S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 9, стр. 460–463 (2012) УДК 511.1
MSC 11A07

WOLSTENHOLME'S THEOREM FOR BINOMIAL COEFFICIENTS

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Abstract. We prove that the numerator of $\sum_{i=k}^{p-1} \binom{i}{k}^{-1}$ is divisible by p^2 for infinitely many primes p if and only if $k = 1$.

Keywords: Binomial coefficient, Wolstenholme's theorem.

For rational numbers $\alpha = a/b$ and $\beta = c/d$ write $\alpha \equiv \beta \pmod{n}$, if $ad - bc$ is divisible by n and denominators b, d are relatively prime with n.

The following result was proved by Wolstenholme in [1]. For any prime number $p > 3$,

$$\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \pmod{p^2}.$$

In our paper we consider the extension of Wolstenholme theorem to the class of binomial coefficients. Does there exist a number k, $k > 1$ such that

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}$$

for any $p > k$? We show that it is impossible.

Theorem 1. *For a given $k \geq 1$ the congruence*

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}, \quad 1 \leq k < p,$$

takes place for infinitely many primes p if and only if $k = 1$.

Dzhumadil'daev A.S., Yeliussizov D.A., Wolstenholme's theorem for binomial coefficients.

Received February, 14, 2012, published October, 23, 2012.

Before the proof of our main result let us introduce some notations

$$F_k(n) = \sum_{i=k}^{n} \frac{1}{\binom{i}{k}},$$

$$H_{a,b} = \sum_{1\le i_1<\cdots<i_b\le a} \frac{1}{i_1\cdots i_b}, \qquad b>0,$$

$$H_{a,0} = 1.$$

We establish the following slightly more general result.

Lemma 1. *For any integer p (not necessary prime) and $1 < k < p$,*

$$F_k(p-1) = \frac{(1+(-1)^k)k}{k-1} + \frac{k}{(k-1)\binom{p-1}{k-1}} \sum_{i=1}^{k-1} (-p)^i H_{k-1,i}. \tag{1}$$

Proof. Let ∇ be a difference operator

$$\nabla f(x) = f(x) - f(x-1).$$

Then for $\phi(x) = \frac{1}{\binom{x}{k-1}}$ we have

$$\frac{1}{\binom{i}{k}} = \frac{k}{k-1}\left(\frac{1}{\binom{i-1}{k-1}} - \frac{1}{\binom{i}{k-1}}\right) = -\frac{k}{k-1}\nabla\phi(i).$$

Since

$$F_k(p-1) = -\frac{k}{k-1}\sum_{i=k}^{p-1}\nabla\phi(i) = \frac{k}{k-1}(\phi(k-1)-\phi(p-1)) = \frac{k}{k-1}\left(1-\frac{1}{\binom{p-1}{k-1}}\right),$$

we obtain

$$F_k(p-1) = \frac{k}{(k-1)\binom{p-1}{k-1}}\left(\binom{p-1}{k-1}-1\right). \tag{2}$$

Let $s_{n,i}$ be Stirling numbers of the first kind. The following relations are well-known

$$x(x-1)\cdots(x-k+1) = \sum_{i=1}^{k} s_{k,i}x^i, \tag{3}$$

$$s_{n,i} = (-1)^{n+i}(n-1)!H_{n-1,i-1}. \tag{4}$$

By (3) and (4)

$$\binom{p-1}{k-1} = \frac{\sum_{i=1}^{k} s_{k,i}p^{i-1}}{(k-1)!} = \sum_{i=1}^{k}(-1)^{k+i}H_{k-1,i-1}p^{i-1}.$$

Hence,

$$\binom{p-1}{k-1} = (-1)^{k+1} - \sum_{i=1}^{k-1}(-1)^{k+i}H_{k-1,i}p^i. \tag{5}$$

Thus, we obtain

$$F_k(p-1) = -\frac{k}{(k-1)\binom{p-1}{k-1}}\{((-1)^k+1) + \sum_{i=1}^{k-1}(-1)^{k+i}p^iH_{k-1,i}\} \tag{6}$$

It is not difficult to see that (6) is equivalent to (1). If k is odd, it is obvious. If k is even, it follows from (5).

Lemma 2. *Let p be a prime number. If $1 < k < p$ then*

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv \frac{(1+(-1)^k)k}{k-1} \pmod{p}, \tag{7}$$

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv \frac{(1+(-1)^k)k}{k-1} + \frac{(-1)^k pk}{k-1} \sum_{i=1}^{k-1} \frac{1}{i} \pmod{p^2}. \tag{8}$$

Proof. The equalities (7) and (8) are easy consequences of Lemma 1. In case of (8) we use the congruence $\binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod{p}$.

Lemma 3. *Let p be a prime number and $1 < k < p$. Then the following conditions are equivalent*

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}, \tag{9}$$

$$k \textit{ is odd and } \sum_{i=1}^{k-1} \frac{1}{i} \equiv 0 \pmod{p}. \tag{10}$$

Proof. Suppose that (9) is true, $F_k(p-1) \equiv 0 \pmod{p^2}$. Then $F_k(p-1) \equiv 0 \pmod{p}$. Therefore, by (7) k is odd. Hence by Lemma 2, relation (8), we have $H_{k-1,1} \equiv 0 \pmod{p}$. So, (9) implies (10).

Conversely, if (10) is given then by (8) we have $F_k(p-1) \equiv 0 \pmod{p^2}$. From Lemma 2 we get the sufficient part of the lemma.

Proof of Theorem 1. For given $k > 1$ the numerator of the sum $\sum_{i=1}^{k-1} \frac{1}{i}$ has a finite number of prime divisors. Therefore, the congruence

$$\sum_{i=1}^{k-1} \frac{1}{i} \equiv 0 \pmod{p}$$

holds only for a finite number of primes $p \geq k$. So, if $k > 1$, then by Lemma 3 the congruence

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}$$

might be true only for a finite number of primes $p > k$.

If $k = 1$, by Wolstenholme's theorem our statement is true

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}$$

for any $p > 3$. Theorem 1 is proved.

References

[1] J. Wolstenholme, *On certain properties of prime numbers* // Quaterly J. Pure and Applied Math., **5** (1862), 35–39.

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