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ON SELF-DEFINABLE SUBSETS OF $\aleph_0$–CATEGORICAL WEAKLY O-MINIMAL STRUCTURES

B.SH. KULPESHOV

Abstract. The present paper concerns the generalization of the notion of o-minimality: *weak o-minimality* originally studied by D. Macpherson, D. Marker and Ch. Steinhorn in [1]. We study self-definable sets of an $\aleph_0$-categorical weakly o-minimal structure, and the main result is a criterion for goodness of every self-definable subset in an $\aleph_0$-categorical weakly o-minimal structure (Theorem 2.3).

Keywords: weak o-minimality, $\aleph_0$-categoricity, self-definable set.

1. Introduction

In recent years there have been several approaches to generalizing the notion of *o-minimality*. Typically, for a structure, one imposes strong restrictions on the 1-variable definable sets. An o-minimal structure M can be viewed as an L-structure where $L \supset L_0 = \{<\}$, $<$ is a total order on M, and every definable subset of M is quantifier-free L_0-definable. This provides a template for other notions: replace L_0 by some other familiar language, consider L-structures such that the L_0-reduct is of stipulated type (e.g. a total order), and require that every definable subset of M is (quantifier-free) L_0-definable (one may require this for all models of the theory). This route was followed in [2], where notions such as *circularly minimal* and *C-minimal* were proposed and slightly explored. Other notions such as *P-minimal* [3], *Boolean o-minimal* [4], *weakly circularly minimal* [5] and *o-stable* [6], [7] have since been developed. Here the notion of *weak o-minimality* is continued studying. The theory of weakly o-minimal structures was developed in [8, 9, 10, 11, 12] and others.

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Let L be a first-order language. Everywhere in this paper we consider L-structures and assume that L contains a binary relation symbol $<$ that is interpreted as a linear ordering in these structures. For an arbitrary complete 1-type p we denote by $p(M)$ the set of realizations of the type p in M. For arbitrary subsets A, B of a structure M we write $A < B$ if $a < b$ whenever $a \in A$ and $b \in B$. A complete theory T is *binary* if every formula is equivalent to a Boolean combination of formulas in at most two free variables. A subset A of a structure M is *convex* if for all $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that every definable (with parameters) subset of the structure M is a finite union of convex sets in M. Real closed fields with a proper convex valuation ring provide an important example of weakly o-minimal (not o-minimal) structures.

In the following definitions M is a weakly o-minimal structure, $A, B \subseteq M$, types $p, q \in S_1(A)$ are non-algebraic.

Definition 1.1. (B.S. Baizhanov, [8]) Let M be $|A \cup B|^+$-saturated. The *neighborhood of the set B in the type p* is the following set:

$$V_p(B) := \{\gamma \in M | \text{ there exists a formula } H(x, \bar{b}, \bar{a}), \bar{b} \in B, \bar{a} \in A \text{ so that}$$
$$\gamma \in H(M, \bar{b}, \bar{a}) \text{ and there are } \gamma_1, \gamma_2 \in p(M) \text{ such that } \gamma_1 < H(M, \bar{b}, \bar{a}) < \gamma_2\}$$

Definition 1.2. (B.S. Baizhanov, [8]) Let M be $|A|^+$-saturated. We say p is *almost orthogonal to q* ($p \perp^a q$) if there exists $\alpha \in p(M)$ such that $V_q(\alpha) = \emptyset$.

Definition 1.3. (B.S. Baizhanov, [8]) Let M be $|A|^+$-saturated. We say p is *weakly orthogonal to q* ($p \perp^w q$) if for any A-definable formula $H(x, y)$ and $\alpha \in p(M)$ the following holds:

$$H(M, \alpha) \cap q(M) \neq \emptyset \Rightarrow q(M) \subseteq H(M, \alpha)$$

Lemma 1.4. ([8], Corollary 34 (iii)) *The relation of non-weak orthogonality is an equivalence relation on $S_1(A)$.*

Definition 1.5. (B.S. Baizhanov, [8]) We say that a weakly o-minimal theory T is *almost o-minimal* if for every model M of T, every set $A \subseteq M$ and for all non-algebraic types $p, q \in S_1(A)$ the following holds: $p \perp^a q \Leftrightarrow p \perp^w q$.

Fact 1.6. *Every o-minimal theory T is almost o-minimal.*

We say that a type p is not *quite orthogonal to q* ($p \not\perp^q q$) if there exists an A-definable bijection $f : p(M) \to q(M)$. We say that a weakly o-minimal theory is *quite o-minimal* if the notions of weak and quite orthogonality of 1-types coincide. Observe that if M is an o-minimal structure then in case of non-weak orthogonality of types p and q there exists an A-definable bijection $f : p(M) \to q(M)$, i.e. quite o-minimal theories are "quite" o-minimal regarding the property of non-weak orthogonality of types. Theorem 1.11 implies that quite o-minimal theories also "quite" inherit the property of o-minimal theories to be binary in the $\aleph_0$-categorical case.

Fact 1.7. *The relation of non-quite orthogonality is an equivalence relation on $S_1(A)$.*

Example 1.8. [1] Let $M = \langle M, <, P^1, f^1 \rangle$ be a linearly ordered structure. Here P is a unary predicate and f is a unary function with $Dom(f) = \neg P$, $Ran(f) = P$. The universe of the structure M is a disjoint union of P and $\neg P$, where $x < y$ whenever

$x \in P$ and $y \in \neg P$. To define f identify P with Q (where Q is the ordering of rational numbers) and $\neg P$ with $Q \times Q$ (which is lexicographically ordered), and for all $m, n \in Q$ let $f(\langle m, n \rangle) = n$.

It is easy to check that $Th(M)$ is an $\aleph_0$-categorical weakly o-minimal theory. Let $p(x) := \{\neg P(x)\}$, $q(x) := \{P(x)\}$. Obviously $p, q \in S_1(\emptyset)$. The function f maps $p(M)$ on $q(M)$, whereas $p \not\perp^w q$. However f is not bijection between $p(M)$ and $q(M)$, and it can be proved that there is no any $\emptyset$-definable bijections between these sets. Therefore, $p \perp^q q$, i.e. $Th(M)$ is not quite o-minimal. Take an arbitrary element $a \in p(M)$, and let $b = f(a)$. Obviously $b \in dcl(\{a\})$ and $a \notin dcl(\{b\})$, i.e. the Exchange Principle for algebraic closure doesn't hold in M.

Lemma 1.9. *Let T be an $\aleph_0$-categorical quite o-minimal theory. Then the Exchange Principle for algebraic closure holds in any model of T.*

Proof of Lemma 1.9. Let M be a model of T. Take arbitrary elements a, b and tuple $\bar{c} \in M$ such that $a \in dcl(b, \bar{c}) \setminus dcl(\bar{c})$. Prove that $b \in dcl(a, \bar{c})$. Let $p = tp(b/\bar{c})$, $q = tp(a/\bar{c})$. Since $a \in dcl(b, \bar{c})$, there is a formula $\phi(x, y, \bar{c})$ such that $M \models \phi(a, b, \bar{c}) \wedge \exists! \phi(x, b, \bar{c})$. If we suppose that $p = q$ then by considering the formula $\Theta(y) := \exists! \phi(x, y, \bar{c})$ we obtain that $dcl(b, \bar{c})$ is infinite contradicting the $\aleph_0$-categoricity of T. Consequently, $p \neq q$. If $b \in dcl(\bar{c})$ then $a \in dcl(\bar{c})$ that contradicts the hypotheses of the lemma. Consequently, p and q are non-algebraic, and $p \not\perp^w q$. Then by quite o-minimality of T there exists an $\bar{c}$-definable bijection $f : p(M) \to q(M)$. We assert that $f(b) = a$, whereas we will have $b \in dcl(a, \bar{c})$. Assume the contrary: $f(b) \neq a$, i.e. there exists $a' \in q(M)$ such that $a' \neq a$ and $f(b) = a'$. Then we have $a \in dcl(a', \bar{c})$, and consequently $tp(a/\bar{c}) = tp(a'/\bar{c})$ that also contradicts the $\aleph_0$-categoricity of T. □

Definition 1.10. [9] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, and let $\phi(x)$ be an arbitrary M-definable formula with one free variable.
The convexity rank of formula $\phi(x)$ ($RC(\phi(x))$ is defined as follows:
1) $RC(\phi(x)) = -1$ if $M \models \neg \exists x \phi(x)$.
2) $RC(\phi(x)) \geq 0$ if $M \models \exists x \phi(x)$.
3) $RC(\phi(x)) \geq 1$ if $\phi(M)$ is infinite.
4) $RC(\phi(x)) \geq \alpha + 1$ if there is a parametrically definable equivalence relation $E(x, y)$ such that there are $b_i, i \in \omega$ which satisfy the following:

- For every $i, j \in \omega$, whenever $i \neq j$ then $M \models \neg E(b_i, b_j)$
- For every $i \in \omega$ $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of $\phi(M)$

5) $RC(\phi(x)) \geq \delta$ if $RC(\phi(x)) \geq \alpha$ for all $\alpha \leq \delta$ (δ is limit)
If $RC(\phi(x)) = \alpha$ for some α we say that $RC(\phi(x))$ is defined. Otherwise (i.e. if $RC(\phi(x)) \geq \alpha$ for all α) we put $RC(\phi(x)) = \infty$.

Let $A \subseteq M$ and $p \in S_1(A)$ be non-algebraic. The *convexity rank of 1-type p* ($RC(p)$) is the infumum of the set $\{RC(\phi(x)) | \phi(x) \in p\}$.

The following theorem completely characterizes $\aleph_0$-categorical quite o-minimal theories:

Theorem 1.11. [13] *Let T be an $\aleph_0$-categorical quite o-minimal theory, $M \models T$, $|M| = \aleph_0$. Then* *(i) there exists a finite set $C = \{c_0, \dots, c_n\} \subseteq M$ ($M \cup \{-\infty, +\infty\}$ if M has no first or last element) consisting of all $\emptyset$-definable elements in M (except possibly for $-\infty$ and $+\infty$) so that $M \models c_i < c_j$ for all $i < j \leq n$ and for every $j \in \{1, \dots, n\}$ either $M \models \neg(\exists x) c_{j-1} < x < c_j$ or $I_j = \{x \in M : M \models c_{j-1} < x < c_j\}$ is a dense linear ordering without endpoints and there are $k_j \in \omega$ and $p_1^j, \dots, p_{k_j}^j \in S_1(\emptyset)$ so that $I_j = \bigcup_{s=1}^{k_j} p_s^j(M)$;*

(ii) for every non-algebraic $p \in S_1(\emptyset)$ there is $n_p \in \omega$ such that $RC(p) = n_p$, i.e. there exist $\emptyset$-definable equivalence relations $E_1^p(x, y), \dots, E_{n_p-1}^p(x, y)$ such that

- *$E_{n_{p-1}}^p$ partitions $p(M)$ into infinitely many $E_{n_{p-1}}^p$-classes, every $E_{n_{p-1}}^p$-class is convex and open so that the induced order on classes is a dense linear order without endpoints;*
- *for every $i \in \{1, \dots, n_{p-2}\}$ E_i^p partitions each E_{i+1}^p-class into infinitely many E_i^p-classes, every E_i^p-class is convex and open so that E_i^p-subclasses of each E_{i+1}^p-class are densely ordered without endpoints;*

(iii) there exists an equivalence relation $\varepsilon \subseteq (s : 1 \leq s \leq k)^2$, where $\{p_s | s \leq k < \omega\}$ is an arbitrary enumeration of all non-algebraic 1-types over $\emptyset$ such that for every $(i, j) \in \varepsilon$ there exists a unique $\emptyset$-definable locally monotonic bijection $f_{i,j} : p_i(M) \to p_j(M)$ with $RC(p_i) = RC(p_j)$, $f_{i,i} = id_{p_i(M)}$ and $f_{j,l} f_{i,j} = f_{i,l}$ for all $(i, j), (j, l) \in \varepsilon$ so that T admits elimination of quantifiers to the language

$$\{=, <\} \cup \{c_i : i \leq n\} \cup \{U_s(x) : s \leq k\} \cup \{E_l^{p_s}(x, y) : s \leq k, l \leq n_{p_s}\} \cup \{f_{i,j} : (i, j) \in \varepsilon\}$$

where $U_s(x)$ isolates the type p_s for each $s \leq k$. In particular, T is binary.

Moreover, any ordering with distinguished elements as in (i)-(ii) and any suitable equivalence relation as in (iii) corresponds an $\aleph_0$-categorical quite o-minimal theory.

2. Main result

Definition 2.1. [2] Let $A \subseteq M$ where M is a structure. The set A is *self-definable* if it is definable in M by parameters that are elements of A.

Recall that a permutation group $(G; X)$ is *closed* if G is the full automorphism group of a first-order structure on X. The notion of self-definable set is connected with the following result:

Theorem 2.2. ([2], Theorem 2.1) *Let M be a countable $\aleph_0$-categorical structure, and $A \subset M$ be self-definable. The following are equivalent:*

(1) $(Aut(M)_A; A)$ is closed;

(2) for every $k < \omega$ and formula $\psi(\bar{x}, \bar{c})$ with $lh(\bar{x}) = k$ and $\bar{c} \subset M$, the set $\psi(M^k, \bar{c}) \cap A^k$ is A-definable;

(3) for all $n < \omega$, every n-type over A realized in M is isolated;

(4) M is prime over A.

We call a self-definable subset of an $\aleph_0$-categorical structure *good* if it satisfies any, hence all, of the conditions (1)–(4) above.

Here we present a criterion for goodness of every self-definable subset of an $\aleph_0$-categorical weakly o-minimal structure.

Theorem 2.3. *Let T be an $\aleph_0$-categorical weakly o-minimal theory. Then the following are equivalent:*

(1) T is quite o-minimal

(2) For any model M of T every self-definable subset $A \subseteq M$ is good.

Proof of Theorem 2.3. (1) $\Rightarrow$ (2). Since A is self-definable, there exist a finite set $A_1 \subseteq A$ and an A_1-definable formula $\phi(x)$ such that $A = \phi(M)$. By the $\aleph_0$-categoricity of T $M = dcl(A_1) \cup \bigcup_{i=1}^{k} p_i(M)$ where $p_i \in S_1(A_1)$ is non-algebraic for each $i \leq k$. Without loss of generality, we can assume that $dcl(A_1) \subseteq A$. Let for definiteness $p_i(M) \subseteq A$ for every $1 \leq i \leq s$ и $p_j(M) \cap A = \emptyset$ for every $s+1 \leq j \leq k$. By binarity of T it is sufficiently to show that every 2-type over A realized in M is isolated. Take an arbitrary tuple $\langle a_1, a_2 \rangle \in (M \setminus A)^2$ with $a_1 < a_2$. The following cases are possible:

(a) $a_1, a_2 \in p_j(M)$ for some $s+1 \leq j \leq k$ and there is $i \leq s$ such that p_i is not weakly orthogonal to p_j.

(b) $a_1, a_2 \in p_j(M)$ for some $s+1 \leq j \leq k$ and for every $i \leq s$ p_i is weakly orthogonal to p_j.

(c) $a_1 \in p_j(M)$, $a_2 \in p_r(M)$ for some $s+1 \leq j$, $r \leq k$, p_j is weakly orthogonal to p_r and there is $i \leq s$ such that p_i is not weakly orthogonal to p_j.

(d) $a_1 \in p_j(M)$, $a_2 \in p_r(M)$ for some $s+1 \leq j, r \leq k$, p_j is weakly orthogonal to p_r and for every $i \leq s$ p_i is weakly orthogonal to both p_j and p_r.

(e) $a_1 \in p_j(M)$, $a_2 \in p_r(M)$ for some $s+1 \leq j, r \leq k$, p_j is not weakly orthogonal to p_r and there is $i \leq s$ such that p_i is not weakly orthogonal to p_j.

(f) $a_1 \in p_j(M)$, $a_2 \in p_r(M)$ for some $s+1 \leq j, r \leq k$, p_j is not weakly orthogonal to p_r and for every $i \leq s$ p_i is weakly orthogonal to p_j.

(a) Since p_i is not weakly orthogonal to p_j, by Theorem 11 there exists a unique A_1-definable locally monotone bijection $f : p_i(M) \to p_j(M)$. Then there are $b_1, b_2 \in p_i(M)$ such that $f(b_1) = a_1$, $f(b_2) = a_2$. Obviously that the formula $f(b_1) = x_1 \wedge f(b_2) = x_2$ isolates the type of $\langle a_1, a_2 \rangle$ over A.

(b) Let the formula $U_j(x)$ isolate the type p_j, and let $RC(p_j) = m$. Then there exist A_1-definable equivalence relations $E_1(x, y)$, $E_2(x, y)$, $\ldots, E_{m-1}(x, y)$ partitioning $p_j(M)$ into infinite convex classes so that $E_1(M, a_1) \subset E_2(M, a_1) \subset \ldots \subset E_{m-1}(M, a_1)$. Suppose for definiteness that $M \models E_l(a_1, a_2) \wedge \neg E_{l+1}(a_1, a_2)$ for some $1 \leq l \leq m-2$. Then the formula $U_j(x_1) \wedge U_j(x_2) \wedge x_1 < x_2 \wedge E_l(x_1, x_2) \wedge \neg E_{l+1}(x_1, x_2)$ isolates the type of $\langle a_1, a_2 \rangle$ over A.

The rest cases are considered analogously.

(2)$\Rightarrow$ (1). Suppose that T is not quite o-minimal. Then there exist a finite set $A \subseteq M$, non-algebraic types $p_1, p_2 \in S_1(A)$ such that p_1 is not weakly orthogonal, but is quite orthogonal to p_2. Let $A_1 := dcl(A) \cup p_1(M)$. Obviously that A_1 is self-definable. Take an arbitrary element $b \in p_2(M)$. Then the type $tp(b/A_1)$ is non-isolated, and the condition (2) doesn't hold, contradicting the hypothesis. Thus, T is quite o-minimal. $\square$

Observe that in the hypotheses of Theorem 14 we cannot weaken the condition of quite o-minimality to almost o-minimality of a theory. Indeed, consider the following example:

Example 2.4. Let $M = \langle M, =, <, U_1^1, U_2^1, E^2, R^2 \rangle$, where $\langle M, < \rangle$ has the ordering type Q. The universe of M is a disjoint union of U_1 and U_2 such that $a < b$ whenever $a \in U_1, b \in U_2$, and every predicate U_i has no endpoints in M. E is an equivalence relation partitioning $U_1(M)$ into infinite convex classes so that the induced ordering on E-classes is a dense order without endpoints. To define R, identify U_i with Q for every $i \leq 2$, and for all $a \in U_1$ and $b \in U_2$ we have $R(a, b) \Leftrightarrow b < a + \sqrt{2}$.

It is not difficult to prove that the theory $Th(M)$ admits quantifier elimination and is $\aleph_0$-categorical almost o-minimal (non-quite o-minimal). Let $A := U_1(M)$. Obviously, A is self-definable. Take an arbitrary element $b \in U_2(M)$. It can understand that $tp(b/A)$ is not isolated.

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Beibut Shaiykovich Kulpeshov
International Information Technology University,
ul. Manasa 34 A / ugol ul. Zhandosova 8 A,
050040, Almaty, Kazakhstan
E-mail address: kulpesh@mail.ru