g$, $h_n \to h$. This convergence automatically turns out uniform, since convex functions always converge uniformly.[4]

We consider no convergence other than the strong convergence as defined. Accordingly, we can speak of the strong convergence of DC surfaces.

Since convex surfaces admit approximation by analytic and polyhedral convex surfaces, it follows that we have

Theorem 8. *Given a DC surface, there exists a sequence of analytic (polyhedral) DC surfaces converging strongly to it.*

This opens up the possibility of studying a DC surface by passing to the limit from analytic or polyhedral surfaces.

Theorem 9. *Consider some surfaces F_n strongly converging to a surface F, and take some points X_n on F_n converging to a point X on F. If F has tangent plane P at X then the tangent cones of F_n at X_n converge to P. But if the tangent cone at X is a dihedral angle V then the limit of every converging sequence of tangent cones at X_n is a dihedral angle with the same edge as V (but it need not coincide with V and can be different for different sequences; it is not excluded, and we should*

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[3]This is based on the following remark. If a function $h(x)$ is convex and so is $h(x) - f(x)$ then $h(x) - g_1(x)$ is convex as well. Indeed, wherever $g_1(x) \neq f(x)$, i.e., the curve g_1 goes below f, the curve g_1 reduces to a segment, i.e., there $g_1(x)$ is a linear function, and therefore, $h(x) - g_1(x)$ is certainly convex. Also, wherever $g_1(x) = f(x)$, we have $h(x) - g_1(x) = h(x) - f(x)$, and therefore the latter is convex as well. It is easy to verify that at the points of transition from $g_1 = f$ to $g_1 < f$ the convexity is preserved.

Suppose now that $f(x) = h(x) - k(x)$, where h and k are convex functions and, moreover, the minimal ones. Then $h(x) - g_1(x)$ is convex by the remark just established.

Furthermore, we verify that $h(x) - g_1(x) + f(x)$ is convex and so $h - g_1 + g_2$ is convex as well, and so on.

[4]Convergence of convex functions is equivalent to convergence of the convex surfaces they represent, or, which is the same, of the corresponding convex bodies. Convergence of bounded closed sets consists in their deviation tending to zero, and this precisely means uniform convergence.

take this more as a rule, that it is a plane). In addition, in every closed domain inside the projection of the surfaces F and F_n onto the plane (x, y) the slopes of the tangent lines are uniformly bounded for all surfaces F_n.

This theorem follows from the fact that converging convex surfaces enjoy these properties; for convex surfaces we can deduce these properties easily since the limit of supporting planes is a supporting plane. In particular, at the points where the tangent cone is a dihedral angle, all supporting planes pass through its edge, and consequently, all limits of supporting planes, and together with them the limits of tangent cones, must pass through the same edge.

In the subsequent theorems we consider not the whole surface F, but only a domain on it whose boundary avoids the boundary of F itself. This domain can, of course, cover almost the whole surface with the exception of an arbitrarily narrow strip along the boundary (considering this domain is equivalent to considering a surface which admits an extension in all directions as a DC surface).

Restricting expositions to the above-indicated type of domains, by Theorem 4 (C) and the last part of Theorem 9 we guarantee bounded derivatives and avoid the complications arising as we approach the boundary of the surface since on the boundary it can completely lose its regularity.

Theorem 10. *Consider some surfaces F_n strongly converging to a surface F. Consider a rectifiable curve L on F (not approaching the boundary of F), and some curves L_n on F_n sharing with L the projections onto the plane (x, y). Then L_n are rectifiable curves and their lengths converge to the length of L.*

Proof. Take the equations

$$z = f(x, y) = g(x, y) - h(x, y), \quad z = f_n(x, y) = g_n(x, y) - h_n(x, y).$$

for F and F_n, with convex functions $g, \dots, h_n$ such that $g_n \to g$ and $h_n \to h$. Denote by λ the common projection of L and L_n. By hypotheses, λ stays away from the boundary of the common projection of the surfaces F and F_n. Therefore, at the points lying under λ the slopes of the supporting planes of the convex surfaces $z = g_n(x, y)$ and $z = h_n(x, y)$ are bounded (otherwise, in the limit we would obtain a vertical supporting plane inside the limit surfaces G and H, which is impossible). Together with the slopes of the supporting planes, the slopes of chords are bounded, i.e., the ratios of the form $\frac{\Delta g_n}{\sqrt{\Delta x^2 + y^2}}$, and thus so are the ratios $\frac{\Delta f_n}{\sqrt{\Delta x^2 + y^2}}$.

The curve λ, as the projection of a rectifiable curve L, is rectifiable. Denote by ζ the arc length of λ. If the equation of λ is $x = x(\zeta)$, $y = y(\zeta)$, then take $z = f_n\big(x(\zeta), y(\zeta)\big)$ as an equation for L_n. By what we have already proved, the ratios $\frac{\Delta f_n}{\Delta \zeta}$ are bounded on λ. And then, as it is known, (5) the curve L_n is rectifiable and its length is expressed as

$$s(L_n) = \int \sqrt{1 + \left(\frac{df_n}{d\zeta}\right)^2}\, d\zeta. \tag{9}$$

By analogy for L we have

$$s(L) = \int \sqrt{1 + \left(\frac{df}{d\zeta}\right)^2}\, d\zeta. \tag{10}$$

The derivatives $\frac{df_n}{d\zeta}$ and $\frac{df}{d\zeta}$ exist almost everywhere (in the sense of the measure by the arc length ζ on λ), and are clearly bounded since so are the ratios $\frac{\Delta f_n}{\Delta\zeta}$. In addition, λ, as every rectifiable curve, almost everywhere has tangent line. Consequently, L_n and L have tangent lines almost everywhere, and their slopes are bounded. Since the union of countably many sets of measure zero has measure zero, all curves L_n and L simultaneously have tangents almost everywhere.

According to Theorem 4, the set of conical points of F_n and F is at most countable. Therefore, we can exclude the consideration of all conical points lying on L_n and L either. Moreover, under this condition at every point where $L(L_n)$ has tangent line $F(F_n)$ either has tangent plane or its tangent cone reduces to a dihedral angle and, furthermore, the tangent curve goes along the edge of this angle.

Take the points X and X_n on L and L_n lying over the same point ξ of λ. The points X_n converge to X. Denote by P the vertical plane (parallel to the z-axis) passing through the tangent line to λ at ξ. The tangent lines to L and L_n are the intersections of P with the tangent cones of F and F_n. As we pointed out, each of these tangent cones can be either a plane or a dihedral angle. In the latter case its edge lies in P.

By Theorem 9, if the tangent plane exists at X then the tangent cones at X_n converge to this plane. Therefore, the tangent lines to L_n at X_n converge to the tangent line to L at X.

But if the tangent cone at X is a dihedral angle then its edge q lies in every limit of tangent cones at X_n. Together with that, it lies in P. Thus, the intersection of the tangent cones at X_n with P converges to q. These intersections are precisely the tangent lines to L_n.

Consequently, in this case too the tangent lines to L_n at X_n converge to the tangent line to L at X.

Thus, we have proved that the tangent lines to L_n converge to the tangent lines to L everywhere but a set of measure zero in the sense of the arc length ζ on λ. This means that the derivatives $\frac{df_n}{d\zeta}$ converge to $\frac{df}{d\zeta}$ almost everywhere. In addition, we saw that these derivatives are bounded. Thus, resting on a theorem on the convergence of integrals, we can conclude that the integrals in (9) converge to the integral in (10), and so we have the convergence of lengths: $s(L_n)$ to $s(L)$.

Therefore, the proof of the theorem is complete.

From this theorem we deduce a theorem on the convergence of intrinsic metrics of strongly converging DC surfaces. The intrinsic metric of a surface F is determined as follows. On assuming that every pair of points on F can be connected by a rectifiable curve on F, take as the distance $\rho_F(XY)$ between two points X and Y on F the greatest lower bound for the lengths of curves lying on F and connecting X and Y. By the intrinsic metric we understand the function ρ_F of pairs of points defined in this way whose values yield intrinsic distances between points on the surface.

Theorem 11. *Every DC surface possesses intrinsic metric.*

Take two points X and Y on F. Connect their projections onto the plane (x, y) by a broken line. To each segment of the broken line there corresponds an arc of a plane section of F. It is rectifiable since by Theorem 7 the swerve of its tangent line is bounded. In result, the points X and Y are connected on F by a rectifiable curve,[5] and then $\rho_F(XY)$ exists as the greatest lower bound for the lengths of these curves.

Theorem 12. *The intrinsic metrics of strongly converging surfaces uniformly converge to the metric of the limit surface. In more detail: Consider some surfaces F_n strongly converging to a F; moreover, according to the condition indicated above, the surfaces F_n and F are domains inside some DC surfaces. Take some points X and Y on F and some points X_n and Y_n on F_n with the same projections ξ and η onto the plane (x, y). Then we may regard the distances $\rho_{F_n}(X_nY_n)$ and $\rho_F(XY)$ as functions of the pair of points ξ and η. These functions converge uniformly.*

Proof. Given two points X and Y on F, take X_n and Y_n on F_n with the same projections ξ and η onto the plane (x, y). By the definition of $\rho_F(XY)$, for every $\varepsilon > 0$ there exists a curve L on F connecting X and Y of length $s(L)$ satisfying

$$s(L) < \rho_F(XY) + \varepsilon. \tag{11}$$

The curves L_n on surfaces F_n with the same projections onto the plane (x, y) are rectifiable by Theorem 10, and

$$s(L) = \lim_{n\to\infty} s(L_n). \tag{12}$$

Together with that, these curves join X_n and Y_n, and the definition of the distances $\rho_{F_n}(X_nY_n)$ ensures that

$$s(L_n) \geq \rho_{F_n}(X_nY_n). \tag{13}$$

Comparing (11)–(13), we obtain

$$\rho_{F_n}(X_nY_n) < \rho_F(XY) + \varepsilon,$$

and since ε is arbitrary, it follows that

$$\limsup_{n\to\infty} \rho_{F_n}(X_nY_n) \leq \rho_F(XY). \tag{14}$$

On the other hand, by the definition of the distances $\rho_{F_n}(X_nY_n)$ on each surface F_n we can indicate a curve L'_n connecting X_n and Y_n and satisfying

$$s(L'_n) < \rho_{F_n}(X_nY_n) + \varepsilon. \tag{15}$$

According to (14), all $\rho_{F'_n}(X_nY_n)$, and so $s(L'_n)$ as well, are uniformly bounded. Therefore, out of the curves L'_n we can choose a converging sequence, and the length of its limit curve L' will satisfy

$$s(L') \leq \liminf_{n\to\infty} s(L'_n). \tag{16}$$

The curve L' joins X and Y, and so[6]

$$s(L') \geq \rho_F(XY). \tag{17}$$

Comparing (15)–(17), and keeping in mind that we can take an arbitrarily small ε, we obtain

$$\rho_F(XY) \leq \liminf_{n\to\infty} \rho_{F_n}(X_nY_n). \tag{18}$$

[5]In Theorem 10 we essentially proved the following much stronger claim: a curve on a DC surface possessing a rectifiable projection is rectifiable itself.

[6]Here we use the assumption that F itself is a closed domain inside some DC surface. Otherwise, the curve L', which is the limit of L'_n, while getting to the boundary of the surface might be leaving F.

Together with (14), this implies that $\lim \rho_{F_n}$ exists, and

$$\rho_F(XY) = \lim_{n\to\infty} \rho_{F_n}(X_nY_n). \tag{19}$$

Thus, the convergence of metrics is established, and it remains to show that this convergence is uniform. But we verify the equicontinuity of metrics, and the equicontinuity of converging functions implies uniform convergence. We have the following statement.

If for a surface F the slopes of the tangent semiaxes (derivatives in every direction) are at most some M then the distance ρ_F between the points $X(x,y)$ and $X'(x+\Delta x,\, y+\Delta y)$ is at most

$$\sqrt{1+M^2}\cdot\sqrt{\Delta x^2+y^2}.$$

Indeed, suppose that L is a plane section of F connecting the points X and X'. Then $\rho_F(XX') \le s(L)$, while (10) yields

$$s(L) = \int \sqrt{1+\left(\frac{df}{d\zeta}\right)^2}\, d\zeta \le \sqrt{1+M^2}\cdot\sqrt{\Delta x^2+\Delta y^2},$$

so that

$$\rho_F(XX') \le \sqrt{1+M^2}\cdot\sqrt{\Delta x^2+\Delta y^2}. \tag{20}$$

Together with that, for all points X, Y, X', and Y' on F we have

$$\left|\rho_F(XY) - \rho_F(X'Y')\right| \le \rho_F(XX') + \rho_F(YY'). \tag{21}$$

Together with (20), this implies that the metric ρ_F satisfies the Cauchy–Lipschitz condition with a constant depending only on the bound M for the slope of tangent lines.

According to Theorem 9, for all surfaces F_n strongly converging to F the slopes of tangent lines are uniformly bounded. Thus, the metrics of these surfaces are equicontinuous, as required.

REMARK. From the convergence and equicontinuity of metrics it is easy to deduce the following supplement to Theorem 12. For every $\varepsilon > 0$ there exist N and $\delta > 0$ such that, as soon as $n > N$ and the distances from the points X_n and Y_n on F_n to the points X and Y on F are less than δ, it follows that $\left|\rho_{F_n}(X_nY_n) - \rho_F(XY)\right| < \varepsilon$.

Theorem 13. *If some surfaces F_n converge strongly then the absolute integral curvatures (i.e., the integrals of the absolute value of the Gaussian curvature) are jointly bounded for all domains with the same projection G not approaching the boundary of the projection of F_n.*

If $z - g_n(x,y) - h_n(x,y)$ are equations for F_n with convex functions g_n and h_n then, as we have already seen in the proof of Theorem 9, the derivatives $\frac{\partial g_n}{\partial x}, \dots, \frac{\partial h_n}{\partial y}$ are bounded in every domain G not approaching the boundary of the projection of F_n.

By this remark, Theorem 13 turns out to be a corollary to

Theorem 13a. *Suppose that $z = f(x,y) = g(x,y) - h(x,y)$ is an equation of a DC surface, where the functions g and h are convex and twice differentiable. Then the absolute curvature Ω of the surface, i.e., the integral of the absolute value of its*

Gaussian curvature, is bounded and the bound depends only on the bound M for the derivatives of g and h. [Namely, $\Omega < 2\pi(1+8M^2)^{3/2}$].

PROOF. The Gaussian curvature is expressed as

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^{3/2}}. \tag{22}$$

Since $f = g = h$, it follows that

$$f_{xx}f_{yy} - f_{xy}^2 = (g_{xx}g_{yy} - g_{xy}^2) + (h_{xx}h_{yy} - h_{xy}^2) - (g_{xx}h_{yy} + g_{yy}h_{xx} - 2g_{xy}h_{xy}),$$

or

$$|f_{xx}f_{yy} - f_{xy}^2| \le |g_{xx}g_{yy} - g_{xy}^2| + |h_{xx}h_{yy} - h_{xy}^2| + |g_{xx}h_{yy} + g_{yy}h_{xx} - 2g_{xy}h_{xy}|. \tag{23}$$

But since g and h are convex functions, their second differentials are nonnegative quadratic forms, and this is known to imply that all three quantities whose absolute values appear here on the right are themselves nonnegative.[7] Therefore, by simple transformations we can replace (23) with

$$|f_{xx}f_{yy} - f_{xy}^2| \le (g_{xx}+h_{xx})(g_{yy}+h_{yy}) - (g_{xy}+h_{xy})^2,$$

or putting $g + h = k$, with

$$|f_{xx}f_{yy} - f_{xy}^2| \le k_{xx}k_{yy} - k_{xy}^2. \tag{24}$$

The integral of the absolute value of the Gaussian curvature is

$$\Omega = \iint |K|\,ds = \iint \frac{|f_{xx}f_{yy} - f_{xy}^2|}{(1+f_x^2+f_y^2)^{3/2}}\,dx\,dy \le \iint (k_{xx}k_{yy} - k_{xy}^2)\,dx\,dy. \tag{25}$$

Since $k = g + h$ and $|g_x|, \dots, |h_y| < M$, it follows that

$$|k_x|, |k_y| < 2M, \quad (1 + k_x^2 + k_y^2) < 1 + 8M^2.$$

Therefore,

$$\iint (k_{xx}k_{yy} - k_{xy}^2)\,dx\,dy < (1+8M^2)^{3/2} \iint \frac{k_{xx}k_{yy} - k_{xy}^2}{(1+k_x^2+k_y^2)}\,dx\,dy. \tag{26}$$

The integral on the right-hand side is nothing but the integral curvature of the surface with the equation $z = k(x, y)$, which is convex since k is the sum of the convex functions g and h.

The spherical image of this surface is included into on a hemisphere since it is uniquely projected onto the plane (x, y). Consequently, its integral curvature is at most 2π. Thus, estimating the integral on the right-hand side of (26), and combining (26) with (25), we obtain

$$\Omega \le 2\pi(1+8M^2)^{3/2}.$$

This completes the proof of Theorem 13a and, therefore, Theorem 13 as well.

Theorems 12 and 13 provide a foundation for studying some deeper properties of DC surfaces.

Namely, resting on Theorems 12 and 13, we can apply to DC surfaces the general theory of intrinsic geometry of surfaces or "manifolds of bounded curvature" whose foundations appeared in my article "Foundations of intrinsic geometry of

[7]For the first terms this is obvious. But if by a linear transformation of the variables x and y we obtain $h_{xy} = 0$ at this point, then the third quantity reduces to $g_{xx}h_{yy} + g_{yy}h_{xx}$, which is nonnegative since $g_{xx}, \dots, h_{yy} \ge 0$.

surfaces" [6] and even earlier were sketched in the last section of my book [1]. Thus, Theorems 12 and 13 open up a path to thoroughly studying the intrinsic geometry of DC surfaces. Here belong, apart from the main results of [6], some results of the notes [7, 8], as well as many others so far unpublished. In addition, we obtain the possibility of studying the relation of the intrinsic geometry of a DC surface to its "extrinsic" geometry, i.e., the properties of itself and figures on it as figures in space.

Theorem 14, proved in the next section, is a simple example from this area.

5. On the Tangent Cone

We defined the tangent cone of a surface F at A as the cone formed by the limits of secant semiaxes emanating from A. It is known that this definition is equivalent to the following: the tangent cone of a surface F at A is the limit of surfaces obtained from F as dilations centered at A as the coefficient of dilation increases indefinitely.[8]

This view of the tangent cone turns out more advantageous in a series of questions. Using it and resting on Theorem 12 on the convergence of metrics, we verify that a DC surface is "infinitesimally isometric to its tangent cone".

By this we mean, more exactly, the following result:

Theorem 14. *Consider a DC surface F and its tangent cone K at some point A. Project F onto K in the direction of the z-axis. Given $\varepsilon > 0$ there is $\delta > 0$ such that, as soon as the distance from two points X and Y on F from A is less than δ, it follows that*

$$\left|\rho_F(XY) - \rho_K(X'Y')\right| < \varepsilon \max\left[\rho_F(AX), \rho_F(AY)\right], \tag{27}$$

where ρ_f and ρ_K are the distances measured on F and K, while X' and Y' are the projections of X and Y onto the cone K.

Speaking intuitively, (27) means that the projection under consideration in an infinitely small neighborhood of A is an isometric mapping; according to (27), the difference of the distances on F and K is infinitely small relative to the distances to A.

Proof[9]. Subject the surface F to a dilation centered at A. Denote the coefficient of dilation by λ. Denote the resulting surfaces by λF, and the points obtained from some point X, by λX.

Take the circle of radius 1 centered at the projection of A onto the plane (x, y) and consider only the parts of surfaces lying under this circle. Project the surfaces λF onto K along the z-axis.

As $\lambda \to \infty$, the surfaces λF converge to the tangent cone K at A. Therefore, by Theorem 12 for every $\varepsilon > 0$ there is λ_0 such that, as soon as $\lambda > \lambda_0$, for all points B and C on λF and the corresponding projections of the points B' and C' on K we have

$$\left|\rho_{\lambda F}(BC) - \rho_K(B'C')\right| < \varepsilon. \tag{28}$$

Now take the points X and Y on F going into B and C:

$$B = \lambda X, \quad C = \lambda Y. \tag{29}$$

[8] The equivalence of both definitions can be established in an obvious fashion.

[9] This proof verbatim repeats the proof of the same theorem for convex surfaces in [1].

Under the dilation centered at A the cone K goes into itself, while the projections of X and Y, into the projections of A and B:

$$B' = \lambda X', \quad C' = \lambda Y'. \tag{30}$$

The dilation under consideration increases all distances by a factor of λ, and so the distances on λF and K satisfy

$$\rho_{\lambda F}(BC) = \lambda_{\rho_F}(XY), \quad \rho_K(B'C') = \lambda_{\rho_K}(X'Y'). \tag{31}$$

Inserting this into (28), we obtain

$$\bigl|\rho_F(XY) - \rho_K(X'Y')\bigr| < \frac{\varepsilon}{\lambda}. \tag{32}$$

Take X and Y so close to A that

$$\max\bigl[\rho_F(AX), \rho_F(AY)\bigr] < \frac{1}{\lambda_0} \tag{33}$$

and put

$$\max\bigl[\rho_F(AX), \rho_F(AY)\bigr] = \frac{1}{\lambda}, \tag{34}$$

so that $\lambda > \lambda_0$.

Since dilation with coefficient λ increases distances by a factor of λ, for B and C this means that

$$\max\bigl[\rho_{\lambda F}(AB), \rho_{\lambda F}(AC)\bigr] = 1.$$

Therefore, B and C remain in the domain under consideration (lie inside the circle of unit radius), and so our conclusion applies to them: for X and Y we have (32) with λ defined in (34). Thus,

$$\bigl|\rho_F(XY) - \rho_K(X'Y')\bigr| < \varepsilon \max\bigl(\rho_F(AX), \rho_F(AY)\bigr). \tag{35}$$

This holds for all points satisfying (33). Therefore, once we only put $\frac{1}{\lambda_0} = \delta$, (33) becomes equivalent to

$$\max\bigl[\rho_F(AX), \rho_F(AY)\bigr] < \delta. \tag{36}$$

Consequently, if (36) holds then so does (35), and this is precisely the claim of the theorem, which is therefore established.

In closing, note without proof some important results established by Theorem 14:

(I) Every geodesic (the shortest curve on every sufficiently small segment) on a DC surface at every point has right and left tangent lines; at every point where the surface has tangent plane, the geodesics have ordinary tangent lines (the right and left tangent lines coincide).

(II) The angle between the geodesics emanating from a point A, defined intrinsically,[10] equals the angle between their tangent lines measured on the tangent cone at A.

(III) In order for a curve on a DC surface to have direction at the initial point A, it is necessary and sufficient that it have tangent line at A. The angle between two curves emanating from A, defined intrinsically, equals the angle between their tangent lines measured on the tangent cone.

The proofs of these statements rest on quite deep conclusions about the intrinsic geometry of DC surfaces.

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[10]For convex surfaces, I gave [1] definitions of an angle between curves and the direction of a curve avoiding the differentiability assumption. These definitions carry over verbatim to DC surfaces, as indicated in [7].

In the case of convex surfaces, their proofs appeared in [1]. Liberman originally proved [9] the first of them in full generality by a beautiful geometric argument which, however, rests crucially on the convexity of surfaces under consideration, and so fails to carry over to other types of surfaces. Our argument is quite different, and so it yields a new proof of Liberman's theorem for geodesics on convex surfaces.

In fact, we directly prove the general claim (III), of which (I) and (II) are only particular cases.

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