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INTERPOLATION FORMULA FOR FUNCTIONS WITH A BOUNDARY LAYER COMPONENT AND ITS APPLICATION TO DERIVATIVES CALCULATION

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Abstract. An interpolation formula for a function of one variable with a boundary layer component is constructed. Such function corresponds to the solution of a singular perturbed problem. The estimate of an accuracy is obtained. On a base of the constructed interpolation formula the difference formulas for derivatives of the function with a boundary layer component are obtained. Numerical resultes are discussed.

Keywords: function, boundary layer, nonpolynomial interpolation, difference formula for a derivative, accuracy estimation.

1. Introduction

Spline interpolation methods are well developed, see e.g. [1], [2]. But when we apply polynomial interpolation methods to functions with large gradients, it can lead to significant errors. In this article we study an interpolation problem for functions with a boundary layer component.

We construct spline interpolation formulas whose interpolation errors do not depend on gradients of the boundary layer component. For a construction of such formulas we can use mesh dense in the boundary layer [3] or we can use the interpolation formulas fitted to the boundary layer component [4]-[7]. In [4]-[7] we constructed the interpolation formulas with two and three nodes. We found an application of constructed formulas in the two-grid method [8] for a nonlinear singular perturbed problem [9], [10].

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In this article we construct the interpolation formula, fitted to the boundary layer component, with any number of interpolation nodes.

Let a function $u(x)$ is smooth enough and has the representation

$$u(x) = p(x) + \gamma\Phi(x), \ x \in [0,1], \tag{1.1}$$

where $\Phi(x)$ is known function with regions of large gradients, the function $p(x)$ is the regular part of $u(x)$, bounded together with some derivatives, the constant γ is unknown. The representation (1.1) holds for solutions of the boundary layer problems [10].

For example, we consider a problem:

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad u(0) = A, \ u(1) = B, \tag{1.2}$$

where

$$a(x) \geq \alpha > 0, b(x) \geq 0, \ \varepsilon \in (0,1],$$

functions a, b, f are smooth enough. According to [10], a solution of this problem has the exponential boundary layer near the point $x = 0$ if $\varepsilon \ll 1$, the representation (1.1) for $u(x)$ holds with

$$\Phi(x) = \exp(-a_0\varepsilon^{-1}x), \gamma = -\varepsilon u'(0)/a_0,$$

where $a_0 = a(0)$, $|p'(x)\Big| \leq C$, C does not depend on ε. Derivatives of the function $\Phi(x)$ are not bounded uniformly in a parameter $\varepsilon \in (0,1]$.

2. Construction and analysis of the spline interpolation formula

Let us a function $u(x)$ of a form (1.1) be given at nodes of a mesh Ω :

$$\Omega = \{x_n : x_n = x_{n-1} + h_n, \ x_0 = 0, \ x_N = 1, \ n = 1, 2, \ldots, N\}, \tag{2.1}$$

$u_n = u(x_n), \ n = 0, 1, 2, \ldots, N$. We construct the interpolation formula using values u_{m+j} at nodes $x_{m+j}, \ j = 0, 1, \ldots, k-1$. We suppose that $0 \leq m \leq N - k + 1$.

Let $L_k(u, x_m, x_{m+1}, \ldots, x_{m+k-1}, x)$ be Lagrange polynomial for the function $u(x)$ with the interpolation conditions at nodes $x_m, x_{m+1}, \ldots, x_{m+k-1}$. It is known the representation of the interpolation error ([1], p. 42):

$$u(x) - L_k(u, x_m, x_{m+1}, \ldots, x_{m+k-1}, x) = \frac{u^{(k)}(s)}{k!} \prod_{j=0}^{k-1} (x - x_{m+j}), \tag{2.2}$$

where $s \in (x_m, x_{m+k-1})$. It follows from (2.2) that the interpolation error is the quantity of the order $O(h^k)$, if the derivative $u^{(k)}(x)$ is uniformly bounded. If a function $u(x)$ has a boundary layer component, this derivative is not uniformly bounded and the interpolation error can be of the order $O(1)$ for small values of the mesh steps [4].

Introduce the interpolation formula:

$$u_{\Phi,k}(x) = L_{k-1}(u, x_m, x_{m+1}, \ldots, x_{m+k-2}, x) +$$

$$+ \frac{[x_m, x_{m+1}, \ldots, x_{m+k-1}]u}{[x_m, x_{m+1}, \ldots, x_{m+k-1}]\Phi} \Big[\Phi(x) - L_{k-1}(\Phi, x_m, x_{m+1}, \ldots, x_{m+k-2}, x)\Big], \tag{2.3}$$

where $[x_m, x_{m+1}, \dots, x_{m+k-1}]u$ is the divided difference ([1], p. 43) for the function $u(x)$. According to ([1], p. 45), for some $s \in (x_m, x_{m+k-1})$

$$[x_m, x_{m+1}, \dots, x_{m+k-1}]\Phi = \Phi^{(k-1)}(s)/(k-1)!\,.$$

Therefore, the expression (2.3) is correct if $\Phi^{(k-1)}(x) \neq 0$ for $x \in (x_m, x_{m+k-1})$.

Lemma 1. *The function $u_{\Phi,k}(x)$ is the interpolant for $u(x)$ with nodes of the interpolation $x_j,\ j = m, m+1, \dots, m+k-1$.*

Proof. For $j = m, m+1, \dots, m+k-2$ $u_{\Phi,k}(x_j) = u_j$ according to interpolation conditions for Lagrange polynomial L_{k-1}.

Now we'll prove that $u_{\Phi,k}(x_{m+k-1}) = u_{m+k-1}$. According to ([1], p.44)

$$\Phi(x) - L_{k-1}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x) =$$
$$= [x_m, x_{m+1}, \dots, x_{m+k-2}, x]\Phi \prod_{j=0}^{k-2}(x - x_{m+j}). \tag{2.4}$$

Therefore,

$$u_{\Phi,k}(x_{m+k-1}) = L_{k-1}(u, x_m, x_{m+1}, \dots, x_{m+k-2}, x_{m+k-1}) +$$
$$+ [x_m, x_{m+1}, \dots, x_{m+k-1}]u \prod_{j=0}^{k-2}(x_{m+k-1} - x_{m+j}).$$

We take into account the relation (2.4) for the function $u(x)$ and obtain

$$u_{\Phi,k}(x_{m+k-1}) = u(x_{m+k-1}).$$

The lemma is proved.

It is obvious that the interpolation formula (2.3) is exact for the function $\Phi(x)$.

Further we'll suppose that the mesh Ω is uniform and for any j $x_j - x_{j-1} = h$. In the following lemma we estimate the interpolation error of the formula (2.3).

Lemma 2. *Let*

$$M_k(\Phi, x) = \frac{\Phi(x) - L_{k-1}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x)}{\Phi(x_{m+k-1}) - L_{k-1}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x_{m+k-1})}. \tag{2.5}$$

Then for any $x \in [x_m, x_{m+k-1}]$

$$\Big| u_{\Phi,k}(x) - u(x) \Big| \le \max_s |p^{(k-1)}(s)| h^{k-1} \Big[|M_k(\Phi, x)| + 1 \Big], \tag{2.6}$$

where $s \in [x_m, x_{m+k-1}]$.

Proof. The interpolation (2.3) is exact for the boundary layer component $\Phi(x)$, then

$$u_{\Phi,k}(x) - u(x) = L_{k-1}(p, x_m, x_{m+1}, \dots, x_{m+k-2}, x) - p(x) +$$
$$+ \frac{[x_m, x_{m+1}, \dots, x_{m+k-1}]p}{[x_m, x_{m+1}, \dots, x_{m+k-1}]\Phi} \Big[\Phi(x) - L_{k-1}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x) \Big].$$

We have from (2.4)

$$[x_m, x_{m+1}, \dots, x_{m+k-1}]\Phi = \Big[\Phi(x_{m+k-1}) - L_{k-1}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x_{m+k-1}) \Big]$$
$$\Big/ \prod_{j=0}^{k-2}(x_{m+k-1} - x_{m+j}). \tag{2.7}$$

It follows from the relation (2.2) that for any $x \in [x_m, x_{m+k-1}]$

$$\Big| L_{k-1}(p, x_m, x_{m+1}, \dots, x_{m+k-2}, x) - p(x) \Big| \le \max_s |p^{(k-1)}(s)|\, h^{k-1}, \tag{2.8}$$

where $s \in [x_m, x_{m+k-1}]$. Therefore,

$$\Big| u_{\Phi,k}(x) - u(x) \Big| \le \max_s |p^{(k-1)}(s)|\, h^{k-1} +$$

$$+ \Big| [x_m, x_{m+1}, \dots, x_{m+k-1}]p \Big| \prod_{j=0}^{k-2} (x_{m+k-1} - x_{m+j}) |M_k(\Phi, x)|. \tag{2.9}$$

According to ([1], p. 45), for some $s \in (x_m, x_{m+k-1})$

$$[x_m, x_{m+1}, \dots, x_{m+k-1}]p = p^{(k-1)}(s)/(k-1)!.$$

Now we obtain (2.6) from (2.9). The lemma is proved.

According to (2.6), the interpolation error depends only on derivatives of a regular component $p(x)$. We need the function $M_k(\Phi, x)$ be bounded.

The expression (2.5) has the following interpretation: the numerator corresponds to the interpolation error of Lagrange polynomial for the function $\Phi(x)$ at a point x, $x \le x_{m+k-1}$ and the denominator corresponds to the same interpolation error at the point x_{m+k-1}. Note that for given function $\Phi(x)$ the expression $M_k(\Phi, x)$ can be calculated for any x.

In [6], [7] we got the formula (2.3) in the cases $k = 2$ and $k = 3$.

In a case $k = 2$ we proved that the interpolation error is the quantity of the order $O(h)$ for any mesh interval $[x_{n-1}, x_n]$, uniformly in a boundary layer component $\Phi(x)$ and its derivatives. We supposed that $\Phi'(x) \ne 0$ if $x \in (x_{n-1}, x_n)$.

In a case $k = 3$ we constructed the interpolation formula in the interval $[x_{n-1}, x_{n+1}]$. We used the condition $\Phi''(x) \ne 0$ if $x \in (x_{n-1}, x_{n+1})$. Under some conditions on $\Phi(x)$ we proved that the interpolation error is the quantity of the order $O(h^2)$ uniformly in a boundary layer component. In a case $\Phi(x) = \exp(-a_0 \varepsilon^{-1} x)$, corresponding to the solution of the problem (1.2), we proved that the interpolation error is the quantity of the order $O(h^2)$ uniformly in ε.

Now we prove that the interpolant (2.3) has the same order of an accuracy as Lagrange polynomial $L_k(u, x_m, x_{m+1}, \dots, x_{m+k-1}, x)$, if we don't require the uniform accuracy on $\Phi(x)$ and its derivatives.

We consider the polynomial $L_k(u, x_m, x_{m+1}, \dots, x_{m+k-1}, x)$ as the interpolant for the function $u_{\Phi,k}(x)$, then according to (2.2) we write

$$u_{\Phi,k}(x) - L_k(u, x_m, x_{m+1}, \dots, x_{m+k-1}, x) = \frac{u^{(k)}_{\Phi,k}(s)}{k!} \prod_{j=0}^{k-1} (x - x_{m+j}).$$

Now we take into account (2.2) and obtain

$$|u_{\Phi,k}(x) - u(x)| \le \frac{1}{k!} \Big[\max_s |u^{(k)}_{\Phi,k}(s)| + \max_s |u^{(k)}(s)| \Big] \Big| \prod_{j=0}^{k-1} (x - x_{m+j}) \Big|.$$

It follows that for any $x \in [x_m, x_{m+k-1}]$

$$|u_{\Phi,k}(x) - u(x)| \le \frac{1}{4k} \Big[\max_s |u^{(k)}_{\Phi,k}(s)| + \max_s |u^{(k)}(s)| \Big] h^k, \quad s \in [x_m, x_{m+k-1}].$$

Thus, if a function $u(x)$ has not large gradients, the interpolant (2.3) has the error of the order $O(h^k)$.

Difference formulas for derivatives. On a base of the formula (2.3) we can obtain difference formulas for derivatives of the function $u(x)$ with a boundary layer component.

We differentiate (2.3) and obtain

$$u_{\Phi,k}^{(j)}(x) = L_{k-1}^{(j)}(u, x_m, x_{m+1}, \dots, x_{m+k-2}, x) +$$
$$+\frac{[x_m, x_{m+1}, \dots, x_{m+k-1}]u}{[x_m, x_{m+1}, \dots, x_{m+k-1}]\Phi}\Big[\Phi^{(j)}(x) - L_{k-1}^{(j)}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x)\Big], j > 0. \tag{2.10}$$

Lemma 3. *Let*

$$R_k(\Phi, x) = \frac{\Phi^{(j)}(x) - L_{k-1}^{(j)}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x)}{\Phi(x_{m+k-1}) - L_{k-1}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x_{m+k-1})}. \tag{2.11}$$

Then for any $x \in [x_m, x_{m+k-1}]$

$$\Big|u^{(j)}(x) - u_{\Phi,k}^{(j)}(x)\Big| \le \max_s |p^{(k-1)}(s)| \Big[\frac{1}{(k-1)!}\Big|\frac{d^j}{d\,x^j}\prod_{j=0}^{k-2}(x - x_{m+j})\Big| +$$
$$+h^{k-1}|R_k(\Phi, x)|\Big], \ s \in [x_m, x_{m+k-1}]. \tag{2.12}$$

Proof. The formula (2.10) is exact for the function $\Phi(x)$, then

$$\Big|u^{(j)}(x) - u_{\Phi,k}^{(j)}(x)\Big| \le \Big|p^{(j)}(x) - L_{k-1}^{(j)}(p, x_m, x_{m+1}, \dots, x_{m+k-2}, x)\Big| +$$
$$+\Big|\frac{[x_m, x_{m+1}, \dots, x_{m+k-1}]p}{[x_m, x_{m+1}, \dots, x_{m+k-1}]\Phi}\Big| \ \Big|\Phi^{(j)}(x) - L_{k-1}^{(j)}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x)\Big|. \tag{2.13}$$

Taking into account (2.7), we obtain

$$\Big|u^{(j)}(x) - u_{\Phi,k}^{(j)}(x)\Big| \le \Big|p^{(j)}(x) - L_{k-1}^{(j)}(p, x_m, x_{m+1}, \dots, x_{m+k-2}, x)\Big| +$$
$$+\Big|p(x_{m+k-1}) - L_{k-1}(p, x_m, x_{m+1}, \dots, x_{m+k-2}, x_{m+k-1})\Big| \times$$
$$\Big|\Phi^{(j)}(x) - L_{k-1}^{(j)}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x)\Big| \Big/$$
$$\Big|\Phi(x_{m+k-1}) - L_{k-1}(\Phi, x_m, x_{m+1}, \dots, x_{m+k-2}, x_{m+k-1})\Big|. \tag{2.14}$$

Using (2.2), (2.11) in (2.14), we obtain

$$\Big|u^{(j)}(x) - u_{\Phi,k}^{(j)}(x)\Big| \le \Big|p^{(j)}(x) - L_{k-1}^{(j)}(p, x_m, x_{m+1}, \dots, x_{m+k-2}, x)\Big| +$$
$$+h^{k-1} \max_s |p^{(k-1)}(s)| \ |R_k(\Phi, x)|. \tag{2.15}$$

According to ([11], p. 52)

$$\Big|p^{(j)}(x) - L_{k-1}^{(j)}(p, x_m, x_{m+1}, \dots, x_{m+k-2}, x)\Big| \le$$
$$\frac{1}{(k-1)!}\max_s |p^{(k-1)}(s)| \Big|\frac{d^j}{d\,x^j}\prod_{j=0}^{k-2}(x - x_{m+j})\Big|. \tag{2.16}$$

Now we obtain (2.12) from (2.15) and (2.16). The lemma is proved.

3. Particular cases of constructed formulas

Interpolation formulas.

Let $k=2$. Then the formula (2.3) for the interval $[x_{n-1},x_n]$ has a form:

$$u_{\Phi,2}(x)=u_{n-1}+(u_n-u_{n-1})\frac{\Phi(x)-\Phi_{n-1}}{\Phi_n-\Phi_{n-1}},\quad \Phi_n=\Phi(x_n). \tag{3.1}$$

Let $k=3$. Then the formula (2.3) can be written in a form:

$$u_{\Phi,3}(x)=u_{n-1}+\frac{u_n-u_{n-1}}{h}(x-x_{n-1})+\frac{u_{n+1}-2u_n+u_{n-1}}{\Phi_{n+1}-2\Phi_n+\Phi_{n-1}}\times$$
$$\times\Big[\Phi(x)-\Phi_{n-1}-\frac{\Phi_n-\Phi_{n-1}}{h}(x-x_{n-1})\Big], x\in[x_{n-1},x_{n+1}]. \tag{3.2}$$

Let $k=4$. The formula (2.3) for the interval $[x_{n-1},x_{n+2}]$ has a form:

$$u_{\Phi,4}(x)=u_{n-1}+\frac{u_n-u_{n-1}}{h}(x-x_{n-1})+\frac{u_{n+1}-2u_n+u_{n-1}}{2h^2}(x-x_{n-1})(x-x_n)+$$
$$+G_4\Big[\Phi(x)-\Phi_{n-1}-\frac{\Phi_n-\Phi_{n-1}}{h}(x-x_{n-1})-\frac{\Phi_{n+1}-2\Phi_n+\Phi_{n-1}}{2h^2}\times$$
$$(x-x_{n-1})(x-x_n)\Big], \tag{3.3}$$

where

$$G_4=\frac{u_{n+2}-3u_{n+1}+3u_n-u_{n-1}}{\Phi_{n+2}-3\Phi_{n+1}+3\Phi_n-\Phi_{n-1}}.$$

Let $k=5$. The formula (2.3) for the interval $[x_{n-2},x_{n+2}]$ has a form:

$$u_{\Phi,5}(x)=u_{n-1}+\frac{u_n-u_{n-1}}{h}(x-x_{n-1})+\frac{u_{n+1}-2u_n+u_{n-1}}{2h^2}(x-x_{n-1})(x-x_n)+$$
$$+\frac{u_{n+2}-3u_{n+1}+3u_n-u_{n-1}}{3h^3}(x-x_{n-2})(x-x_{n-1})(x-x_n)+$$
$$+G_5\Big[\Phi(x)-\Phi_{n-1}-\frac{\Phi_n-\Phi_{n-1}}{h}(x-x_{n-1})-\frac{\Phi_{n+1}-2\Phi_n+\Phi_{n-1}}{2h^2}(x-x_{n-1})\times$$
$$(x-x_n)-\frac{\Phi_{n+2}-3\Phi_{n+1}+3\Phi_n-\Phi_{n-1}}{3h^3}(x-x_{n-2})(x-x_{n-1})(x-x_n)\Big], \tag{3.4}$$

where

$$G_5=\frac{u_{n+2}-4u_{n+1}+6u_n-4u_{n-1}+u_{n-2}}{\Phi_{n+2}-4\Phi_{n+1}+6\Phi_n-4\Phi_{n-1}+\Phi_{n-2}}.$$

Difference formulas for a derivative. Now we consider difference formulas for the first derivative, obtained from (2.10).

Let $k=2$. Then

$$u'_{\Phi,2}(x)=\frac{u_n-u_{n-1}}{\Phi_n-\Phi_{n-1}}\Phi'(x),\ x\in[x_{n-1},x_n]. \tag{3.5a}$$

Taking $\Phi(x)=x$, we obtain

$$L'_2(x)=\frac{u_n-u_{n-1}}{h}.\ \ x\in[x_{n-1},x_n],\ 1\le n\le N. \tag{3.5b}$$

We proved in [4] that if $\Phi(x)=\exp(-a_0\varepsilon^{-1}x)$ and $\Phi(x)$ correspods to the problem (1.2), then

$$\varepsilon|u'_{\Phi,2}(x)-u'(x)|\le Ch,\ x\in[x_{n-1},x_n],\ 1\le n\le N, \tag{3.6}$$

where constant C does not depend on ε.

If we use the classical formula (3.5b), then for the function $u(x) = \exp(-\varepsilon^{-1}x)$ and $\varepsilon = h$ we have

$$\varepsilon|L_2'(0) - u'(0)| = e^{-1}.$$

The relative error does not decrease, when $h \to 0$.

Let $k = 3$. Then in the interval $[x_{n-1}, x_{n+1}]$ we have

$$u'_{\Phi,3}(x) = \frac{u_n - u_{n-1}}{h} + \frac{u_{n+1} - 2u_n + u_{n-1}}{\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}}\Big[\Phi'(x) - \frac{\Phi_n - \Phi_{n-1}}{h}\Big]. \quad (3.7a)$$

Taking $\Phi(x) = x^2$, we obtain

$$L_3'(x) = \frac{u_n - u_{n-1}}{h} + \frac{u_{n+1} - 2u_n + u_{n-1}}{2h^2}(2x - x_n - x_{n-1}). \quad (3.7b)$$

Let $k = 4$. Then in the interval $[x_{n-1}, x_{n+2}]$ we have

$$u'_{\Phi,4}(x) = \frac{u_n - u_{n-1}}{h} + \frac{u_{n+1} - 2u_n + u_{n-1}}{2h^2}(2x - x_{n-1} - x_n) +$$

$$+ G_4\Big[\Phi'(x) - \frac{\Phi_n - \Phi_{n-1}}{h} - \frac{\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}}{2h^2}(2x - x_{n-1} - x_n)\Big], \quad (3.8)$$

where G_4 corresponds to (3.3).

Let $k = 5$. Then in the interval $[x_{n-2}, x_{n+2}]$ we obtain

$$u'_{\Phi,5}(x) = \frac{u_n - u_{n-1}}{h} + \frac{u_{n+1} - 2u_n + u_{n-1}}{2h^2}(2x - x_{n-1} - x_n) +$$

$$+ \frac{u_{n+2} - 3u_{n+1} + 3u_n - u_{n-1}}{3h^3} r(x) +$$

$$+ G_5\Big[\Phi'(x) - \Big\{\frac{\Phi_n - \Phi_{n-1}}{h} + \frac{\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}}{2h^2}(2x - x_{n-1} - x_n) +$$

$$+ \frac{\Phi_{n+2} - 3\Phi_{n+1} + 3\Phi_n - \Phi_{n-1}}{3h^3} r(x)\Big\}\Big], \quad (3.9)$$

where $r(x) = ((x - x_{n-2})(x - x_{n-1})(x - x_n))'$ and G_5 corresponds to (3.4).

Using (2.10), we can write difference formulas for high derivatives.

4. Numerical results

Let a function with a boundary layer component

$$u(x) = cos(\pi x) + \exp(-x/\varepsilon), \quad 0 \le x \le 1, \ \varepsilon > 0$$

is given at nodes of an uniform grid. Define the interpolation error:

$$\Delta = \max_{1 \le n \le N} |u(\tilde{x}_n) - u_{int}(\tilde{x}_n)|, \quad \tilde{x}_n = (x_n + x_{n-1})/2,$$

where $u_{int}(x)$ is the tested spline-interpolation.

In numerical experiments we divide the interval $[0, 1]$ into equal intervals of the length h, $2h$, $3h$ and $4h$ accordingly, on each of them we perform the interpolation. In tables $e \pm m$ means $10^{\pm m}$.

Table 1 presents the error Δ of the formula (3.1) for different ε and N. The interpolation error is the quantity of the order $O(h^2)$ for $\varepsilon = 1$ and of the order $O(h)$ for small values of ε.

Table 2 presents the error Δ of the formula (3.2) depending on ε and N. The interpolation error of the order $O(h^3)$ if $\varepsilon = 1$ and of the order $O(h^2)$ for small values of ε.

Table 3 similar presents the error Δ of the formula (3.3). The interpolation error of the order $O(h^4)$ if $\varepsilon = 1$ and of the order $O(h^3)$ for small values of ε.

Table 4 presents the error Δ of the formula (3.4) depending on ε and N. The interpolation error of the order $O(h^5)$ if $\varepsilon = 1$ and of the order $O(h^4)$ for small values of ε.

Tables 1-4 confirm the estimate

$$\Big|u_{\Phi,k}(x) - u(x)\Big| \le Ch^{k-1},$$

corresponding to Lemma 2. When $u(x)$ has not large derivatives ($\varepsilon = 1$), the order of the accuracy increases to $O(h^k)$. It corresponds to the estimate before (2.10).

Now we consider results of experiments with using of Lagrange polynomials.

Table 5 presents the error Δ of the linear interpolation $L_2(x)$ constructed in every mesh interval $[x_{n-1}, x_n]$. In Table 6 is presented the error Δ of the quadratic interpolation $L_3(x)$ in every interval $[x_{n-1}, x_{n+1}]$, $n = 1, 3, \ldots, N-1$. We see the loss of the accuracy for small values of the parameter ε. We have similar results for polynomials $L_4(x), L_5(x)$.

Now we consider results of calculations with difference formulas for the first derivative. The derivative $u'(x)$ is a quantity of the order $O(\varepsilon^{-1})$, therefore, we calculate the relative error.

Table 7 presents the error

$$\Delta = \max_n \varepsilon |u'(x_n) - u'_{\Phi,3}(x_n)|,\ 0 \le n \le N,$$

where $u'_{\Phi,3}(x)$ corresponds to (3.7a).

Table 8 presents the error

$$\Delta = \max_n \varepsilon |u'(x_n) - L'_3(x_n)|,\ 0 \le n \le N,$$

for the classical difference formula (3.7b). We have the essential error if $\varepsilon = h$. We have similar results for difference formulas with other number of nodes. Formulas, based on Lagrange polynomials, lead to essential errors if $\varepsilon = h$.

Table 1. The error of the interpolant $u_{\Phi,2}(x)$

ε	N					
	$3*2^3$	$3*2^4$	$3*2^5$	$3*2^6$	$3*2^7$	$3*2^8$
1	$6.63e-3$	$1.67e-3$	$4.20e-4$	$1.05e-4$	$2.63e-5$	$6.58e-6$
10^{-1}	$1.87e-2$	$5.00e-3$	$1.29e-3$	$3.29e-4$	$8.30e-5$	$2.08e-5$
10^{-2}	$6.36e-2$	$2.67e-2$	$9.12e-3$	$2.69e-3$	$7.33e-4$	$1.91e-4$
10^{-3}	$6.54e-2$	$3.27e-2$	$1.64e-2$	$8.10e-3$	$3.61e-3$	$1.31e-3$
10^{-4}	$6.54e-2$	$3.27e-2$	$1.64e-2$	$8.18e-3$	$4.09e-3$	$2.05e-3$
10^{-5}	$6.54e-2$	$3.27e-2$	$1.64e-2$	$8.18e-3$	$4.09e-3$	$2.04e-3$

TABLE 2. The error of the interpolant $u_{\Phi,3}(x)$

ε	N					
	$3*2^3$	$3*2^4$	$3*2^5$	$3*2^6$	$3*2^7$	$3*2^8$
1	$1.47e-4$	$1.84e-5$	$2.30e-6$	$2.87e-7$	$3.59e-8$	$4.49e-9$
10^{-1}	$4.87e-4$	$6.00e-5$	$7.40e-6$	$9.19e-7$	$1.15e-7$	$1.43e-8$
10^{-2}	$4.61e-3$	$6.34e-4$	$7.69e-5$	$9.23e-6$	$1.12e-6$	$1.38e-7$
10^{-3}	$6.38e-3$	$1.60e-3$	$3.96e-4$	$8.26e-5$	$1.23e-5$	$1.52e-6$
10^{-4}	$6.38e-3$	$1.60e-3$	$4.01e-4$	$1.00e-4$	$2.51e-5$	$6.25e-6$
10^{-5}	$6.38e-3$	$1.60e-3$	$4.01e-4$	$1.00e-4$	$2.51e-5$	$6.27e-6$

TABLE 3. The error of the interpolant $u_{\Phi,4}(x)$

ε	N					
	$3*2^3$	$3*2^4$	$3*2^5$	$3*2^6$	$3*2^7$	$3*2^8$
1	$1.20e-5$	$7.55e-7$	$4.71e-8$	$2.94e-9$	$1.84e-10$	$1.15e-11$
10^{-1}	$4.12e-5$	$2.50e-6$	$1.52e-7$	$9.44e-9$	$5.87e-10$	$3.66e-11$
10^{-2}	$4.68e-4$	$2.99e-5$	$1.70e-6$	$9.81e-8$	$5.86e-9$	$3.57e-10$
10^{-3}	$6.89e-4$	$8.72e-5$	$1.08e-5$	$1.08e-6$	$7.46e-8$	$4.28e-9$
10^{-4}	$6.89e-4$	$8.72e-5$	$1.09e-5$	$1.37e-6$	$1.71e-7$	$2.13e-8$
10^{-5}	$6.89e-4$	$8.72e-5$	$1.09e-5$	$1.37e-6$	$1.71e-7$	$2.14e-8$

TABLE 4. The error of the interpolant $u_{\Phi,5}(x)$

ε	N					
	$3*2^3$	$3*2^4$	$3*2^5$	$3*2^6$	$3*2^7$	$3*2^8$
1	$1.11e-6$	$3.45e-8$	$1.08e-9$	$3.37e-11$	$1.05e-12$	$3.46e-14$
10^{-1}	$3.86e-6$	$1.15e-7$	$3.51e-9$	$1.08e-10$	$3.37e-12$	$1.05e-13$
10^{-2}	$5.02e-5$	$1.51e-6$	$4.10e-8$	$1.15e-9$	$3.40e-11$	$1.03e-12$
10^{-3}	$7.76e-5$	$4.98e-6$	$3.07e-7$	$1.50e-8$	$4.84e-10$	$1.31e-11$
10^{-4}	$7.76e-5$	$4.98e-6$	$3.13e-7$	$1.96e-8$	$1.22e-9$	$7.61e-11$
10^{-5}	$7.76e-5$	$4.98e-6$	$3.13e-7$	$1.96e-8$	$1.22e-9$	$7.66e-11$

TABLE 5. The interpolation error of the polynomial $L_2(x)$

ε	N					
	$3*2^3$	$3*2^4$	$3*2^5$	$3*2^6$	$3*2^7$	$3*2^8$
1	$2.21e-3$	$5.56e-4$	$1.39e-4$	$3.47e-5$	$8.68e-6$	$2.17e-8$
10^{-1}	$1.55e-2$	$4.36e-3$	$1.16e-3$	$2.97e-4$	$7.53e-5$	$1.90e-5$
10^{-2}	$3.81e-1$	$2.08e-1$	$8.23e-2$	$2.62e-2$	$7.44e-3$	$1.98e-3$
10^{-3}	$4.98e-1$	$5.00e-1$	$4.94e-1$	$4.29e-1$	$2.65e-1$	$1.14e-1$
10^{-4}	$4.98e-1$	$4.99e-1$	$5.00e-1$	$5.00e-1$	$5.00e-1$	$4.99e-1$

TABLE 6. The interpolation error of the polynomial $L_3(x)$

ε	N					
	$3*2^3$	$3*2^4$	$3*2^5$	$3*2^6$	$3*2^7$	$3*2^8$
1	$1.36e-4$	$1.72e-5$	$2.15e-6$	$2.68e-7$	$3.36e-8$	$4.19e-9$
10^{-1}	$3.15e-3$	$4.71e-4$	$6.45e-5$	$8.43e-6$	$1.08e-6$	$1.36e-7$
10^{-2}	$2.62e-1$	$1.14e-1$	$3.00e-2$	$5.68e-3$	$8.82e-4$	$1.23e-4$
10^{-3}	$3.75e-1$	$3.75e-1$	$3.70e-1$	$3.05e-1$	$1.58e-1$	$4.82e-2$
10^{-4}	$3.75e-1$	$3.75e-1$	$3.75e-1$	$3.75e-1$	$3.75e-1$	$3.74e-1$

TABLE 7. The error of the difference formula on a base $u'_{\Phi,3}(x)$

ε	N					
	10	10^2	10^3	10^4	10^5	10^6
1	$5.39e-2$	$5.42e-4$	$5.42e-6$	$5.42e-8$	$5.75e-10$	$7.35e-11$
10^{-1}	$1.66e-2$	$1.72e-4$	$1.72e-6$	$1.72e-8$	$1.74e-10$	$3.13e-11$
10^{-2}	$4.80e-3$	$1.59e-4$	$1.64e-6$	$1.65e-8$	$1.65e-10$	$3.80e-12$
10^{-3}	$4.81e-4$	$4.93e-5$	$1.60e-6$	$1.64e-8$	$1.65e-10$	$1.85e-12$
10^{-4}	$4.81e-5$	$4.93e-6$	$4.93e-7$	$1.59e-8$	$1.64e-10$	$1.66e-12$
10^{-5}	$4.81e-6$	$4.93e-7$	$4.93e-8$	$4.93e-9$	$1.59e-10$	$1.65e-12$

TABLE 8. The error of the difference formula on a base $L'_3(x)$

ε	N					
	10	10^2	10^3	10^4	10^5	10^6
1	$5.04e-2$	$5.07e-4$	$5.07e-6$	$5.07e-8$	$5.39e-10$	$3.27e-10$
10^{-1}	$2.06e-2$	$1.36e-3$	$1.63e-5$	$1.66e-7$	$1.67e-9$	$3.65e-11$
10^{-2}	$5.14e-4$	$2.37e-2$	$1.37e-3$	$1.63e-5$	$1.66e-7$	$1.67e-9$
10^{-3}	$5.14e-5$	$2.24e-6$	$2.37e-2$	$1.36e-3$	$1.63e-5$	$1.66e-7$
10^{-4}	$5.14e-6$	$5.17e-8$	$2.27e-6$	$2.37e-2$	$1.37e-3$	$1.63e-5$
10^{-5}	$5.14e-7$	$5.17e-9$	$5.17e-11$	$2.27e-6$	$2.37e-2$	$1.37e-3$

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