S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 9, стр. 456–459 (2012) УДК 515.16
MSC 57N65

ON MORSE THEORY FOR MANIFOLDS WITH CROSS PRODUCTS

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Abstract. We consider the finite-dimensional Morse theory for closed Riemannian manifolds equipped with the vector cross product on the tangent bundle. These are, for example, G_2-manifolds. Under some conditions toric actions generate the Morse–Bott function, whose gradient trajectories are explicit. This allows us to construct the Morse–Bott complex and calculate the real cohomology ring of the manifold.

Keywords: Morse theory, toric action

1. Introduction

It is well-known that the definition of symplectic manifolds takes its origin in classical mechanics. Namely, symplectic manifolds are generalizations of phase spaces of mechanical systems. The dynamics in phase space are described by the Hamilton equations, which imply an existence of the real-valued function called Hamiltonian. Frankel was first to show that the Hamiltonian may serve as the Morse–Bott function [1].

Theorem 1.1 (Frankel). *Let M be a simply-connected connected closed Kähler manifold. Suppose that there exists an isometric S^1-action on M that preserves the Kähler structure. Let X be the Killing vector field generated by this action. Then $f : M \to \mathbb{R}$ such that*

$$df^X = i_X\omega, \quad \textit{or equivalently} \quad -\nabla f^X = JX, \tag{1.1}$$

Egorov, D.V., On Morse theory for manifolds with cross products.

This work is supported in part by the Russian Federation president grants (NSh-544.2012.1 and MK-842.2011.1); RFBR grants 12-01-00873-a, 12-01-31020-mol_a; Federal purpose program "Scientific and pedagogical staff of innovative Russia arrangement 1.2.2.

Received April, 2, 2012, published October, 19, 2012.

is a perfect Morse–Bott function.

Remark 1.2. One can substitute S^1-action with an action of torus. Then the Killing vector field is generated by the 1-parametric dense subgroup of the torus.

Remark 1.3. Frankel notes that his theorem can be easily generalized to symplectic manifolds.

Using (1.1), one can construct the filtered Morse–Bott complex [2]. Under some conditions on the Morse–Bott function the spectral sequence of this complex gives the real cohomology ring of the manifold. For finite-dimensional case this is proved in [3].

Theorem 1.4 (Austin–Braam). *Let M be a closed manifold and let $f : M \to \mathbb{R}$ be a Morse–Bott function. Assume that f satisfies the following conditions.*

(1) *If a gradient trajectory connects critical submanifold α to critical submanifold β, then the index of α is strictly greater than the index of β.*
(2) *Stable and unstable manifolds of any critical submanifold intersect transversely.*
(3) *All critical submanifolds and negative normal bundles over them are oriented.*

Then the cohomology of the Morse–Bott complex of f is isomorphic to the de Rham cohomology of M.

In this paper we consider Morse–Bott functions such that $-\nabla f^{X,Y} = X \times Y$, where $\times$ is the vector cross product, X and Y are vector fields generated by toric actions.

2. The main theorem

Let M be a simply-connected connected Riemannian manifold; let $\times$ be a vector cross product, determined by a real 3-form Ω

$$X \times Y := (i_Y i_X \Omega)^\natural,$$

where $X, Y \in \Gamma(TM)$, i is the inner product, and $\natural$ is the duality between vectors and 1-forms.

Manifolds admitting the vector cross product are Riemannian 3-manifolds and G_2 structure manifolds.

Theorem 2.1. *Suppose that there exist two toric actions on M such that their fixed point sets are non-empty and do not intersect. Suppose that vector fields X and Y generated by these actions are linearly independent everywhere except critical points. If*

$$d i_X i_Y \Omega = 0, \tag{2.1}$$

then $f : M \to \mathbb{R}$ such that

$$df^{X,Y} = i_X i_Y \Omega,$$

or equivalently

$$-\nabla f^{X,Y} = X \times Y$$

is the Morse–Bott function.

Доказательство. The proof follows the one by Frankel. Since M is simply-connected, (2.1) implies existence of $f^{X,Y} : M \to \mathbb{R}$ such that $df^{X,Y} = i_X i_Y \Omega$ or

$$\frac{\partial f}{\partial x_k} = \Omega_{ijk} Y^i X^j.$$

Differentiating, we obtain Hessian $\mathcal{H}$. If for example $Y = 0$, then $\mathcal{H}$ equals to

$$\mathcal{H}_{kl} = \frac{\partial^2 f}{\partial x_k \partial x_l} = \Omega_{ijk} X^i \frac{\partial Y^j}{\partial x_l} = \omega_{jk} S^j_l,$$

where ω is a non-degenerate 2-form on the complement to X and S is an adjoint action generated by Y.

If $SZ = 0$, then Z is tangent to the critical submanifold of f. Hence, $\mathcal{H}$ is non-degenerate on the vectors normal to critical submanifold and f is the Morse–Bott function.

□

Remark 2.2. If X and Y vanish simultaneously or happen to be linearly dependent, then the Hessian degenerates.

The following special cases of the main theorem are more convenient for possible applications.

Proposition 2.3. *If in the statement of the main theorem toric actions preserve Ω and commute, then* (2.1) *can be omitted.*

Доказательство. Since Ω is closed with respect to d, L_X, and L_Y, the RHS of

$$di_X i_Y \Omega = i_{[X,Y]}\Omega - i_X di_Y \Omega + i_Y di_X \Omega - i_Y i_X d\Omega$$

vanishes. Here Cartan's formula $L_X = di_X + i_X d$ is used.

□

By Bochner's theorem, if a manifold is Ricci flat, then any Killing vector field is parallel [4]. Since G_2-structure canonically determines the Riemannian metric, Proposition 2.3 can not be applied to G_2 holonomy manifolds.

Example 2.4. Consider a 3-sphere made of two solid tori T_1 and T_2 glued by the diffeomorphism of boundaries, which maps meridians to parallels and vice versa. Let r_i, φ_i, and ψ_i be the coordinate chart on T_i, where φ_i and ψ_i are along parallels and meridians respectively.

We define two S^1-actions $\mathcal{S}_1$ and $\mathcal{S}_2$. The $\mathcal{S}_1$ is a shift along φ_1 together with a shift along ψ_2. The second action is symmetric to the first. The fixed point set C_i of $\mathcal{S}_i$ is the central parallel $r_i = 0$.

The action $\mathcal{S}_1$ generates smooth vector field $X = \frac{\partial}{\partial \varphi_1}$ and $X = \frac{\partial}{\partial \psi_2}$. The action $\mathcal{S}_2$ generates Y symmetric to X. Then form $i_X i_Y \Omega$ is equal to $r_1 dr_1$ on T_1 and to $-r_2 dr_2$ on T_2. By the main theorem, smooth function $f = (r_1)^2$ on T_1 and $f = -(r_2)^2$ on T_2 is the Morse–Bott function.

The E_1 term of the spectral sequence of the Morse–Bott complex is

$$\begin{matrix} 0 & 0 & 0 & 0 \\ \mathbb{R} & 0 & \mathbb{R} & 0 \\ \mathbb{R} & 0 & \mathbb{R} & 0 \end{matrix}.$$

The differential $d^2_{2,0} : \mathbb{R} \to \mathbb{R}$ is given by integrating the volume form over the moduli space $\widetilde{\mathcal{M}}(C_1, C_2) = T^2$ of gradient trajectories form C_1 to C_2. Therefore, $d_2^{0,1}(\omega) = 1$ and the E_∞ is

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{R} & 0 \\ \mathbb{R} & 0 & 0 & 0 \end{matrix}.$$

We obtain that $H^0(S^3;\mathbb{R}) = H^3(S^3;\mathbb{R}) = \mathbb{R}$ and all other groups vanish.

Proposition 2.5. *If in the statement of the main theorem*

(1) $L_X\Omega = 0$*, where* Ω *is a closed* G_2*-form;*
(2) $L_Y i_X\Omega = 0$*,*

then (2.1) *can be omitted.*

Доказательство.

$$di_Y i_X\Omega = L_Y i_X\Omega - i_Y di_X\Omega = L_Y i_X\Omega - i_Y L_X\Omega = 0.$$

□

Since X is the Killing vector field, the critical point set of the Morse–Bott function coincides with the zero set of Y.

Example 2.6. Consider $M = N \times S^1$, where N is the Calabi–Yau manifold. Then $\Omega_M = \Omega_N + \omega_N \wedge dt$, where Ω_N is a real part of the holomorphic volume form on N and ω_N is a Kähler form. $X = \partial/\partial t$ is the Killing vector field. Then

$$0 = L_Y i_X\Omega_M = L_Y\omega_N.$$

It means that Y preserves the Kähler form on N. Note that, if Y preserves the complex structure, then Y is Killing and parallel.

The author would like to thank the referee for pointing out numerous typos.

References

[1] T. Frankel, *Fixed points and torsion on Kähler manifold* // Ann. of Math., **70**:1 (1959), 1–8. MR0131883

[2] R. Bott, *The stable homotopy of the classical groups* // Ann. of Math., **70**:2 (1959), 313–337. MR0110104

[3] D.M. Austin, P.J. Braam, *Morse–Bott theory and equivariant cohomology* // The Floer Memorial Volume (H. Hofer, C.H. Taubes, A. Weinstein and E. Zehnder, eds.), Progress in Math. 133, Birkhäuser, 1995, 123–183. MR1362827

[4] S. Bochner, *Vector fields and Ricci curvature* // Bull. AMS, **52** (1946), 776–797. MR0018022

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