S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 9, стр. 377–381 (2012) УДК 519.101
MSC 05A05

FINE AND WILF'S THEOREM FOR PERMUTATIONS

A. E. FRID

Abstract. We try to extend to permutations the famous Fine and Wilf's theorem valid for words and see that it is possible to do it only partially: the theorem is valid for coprime periods, but if the periods are not coprime, there is another statement valid instead.

Keywords: Fine and Wilf's theorem, periodicity, permutations, infinite permutations.

1. Introduction

A finite word $w = w_1 \cdots w_n$, where w_i are symbols of an alphabet Σ, is called *m-periodic* if $w_i = w_{i+m}$ for all $1 \leq i \leq n - m$.

The following theorem by Fine and Wilf's [5] is one of keystones of combinatorics on words.

Theorem 1 (Fine and Wilf)**.** *If a word of length at least $p+q-(p,q)$ is p-periodic and q-periodic, then it is (p,q)-periodic. The length $p+q-(p,q)$ is the best possible for non-unary words since there exist binary words of length $p+q-(p,q)-1$ which are p-periodic and q-periodic but not (p,q)-periodic.*

Initially stated for periodic functions [2], this theorem was naturally reformulated for words, and later extended to words with more than two periods, to partial words, to bidimensional words and to other periodicity notions, see [7] for a survey.

In this paper we study its possible extensions to permutations interpreted as linear orders on $\{1, \dots, n\}$. We see that in the case of coprime periods, the theorem is valid on them, but if the periods are not coprime, we have to state another

Frid, A. E., Fine and Wilf's theorem for permutations.

The work is supported by Presidential grant MK-4075.2012.1 and by RFBR grants 10-01-00424 and 12-01-00448.

Received July, 10, 2012, published August, 22, 2012.

theorem instead: there exists arbitrarily long permutations which are p- and q-periodic, but not (p, q)-periodic; however, all its suffucuently long factors must be (p, q)-periodic.

2. Permutations

In this paper we shall use the definition of a permutation α of n elements as a linear ordering on $\{1, \dots, n\}$ which can be different from the natural one. We shall write the fact that i is less than j according to that ordering as $\alpha_i < \alpha_j$ and represent the permutation α as a sequence of abstract elements $\alpha_1, \dots, \alpha_n$, somehow ordered.

This definition fits to the standard definition of a permutation: indeed, the fact that an element α_{i_1} is the least one means exactly that i_1 goes to 1, if an element α_{i_2} is greater than α_{i_1} but less than all other elements, it means exactly that i_2 goes to 2, and so on. At the same time, the definition we use can be naturally extended to infinite sets, which has been done in [3]. Infinite permutations were studied then in [1, 4]; the particular case of infinite permutations arising from infinite words is considered in [6] and other papers.

Another way to define a permutation α is by a *representative* sequence of pairwise distinct real numbers $a = a_1, \dots, a_n$, such that $a_i < a_j$ if and only if $\alpha_i < \alpha_j$. We denote this fact by $\alpha = \overline{a}$. Clearly, each permutation has an uncountable number of representatives.

However, in this paper we use only two notions developped by the theory of infinite permutations, namely, the periodicity of permutations and its factors, defined already in [3]. Applied to the finite case, they look as follows.

A permutation of n elements is called *t-periodic* if $\alpha_i < \alpha_j$ if and only if $\alpha_{i+t} < \alpha_{j+t}$ for all $1 \leq i < j \leq n - t$.

Consider a 1-periodic permutation; then $\alpha_1 < \alpha_2$ if and only if $\alpha_2 < \alpha_3$, if and only if $\alpha_3 < \alpha_4$ and so on; we see that either $\alpha_1 < \alpha_2 < \dots < \alpha_n$, and then the permutation is *increasing*, or $\alpha_1 > \alpha_2 > \dots > \alpha_n$, and then the permutation is *decreasing*. In both cases we say that α is *monotonic*; so, a permutation is 1-periodic if and only if it is monotonic, and there exist only two 1-periodic permutations of each length.

A permutation β of m elements is called a *factor* of a permutation α of n elements, $n \geq m$, if there exists some k, $1 \leq k \leq n-m+1$, such that for all i, j we have $\beta_i < \beta_j$ if and only if $\alpha_{i+k-1} < \alpha_{j+k-1}$. In this case we say that β is a factor of α of length m starting from position k and denote this fact by $\beta = \alpha[k..k+m) = \alpha[k..k+m-1]$.

At last, we define the following convenient notation. Given a permutation $\alpha = \alpha_0 \cdots \alpha_n$, let us denote by γ_{ij} the relation $<$ or $>$ between α_i and α_j, so that by the definition we have $\alpha_i \gamma_{ij} \alpha_j$. For each k let us denote by γ_k the word $\gamma_k = \gamma_{0,k}\gamma_{1,k+1} \cdots \gamma_{n-i,n}$ on the alphabet $\{<, >\}$.

3. The theorems

For the case of coprime periods, we can immediately extend the Fine and Wilf's theorem to permutations as follows.

Theorem 2. *If a permutation α of length at least $p+q$ is p-periodic and q-periodic, where $(p, q) = 1$, then α is 1-periodic, that is, monotonic. The length $p + q$ is the best possible since there exist permutations of length $p + q - 1$ which are p-periodic and q-periodic but not monotonic.*

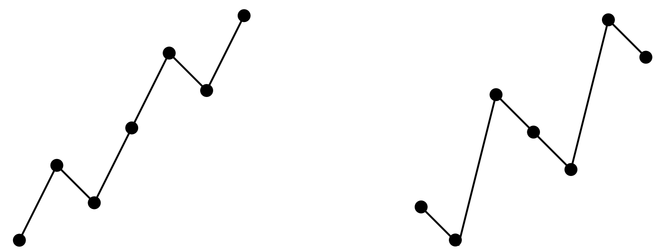

Рис. 1. Non-monotonic 3-periodic and 5-periodic permutations of length 7

Proof. If the permutation of length at least $p+q$ is p- and q-periodic, it means in particular that $\gamma_{i,i+1} = \gamma_{i+p,i+p+1}$ and $\gamma_{i,i+1} = \gamma_{i+q,i+q+1}$ for all i such that these relations are well-defined. It means exactly that the word γ_1 is p-periodic and q-periodic. Its length is equal to the length of α minus one, that is, it is at least $p+q-1$, and the Fine and Wilf's theorem for words is applicable to it. Thus, the word γ_1 is 1-periodic, that is, equal to $<< \cdots <$ or $>> \cdots >$. It means exactly that the permutation α is monotonic.

To prove the second part of the statement, let us consider a word u on the alphabet $\{<,>\}$ of length $p+q-2$ which is p-periodic and q-periodic but not 1-periodic. It exists due to Theorem 1 and contains both symbols $<$ and $>$. Now let us find a permutation α of length $p+q-1$ with $\gamma_1 = u$. Let the greatest number of consecutive symbols $>$ in u be l. We can always define the permutation α by a representative $a = a_0 \cdots a_{p+q-2}$ of the form $a_0 = 0$ and

$$a_{i+1} = \begin{cases} a_i + (2l+1), & \text{if } u_i =<, \\ a_i - 1, & \text{if } u_i =>, \end{cases}$$

for all $i = 0, \dots, p+q-3$. For all $i < j$ we see that $a_i > a_j$ if and only if $u_i \cdots u_{j-1} => \cdots >$ and $a_i < a_j$ if the word $u_i \cdots u_{j-1}$ contains an occurrence of the symbol $<$. So, the permutation whose representative is a is well-defined and p- and q-periodic since u is; at the same time, it is not monotonic. □

Example 1. As we have proven, a permutation of length 8 which is 3-periodic and 5-periodic must be monotonic. To construct a permutation of length 7 which is 3-periodic, 5-periodic but not 1-periodic, consider first the word of length 6 with these properties, that is, the word *abaaba*: it is unique up to renaming of symbols. We can rewrite this word either as $<><<><$, or as $><>><>$. Following our technique, in the first case we construct the permutation α of length 7 given by the representative 0 3 2 5 8 7 10, that is, by the inequalities $\alpha_0 < \alpha_2 < \alpha_1 < \alpha_3 < \alpha_5 < \alpha_4 < \alpha_6$. In the second case we have a representative 0 (-1) 4 3 2 7 6 and α is given by $\alpha_1 < \alpha_0 < \alpha_4 < \alpha_3 < \alpha_2 < \alpha_6 < \alpha_5$. Both permutations are shown at Fig. 1. Note that there exist also other permutations with that property.

Now let us show that for the p and q not coprime, the Fine and Wilf's theorem for permutations does not hold.

Let us call an *ascending saw* a finite permutation α with $\gamma_1 = (><)^{x/2}$ for some $x \in \mathbb{N}$ and $\gamma_2 =<^{x-1}$. Ascending saws give most known examples of "almost periodic" permutations, and here is one of them.

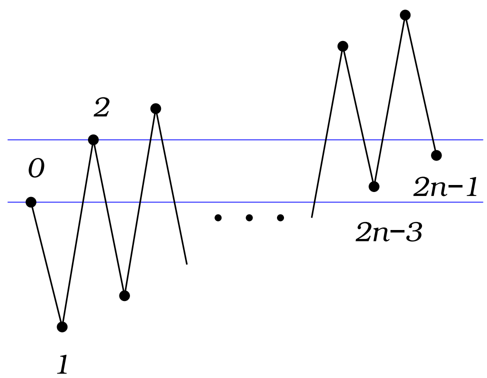

Рис. 2. Arbitrarily long permutation which is 4- and 6-periodic but not 2-periodic

Example 2. Let us consider an ascending saw $\alpha = \alpha_0\alpha_1 \cdots \alpha_{2n-1}$ with $\alpha_{2n-5} < \alpha_0 < \alpha_{2n-3} < \alpha_{2n-1} < \alpha_2$: these conditions uniquely define the saw shown at Fig. 3

This permutation is not 2-periodic since $\gamma_{0,2n-3} =<$ and $\gamma_{2,2n-1} =>$. However, it is not difficult to see that it is $2l$-periodic for all $l > 1$: in particular, it is 4-periodic and 6-periodic. Since its length n can be arbitrarily large, it already disproves the Fine and Wilf's theorem for $(p, q) = 2$.

Example 3. To give a counterexample for any other $(p, q) = k > 1$, we shuffle α with $k - 2$ monotonic permutations. Namely, consider $\beta = \beta_0 \cdots \beta_{k(n-1)+1}$ defined by $\beta_{ki} = \alpha_{2i}$ and $\beta_{ki+1} = \alpha_{2i+1}$ for all i; $\beta_{ki+j} < \beta_{k(i+1)+j}$ for all i and all $j \neq 0, 1 \bmod k$; and $\beta_{ki_1+j_1} < \beta_{ki_2+j_2}$ whenever $j_1 < j_2$ for $j_1, j_2 \in \{0, \dots, k-1\}$, except for the case of $\{j_1, j_2\} = \{0, 1\}$. Clearly these conditions uniquely define the permutation which is not k-periodic (since $\gamma_{0,k(n-2)+1} =<$ and $\gamma_{k,k(n-1)+1} =>$) but is kl-periodic for all $l > 1$.

However, the following statement can be considered as the Fine and Wilf's theorem for general permutations.

Theorem 3. *Suppose that a finite permutation α of length n is p-periodic and q-periodic. Then each its factor of length at most $n - p - q + 2(p, q) + 1$ is (p, q)-periodic.*

Proof. If α is p-periodic and q-periodic, then so are all its strings γ_i on the alphabet $\{<, >\}$, $i = 1, \dots, n-1$. The length of γ_i for each i is equal to $n - i$, and whenever the length of γ_i at least $p+q-(p, q)$ (that is, i is at most $n-p-q+(p, q)$), from the Fine and Wilf's theorem for words we get that γ_i is (p, q)-periodic. But a factor β of α of length $l \leq n-p-q+2(p, q)+1$, say, $\beta = \alpha_m \cdots \alpha_{m+l-1}$ is uniquely defined by the list of underlying relations $\gamma_{k,k+i}$ with $m \leq k < k+i \leq m+l-1$. They all are elements either of some γ_i for $i \leq l \leq n-p-q+(p, q)$, which have been shown to be (p, q)-periodic, or of some γ_i for $n-p-q+(p, q)+1 \leq i \leq n-p-q+2(p, q)$, and the factor of γ_i involved into the latter case is of length at most (p, q). So, all involved factors of sequences γ_i are (p, q)-periodic, and so is β. □

Example 4. Let us consider the permutation of length 10 defined by the inequalities

$$\alpha_1 < \alpha_3 < \alpha_2 < \alpha_5 < \alpha_7 < \alpha_4 < \alpha_6 < \alpha_9 < \alpha_8 < \alpha_{10}$$

Рис. 3. Factors of length at most 5 of this permutation are 2-periodic

and depicted at Fig. 3. It is 4-periodic and 6-periodic but not 2-periodic since $\gamma_3 =<<<><<<$. Nevertheless, all its factors of length at most $5 = 10-6-4+2*2+1$ are 2-periodic, and the shortest factors which are not 2-periodic are of length 6: these are exactly factors covering three elements of γ_3 including the element $>$, that is, the factors $\alpha[2..7]$, $\alpha[3..8]$, $\alpha[4..9]$.

Acknowledgements

The author is grateful to Julien Cassaigne and Sergey Avgustinovich for useful discussions.

References

[1] S. V. Avgustinovich, A. Frid, T. Kamae and P. Salimov, *Infinite permutations of lowest maximal pattern complexity*, Theoret. Comput. Sci. **412** (2011), 2911–2921. MR2830255

[2] N. J. Fine and H. S. Wilf, *Uniqueness theorems for periodic functions*, Proc. Amer. Math. Soc., **16** (1965), 109–114. MR0174934

[3] D. G. Fon-Der-Flaass, A. E. Frid, *On periodicity and low complexity of infinite permutations*, European J. Combin. **28** (2007), 2106–2114. MR2351513

[4] A. Frid and L. Zamboni, *On automatic infinite permutations*, Theoret. Inf. Appl. **46** (2012), 77–85.

[5] M. Lothaire, *Combinatorics on words*, Addison-Wesley Publishing Co., 1983. MR0675953

[6] M. A. Makarov, *On permutations generated by infinite binary words*, Sib. Electron. Mat. Izv. **3** (2006), 304–311 [in Russian, English abstract]. MR2276028

[7] J. O. Shallit, *50 Years of Fine and Wilf*, invited talk at University of Turku, Finland, November 2011; slides available at `http://www.cs.uwaterloo.ca/ shallit/Talks/wilf5.pdf`.

Anna E. Frid
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
E-mail address: anna.e.frid@gmail.com