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ON RECTIFIABLE CURVES, ADDITIVE VECTOR FUNCTIONS, AND THE MINKOWSKI SUM OF STRAIGHT LINE SEGMENTS

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Abstract. This is a translation into English of the classsical paper of Yu. G Reshetnyak and V. A. Zalgaller “On rectifiable curves, additive vector functions, and the Minkowski sum of straight line segments”.

1. Introduction

1. The operation of the Minkowski sum of bodies was introduced into geometry by Brunn and Minkowski. Recall the appropriate definition. Given some bodies P_i and nonnegative numbers λ_i for $i = 1, 2, \dots, m$, the Minkowski sum, or linear combination, of P_i with coefficients λ_i is the body P traced by the endpoint of the variable vector

$$\mathbf{r} = \sum_{i=1}^{m} \lambda_i \mathbf{r}_i$$

when the endpoints of the vectors $\mathbf{r}_i$ run over the corresponding P_i independently of each other.

The Minkowski sum of convex bodies is again a convex body. Parallel translation of some P_i yields a parallel translation of P.

Everything under discussion in this article refers to the $n$dimensional Euclidean space. As a rule, we consider bodies up to parallel translation, without further specification, and only if need be we mark the location of bodies with respect to the origin.

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2. Refer as a zonotope[1] to any body expressible as the Minkowski sum of finitely many straight line segments.[2] Every zonotope turns out a convex centrally symmetric polyhedron, whose faces (of all dimensions) have centers of symmetry. The Minkowski sum of straight line segments appeared in a series of articles (see [1, pp. 28–29] for instance). Chuĭkina [2] used the term "parallelohedron" in the same sense as above with the difference that she also included in the definition the containment of the origin in a parallelohedron.

3. The goal of this article is to deal with the bodies expressible as limits of parallelohedra, as well as to elucidate their connection with rectifiable curves and use it for clearing up some questions of the theory of curves and additive vector functions.

The Minkowski sum of finitely many straight line segments, out of which we can compose broken lines by parallel translation, is not difficult to generalize to the Minkowski sum of line elements of a rectifiable curve. Namely, the Minkowski sum of straight line segments $\mathbf{r}_i$, considered up to parallel translation, is the body traced by the endpoint of the vector

$$\mathbf{r} = \sum_{i=1}^{m} \mu_i \, \mathbf{r}_i,$$

when μ_i run over $-\frac{1}{2} \le \mu_i \le \frac{1}{2}$ independently of each other. (This becomes obvious, if beforehand we translate the midpoints of $\mathbf{r}_i$ to the origin.) It is natural to define the integral Minkowski sum of line elements of a rectifiable curve $K = \mathbf{x}(s)$ as the body P filled by the endpoints of the vectors of the form

$$\mathbf{r} = \int_0^l \mu \, \mathbf{x}' \, ds,$$

where $\mu(s)$ is an arbitrary measurable function with $-\frac{1}{2} \le \mu(s) \le \frac{1}{2}$ and l stands for the total length of the curve, while s, for the arc length counted from the beginning of the curve. The integral is understood in the sense of Lebesgue.

Most further results rest on this natural generalization and some theorem on the sequence of curves in Section 1.

2. Rectification of Curves by Rearranging Their Fragments

Theorem 1. *Given a sequence of rectifiable curves $K_1', K_2', \dots$, we can transform each of these curves by rearranging its fragments so that every converging subsequence of transformed curves converges in length.*

We refer as a rearrangement of fragments to the following transformation. Subdivide the curve into finitely many fragments, translate them in the space, and then make up a new curve. Separate fragments of the new curve may be traversed in the opposite direction to the original curve.

But in Theorem 1 it suffices to use rearrangements that preserve the direction of traversal of all fragments.

Firstly, let us prove Theorem 1 for broken lines.

1. Fix a sequence of partitions $\xi_1, \xi_2, \dots$ of a unit sphere Ω (in the $n$dimensional space) into finitely many disjoint Borel sets enumerated somehow and such that:

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[1]Editor of Translation: The author used the term "parallelohedron" which is obsolete now.

[2]They should not be confused with another class of polyhedra, also called parallelohedra, which are capable of filling the entire space without holes and overlaps in parallel lattice-like arrangements.

(1) the maximal diameter of a set in ξ_k tends to zero with the growth of k; (2) ξ_{k+1} is obtained by dividing each set in ξ_k into two parts; (3) the transition from ξ_k to ξ_{k+1} preserves the order of sets in ξ_k, and we only have to determine the order of the two halves of each set of ξ_k.

2. Consider a sequence $L'_1, L'_2, \dots$ of broken lines. The straight line segments of every L'_k are vectors. (In this article we always assume the directions of traversal of all curves fixed.) Take the partition ξ_k and rearrange the straight line segments of L'_k, putting first those whose directions lie in the first set of ξ_k, then the straight line segments directed into the second set of this partition, and so on. Doing likewise for all broken lines L'_k ($k = 1, 2, \dots$), we obtain a sequence $L_1, L_2, \dots$ of broken lines. Verify that it enjoys the required property.

3. Take an arbitrarily small $\delta > 0$, with $\delta < \frac{\pi}{2}$, and fix k_0 for which every pair of vectors directed into the same set of the partition ξ_{k_0} of Ω form an angle less than δ. On every broken line L_k for $k \geq k_0$ we can mark a sequence of vertices A_1, $A_2, \dots, A_m$ such that on the fragment A_iA_{i+1} the straight line segments of the broken line are directed into the ith the set of the partition.

The number of these vertices is $m = 2^{k_0} + 1$, including the endpoints of the curve. (Of course, some points $A_1, \dots, A_m$ may coincide if no straight line segment was directed into a set of ξ_{k_0}.)

The length of every fragment A_iA_{i+1} of the broken line L_k obviously exceeds in at most $\frac{1}{\cos\delta}$ times the distance A_iA_{i+1}[3], whence

$$\text{len. } L_k \leq \frac{1}{\cos\delta} \text{ len. } A_1A_2\dots A_m. \tag{1}$$

4. Suppose that among the broken lines L_1, $L_2, \dots$ there is a subsequence L_{k_1}, $L_{k_2}, \dots$ converging to a curve K. Select a subsequence for which the lengths converge to the upper limit and then select from the latter a subsequence for which the points $A_1, \dots, A_m$ lying on the broken lines converge to some points $B_1, \dots, B_m$ on K.

It is obvious that the lengths of the broken lines $A_1A_2\dots A_m$ converge to the length of the broken line $B_1B_2\dots B_m$. Since the latter is finite, the lengths of $A_1A_2\dots A_m$, and with them, by (1), the lengths of the chosen broken lines L_{k_i}, are jointly bounded. This enables us to claim that K is rectifiable. In addition, the convergence $L_{k_i} \to K$ implies that

$$\text{len. } K \leq \underline{\lim} \text{ len. } L_{k_i}. \tag{2}$$

5. Since K is not shorter than the inscribed broken line $B_1\dots B_m$, whose length is however well approximated by the lengths of $A_1\dots A_m$, from which according to (1) for small δ the lengths of L_{k_i} differ little, by the arbitrariness of δ we may conclude that

$$\text{len. } K > \overline{\lim} \text{ len. } L_{k_i}. \tag{3}$$

It follows from (2) and (3) that len. $L_{k_i} \to$ len. K.

Now we can prove Theorem 1 for curves.

[3]This follows from the fact that the sum of the vectors $\mathbf{a}_i$ forming with the direction $\mathbf{e}$ an angle less than $\delta < \frac{\pi}{2}$, also forms with $\mathbf{e}$ an angle less than δ. Indeed, if $\mathbf{c} = \Sigma\,\mathbf{a}_i$ then

$$\cos(\,\mathbf{c}\,\mathbf{e}) = \frac{\Sigma\,\mathbf{a}_i\,\mathbf{e}}{|\Sigma\,\mathbf{a}_i|} > \frac{\Sigma|\,\mathbf{a}_i|\cos\delta}{|\Sigma\,\mathbf{a}_i|} \geq \cos\delta.$$

6. Given a sequence K'_k of rectifiable curves ($k=1, 2, \ldots$), inscribe a broken line L'_k into every K'_k so that L'_k for large k approach arbitrarily closely in length to K'_k. Transform L'_k, as described in subsection 3, into broken lines L_k, rearranging together with their fragments the corresponding fragments of K'_k. This yields certain curves K_k circumscribed about L_k. They satisfy the requirements of Theorem 1.

Indeed, if $K_{k_i} \to K$ then $L_{k_i} \to K$. By Theorem 1, established for broken lines, len. $L_{k_i} \to$ len. K, which implies that len. $K_{k_i} \to$ len. K as well. The proof of Theorem 1 is complete.

3. The Minkowski Sum of Line Elements

1. Above we defined the Minkowski sum of line elements $\mathbf{x}'\,ds$ of a rectifiable curve $K = \mathbf{x}(s)$ as the body $P(K)$ filled by the endpoints of the vectors

$$\mathbf{r} = \int_0^l \mu x'\,ds, \tag{4}$$

where $\mu(s)$ is an arbitrary measurable function with $-\frac{1}{2} \leq \mu(s) \leq \frac{1}{2}$.

Refer as the variation $\nu_{\mathbf{e}}(K)$ of a curve K in some direction $\mathbf{e}$ to the least upper bound over all broken lines inscribed in K of the sum of the lengths of the projections onto $\mathbf{e}$ of the straight line segments of these broken lines. The least upper bound of the sum of the lengths of the projections sharing the direction with $\mathbf{e}$ is called the positive part of the variation of K in the direction $\mathbf{e}$.

Refer to the body $P(K)$ defined in (4) as the *indicatrix of variations* of the curve K, which is justified by the fact that the support function $H(\mathbf{e})$ of this body (i.e., the distance from the origin to the supporting plane with the outer normal $\mathbf{e}$) coincides for every $\mathbf{e}$ with the half of $\nu_{\mathbf{e}}(K)$. (A proof of this almost obvious statement appears in [3, p. 467].[4])

Considering the body $Q(K)$ filled by the endpoints of the vectors

$$\mathbf{r} = \int_0^l \mu\,\mathbf{x}'\,ds, \tag{5}$$

for all possible measurable $0 \leq \mu(s) \leq 1$, it is natural to call this body the *indicatrix of the positive part of variation* of K since the support function coincides in all directions with the positive part of variation of this curve. The body $Q(K)$ results from $P(K)$ by parallel translation by the half of the vector going from the initial point to the terminal point of K.

Subsequently, speaking of the indicatrices $P(K)$ or $Q(K)$, we will understand the body that is determined respectively by the integral (4) or (5), while speaking of the Minkowski sum of line elements of a curve, the same body considered up to parallel translation.

The following lemma is noteworthy.

If a sequence of curves K_i converges to a rectifiable curve K together with the lengths then the bodies satisfy $P(K_i) \to P(K)$ and $Q(K_i) \to Q(K)$. The convergence of bodies is henceforth understood in the sense of the topological limit.

Proof. It is known that the convergence of curves in length implies the convergence in variation in every direction. The latter means the convergence of the support functions of $P(K_i)$ to the support function of $P(K)$, which is equivalent to

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[4] In [3] the indicatrix of variations is a slightly different body (twice as large) satisfying $H(\mathbf{e}) = \nu_{\mathbf{e}}(K)$.

the topological convergence $P(K_i) \to P(K)$ for convex bodies. The initial and terminal points of K_i converge respectively to those of K since on all curves we assume a fixed direction of traversal. This enables us to conclude also that $Q(K_i) \to Q(K)$.

2. **Theorem 2.** *In order for a bounded body to be the Minkowski sum of line elements of some rectifiable curve, it is necessary and sufficient that this body be a limit of zonotopes.*[5] (For a body to be the indicatrix of the positive part of variation of a rectifiable curve, it is necessary and sufficient that this body be a limit of zonotopes and contain the origin, while for it to be the indicatrix of variations, it is necessary and sufficient that it also be centered at the origin.)

Let us prove the main claim of the theorem.

Necessity: Suppose that R is the Minkowski sum of line elements of a curve K. Inscribe into K a sequence of broken lines L_i converging to K in length. The indicatrices $P(L_i)$ are parallelohedra converging to a body $P(K)$, from which R differs only by parallel translation. Hence, R is a limit of parallelohedra as well.

Sufficiency: Suppose that a bounded body R is the limit of parallelohedra R_i. From the straight line segments whose Minkowski sums are R_i we form broken lines L_i, assuming that their straight line segments are rearranged in accordance with Theorem 1, while the broken lines themselves begin at the origin.

The indicatrices $P(L_i)$ are jointly bounded since they differ only by translation from the bodies R_i converging to the bounded body R. This implies the boundedness of all variations, and consequently, of the lengths of L_i ([3, p. 464]). By the well-known theorem on the compactness of the family of curves lying in a bounded part of space and possessing a jointly bounded length, we can choose from L_i a subsequence L_{i_k} converging to a curve K.

Theorem 1 implies that K is a rectifiable curve, and the broken lines L_{i_k} converge to K in length. Therefore, $P(L_{i_k}) \to P(K)$. But the body R is the unique limit of the bodies R_i differing from $P(L_i)$ only by translation. Hence, R differs only by translation from $P(K)$, and so it is the Minkowski sum of line elements of K. The proof of Theorem 2 is complete.

(The additional claims of the theorem can be established similarly. Only while proving sufficiency we must somewhat specialize the choice of directions of the straight line segments in order to construct from a zonotope containing the origin a broken line L for which it is the indicatrix $Q(L)$. Then we can rearrange the straight line segments of this broken line in accordance with Theorem 1, preserving the directions of their traversal.)

3. **Theorem 3.** *The collection of all curves possessing as the indicatrix of variations the same body P as a rectifiable curve K_0 is exhausted by the curves resulting from K_0 by a rearrangement of its fragments, and the curves resulting from these curves by passing to limits with convergence in length.*

Let us sketch a proof of Theorem 3. Consider a curve $K = \mathbf{x}(s)$ with the same indicatrix of variations as $K_0 = \mathbf{x}(s)$. Verify that by rearranging the fragments of K_0 we can obtain a sequence of curves $K_1, K_2, \dots$, converging to K in length. We assume that all curves begin at the origin; the parameter s is the length counted from the beginning, $0 \le s \le l$.[6]

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[5]Editor of Translation: The limits of zonotopes are called zonotopes nowadays.

[6]The lengths of K and K_0 are obviously the same.

As in the proof of Theorem 1, consider a special sequence of partitions $\xi_1, \xi_2, \dots$ of the sphere Ω, adding the condition that every partition ξ_k contains, along with a set M_i, the set $-M_i$, symmetric to M_i about the center of Ω.

Denote by $F(M)$ the measure of the set of those values s for which the tangent vector $\mathbf{x}'(s)$ of K is directed into M. Similarly, let $F_0(M)$ stand for the measure of those s for which the tangent vector $\mathbf{x}_0'(s)$ of K_0 is directed into M. According to Theorem 6, which we prove in Section 4,

$$F(M) + F(-M) = F_0(M) + F_0(-M).$$

For each of the sets $M_1, M_2, \dots, M_m, -M_1, \dots, -M_m$ of the partition ξ_k, the segment $0 \le s \le l$ splits, on the one hand, into the sets $A_i B_i$, where $\mathbf{x}'(s) \in M_i$ and $\mathbf{x}'(s) \in -M_i$, and on the other hand, into the sets $A_i' B_i'$, where $\mathbf{x}_0'(s) \in M_i$ and $\mathbf{x}_0'(s) \in -M_i$.

Since

$$\operatorname{mes}(A_i) + \operatorname{mes}(B_i) = \operatorname{mes}(A_i') + \operatorname{mes}(B_i'),$$

we can always move some set C_i' from A_i' to B_i' or from B_i' to A_i' (assume for definiteness that the first case holds) so that the equalities

$$\operatorname{mes}(A_i) = \operatorname{mes}(A_i' - C_i'), \quad \operatorname{mes}(B_i) = \operatorname{mes}(B_i' + C_i')$$

are separately satisfied.

Replace each of the sets A_i, $B_i(A_i' - C_i')$, B_i', and C_i', as it is done in [3] while proving Theorem 6, by a finite system of intervals, "almost" exhausting in measure the corresponding set and containing an "insignificant amount" of outside points. The words in quotes mean deviation to at most $\frac{\varepsilon_k}{m}$ in measure, where $\varepsilon_k \to 0$ as $k \to \infty$. In addition, we assume that all these intervals are nonoverlapping both for the system of sets A_i and B_i, and for the system of sets $(A_i' - C_i')$, B_i', and C_i'.

Subdivide the curve K_0 into fragments in accordance with the intervals of the systems "covering" the sets $(A_i' - C_i')$, B_i', and C_i'. Upon the rearrangement, the fragments corresponding to C_i' are traversed in the opposite direction, while the remaining fragments, in the proper direction. Rearrange all these fragments (or their parts of necessary length) following the order of fragments "covering" the sets A_i and B_i. This yields a curve K_k.

We can verify that as $k \to \infty$ the curves $K_k = \mathbf{x}_k(s)$ converge to $K = \mathbf{x}(s)$. To this end, we have to make rather simple estimates for the difference

$$\int_0^s \mathbf{x}_k'(s)\, ds - \int_0^s \mathbf{x}'(s)\, ds,$$

which we skip because of their similarity to the estimates presented in [3] while proving Theorem 6.

4. Let us make several remarks on Theorem 3.

(1) Passage to the limit cannot be dropped from statement of the theorem. For instance, two halves of a circle have the same variation in all directions, but cannot be obtained one from the other by rearranging finitely many parallely translated straight line segments. It is less obvious that even for every broken line we can construct a curve possessing the same variation but failing to be a line segment on any of this pieces.

(2) From Theorem 3 we conclude easily that if the indicatrices of variations $P(K_1)$, $P(K_2), \dots$ converge to a bounded body P then we can transform K_1,

$K_2, \ldots$ by rearranging the fragments into curves converging to an arbitrarily specified curve K whose indicatrix of variations is P.

(3) If we form from a rectifiable curve K_0 new curves by rearranging its fragments while preserving the direction of their traversal, and then adding also all curves obtained from these curves by passage to the limit with respect to convergence in length, then we obtain curves with the same indicatrix Q of the positive part of variation, but we fail to exhaust *all* these curves. For instance, the contour of an equilateral triangle traversed in one direction once, or the contour of a parallel triangle of half the size traversed twice, in both proper and opposite directions, are broken lines with the same indicatrix Q; but, these broken lines cannot be obtained one from the other by the above means.

(4) Consider two measurable functions $f(x)$ and $g(x)$ on the segment $[a, b]$. One of them is called a rearrangement of the other (see [4, p. 332]) if for every c the measures of the sets on which $f(x) \geq c$ and $g(x) \geq c$ hold coincide. By analogy with the proof of Theorem 3, in this case we can verify that by rearranging fragments of $[a, b]$ together with the values of $f(x)$, we can always obtain from $f(x)$ of some functions $f_1(x)$, $f_2(x), \ldots$ converging in measure to $g(x)$.

4. On Additive Vector Functions

1. Considering an arbitrary set X, select a system B of subsets of X including the empty set and invariant under complements and finite or countable unions. If to every $E \in B$ there is associated a vector $\varphi(E)$ in the space R^n then we say that the finite-dimensional vector function $\varphi(E)$ is defined on B.

A function $\varphi(E)$ is called *countably additive* if for every countable system of pairwise disjoint $E_i \in B$ we have

$$\sum_{i=1}^{\infty} \varphi(E_i) = \varphi\left(\bigcup_{i=1}^{\infty} E_i\right).$$

A function $\varphi(E)$ is called nonatomic or differ if for every $E \in B$ either $\varphi(E) = 0$ or E includes a subset $E' \in B$ such that $0 \neq \varphi(E') \neq \varphi(E)$.

2. **Theorem 4.** *Every countably additive finitedimensional vector function $\varphi(E)$ on a Bsystem of sets $E \subset X$ can be expressed as the Lebesgue–Stieltjes integral*

$$\varphi(E) = \int_E \mathbf{x}(t)\, d\mu, \tag{6}$$

where $\mathbf{x}(t)$ is the unit vector function defined for all $t \in X$, while $\mu(E)$ is the variation of $\varphi(E)$ on E.[7]

Proof. Consider the coordinates $\varphi_1(E)$, $\varphi_2(E), \ldots, \varphi_n(E)$ of $\varphi(E)$. Since $\bigl|\varphi_k(E)\bigr| \leq \bigl|\varphi(E)\bigr| \leq \mu(E)$, it follows that each scalar function $\varphi_k(E)$ is absolutely continuous with respect to the measure $\mu(E)$, and so, according to the Radon–Nikodým theorem ([5, § 31]; [6, p. 59]), there exist functions $\chi_k(t)$ summable with respect to $\mu(E)$ ans satisfying

$$\chi_k(E) = \int_E \chi_k(t)\, d\mu.$$

[7]The variation of the function $\varphi(E)$ is $\sup \Sigma \bigl|\varphi(E_i)\bigr|$ over all possible finite systems of pairwise disjoint sets $E_i \in B$ lying in E. The variation of a countably additive vector function is itself a countably additive function defined on the sets of the system B. The variation of a nonatomic function is also a nonatomic function.

Therefore, (6) holds for the vector function

$$\mathbf{x}(t) = \big[\chi_1(t), \chi_2(t), \dots, \chi_n(t)\big].$$

Verify that $\big|\,\mathbf{x}(t) = 1\big|$ almost everywhere in the sense of the measure $\mu(E)$. To this end, it suffices to show that every measurable set E satisfies

$$\mu(E) = \int_E \big|x(t)\big|\, d\mu. \tag{7}$$

(Here and for the rest of this section we mean measurability with respect to μ.)

Choose in E, one by one, some finite systems of disjoint subsets E_i with

$$\sum_i \big|\varphi(E_i)\big| \to \mu(E).$$

Since

$$\sum_i \big|\varphi(E_i)\big| = \sum_I \left|\int_{E_i} \mathbf{x}(t)\, d\mu\right| \le \sum \int_{E_i} \big|\,\mathbf{x}(t)\big|\, d\mu = \int_E \big|\,\mathbf{x}(t)\big|\, d\mu,$$

it follows that

$$\mu(E) \le \int_E \big|\,\mathbf{x}(t)\big|\, d\mu.$$

Verify the reverse inequality. To this end, by dividing the space into half-open cubes, divide the unit sphere into parts whose diameters are less than an arbitrary given $\delta > 0$. Take as E_i the part of E within which $\mathbf{x}(t)$ is directed into one part of the partition of the unit sphere. The set E_i is obviously measurable and

$$\left|\int_{E_i} \mathbf{x}(t)\, d\mu\right| \ge \cos\delta \int_{E_i} \big|\,\mathbf{x}(t)\big|\, d\mu.$$

Hence,

$$\mu(E) \ge \sum_i \left|\int_{E_i} \mathbf{x}(t)\, d\mu\right| \ge \cos\delta \int_{E_i} \big|\,\mathbf{x}(t)\big|\, d\mu.$$

Since δ is arbitrarily small, this yields

$$\mu(E) \ge \int_E \big|\,\mathbf{x}(t)\big|\, d\mu,$$

which together with the reverse inequality above implies (7).

The proof of Theorem 4 is complete.

3. To every countably additive vector function of sets admitting, in accordance with Theorem 4, representation as

$$\varphi(E) = \int_E \mathbf{x}(t)\, d\mu,$$

associate the set Q in R^n filled by the endpoints of the vectors of the form

$$\mathbf{r} = \int_X \mathbf{x}(t)\alpha(t)\, d\mu,$$

for all possible measurable (with respect to μ) functions $\alpha(t)$ with $0 \le \alpha(t) \le 1$.

Obviously, Q is a bounded convex body. It is easy to verify that the support function of Q is equal to

$$H(\,\mathbf{e}) = \int_X \frac{\big|\,\mathbf{e}\,\mathbf{x}(t)\big| + \,\mathbf{e}\,\mathbf{x}(t)}{2}\, d\mu.$$

In addition, Q is closed. Indeed, suppose that

$$\mathbf{r}_i = \int_X \mathbf{x}(t)\alpha(t)\,d\mu \to \mathbf{r}_0.$$

On the subsets M of the unit sphere Ω define the measure $\mu(M) = \mu(E)$, where $E = \varepsilon_t[\mathbf{x}(t) \in M]$. It is easy to see that then the Borel sets of M are measurable. Without loss of generality, assume that $\alpha_i(t)$ is constant on every set of nonzero measure whenever $\mathbf{x}(t)$ is constant on it. This enables us to uniquely define the function $\alpha(\mathbf{x}) = \alpha\big(t(\mathbf{x})\big)$ for almost all points $\mathbf{x} \in \Omega$. In addition, put

$$\varphi_i(M) = \int_M \alpha_i(\mathbf{x})\mu(d\omega),$$

where $d\omega$ is the element of the sphere into which $\mathbf{x}$ is directed. In this notation,

$$\mathbf{r}_i = \int_\Omega \mathbf{x}\alpha_i(\chi)\mu(d\omega) = \int_\Omega \mathbf{x}\varphi_i(d\omega).$$

From the bounded functions $\varphi_i(M) \le \mu(M)$ we can choose a weakly converging sequence to some function $\varphi_0(M)$ (see [7, p. 215] for instance). Furthermore, the open subsets M satisfy

$$\varphi_0(M) \le \underline{\lim}\,\varphi_i(M) \le \mu(M),$$

and so $\varphi_0(M) \le \mu(M)$ for all Borel sets M. The latter implies that $\varphi_0(M)$ is absolutely continuous, and by the Radon–Nikodým theorem we can express $\varphi_0(M)$ as

$$\varphi_0(M) = \int_M \alpha_0(\mathbf{x})\mu(d\omega); \quad 0 \le \alpha_0(\mathbf{x}) \le 1.$$

The weak convergence yields

$$\mathbf{r}_0 = \int_\Omega \mathbf{x}\varphi_0(d\omega) = \int_\Omega \mathbf{x}\alpha_0(\mathbf{x})\mu(d\omega) = \int_\Omega \mathbf{x}(t)\alpha_0(t)d\mu \in Q.$$

Therefore $\mathbf{r}_0 \in Q$, and so Q is proved to be closed.

Lemma. *For every nonatomic countably additive finitedimensional vector function $\varphi(E)$ its range, i.e., the body in R^n filled by the endpoints of the vectors $\varphi(E)$, and the body Q coincide.*

The proof of the lemma goes by induction. For the case $n = 1$ the claim holds since the range of $\varphi(E)$ and the body Q reduce to the closed segment $\big[\inf\varphi(E), \sup\varphi(E)\big]$ (see [5, p. 171]). Assume that the claim holds for vector functions of dimensions less than n.

Construct a family of measurable sets X_τ, with parameter $0 \le \tau \le 1$, expanding with the growth of τ and satisfying $\mu(X_\tau) = \tau\mu(X)$.[8]

It is easy to see that the characteristic functions $\chi_\tau(t)$ of the sets $X_\tau(t)$ almost everywhere satisfy $\chi_\tau(t) \to \chi_{\tau_0}(t)$ as $\tau \to \tau_0$.

[8]We can obtain the required family of sets, for instance, as follows. Since $\varphi(E)$ is nonatomic, we can subdivide X into two parts X_0 and X_1 so that $\mu(X_0) = \mu(X_1) = \frac{1}{2}\mu(X)$. Subdivide each of these sets into two, denoted by X_{00}, X_{01}, and X_{10}, X_{11} so that $\mu(X_{00}) = \mu(X_{01}) = \mu(X_{10}) = \mu(X_{11}) = \frac{1}{4}\mu(X)$, and so on. This yields sets of the form $X_{i_1,i_2,\dots,i_k}$ $(i_j = 0; 1)$. Furthermore, take an arbitrary number a, $0 < a < 1$, of the form $a = \frac{m}{2^k}$ with integer m and k. Denote by X_a the union of all sets $X_{i_1,\dots,i_k}$ satisfying $\frac{i_1}{2} + \dots + \frac{i_k}{2^k} < a$. It is easy to see that $\mu(X_a) = a\mu(X)$ and $X_{a_1} \subset X_{a_2}$ provided that $a_1 < a_2$. For every real τ denote by X_τ the sum of all X_a with $a \le \tau$. It is not difficult to see that X_τ constitute the required system.

To every set X_τ associate the body Q_τ filled by the endpoints of the vectors

$$\mathbf{r} = \int_{X_\tau} \mathbf{x}(t)\alpha(t)\,d\mu = \int_X \mathbf{x}(t)\alpha(t)\chi_\tau(t)\,d\mu$$

where $0 \le \alpha \le 1$ is an arbitrary measurable function, while $\mathbf{x}(t)$ is from the representation (6) of the function $\varphi(E)$ under consideration. It is easy to see that the support function of Q_τ is equal to

$$H_\tau(\mathbf{e}) = \int_X \frac{|\mathbf{e}\,\mathbf{x}(t)| + \mathbf{e}\,\mathbf{x}(t)}{2}\chi_\tau(t)\,d\mu,$$

which implies that $H_\tau(\mathbf{e}) \to H_{\tau_0}(\mathbf{e})$ as $\tau \to \tau_0$. In other words, the bodies Q_τ continuously expand with the growth of τ. Furthermore, Q_0 is a point (the origin), while $Q_1 = Q$.

Denote the range of $\varphi(E)$ by Θ. The representation (6) and the definition of Q directly imply that $\Theta \subset Q$. It remains to prove the reverse inclusion.

Take $\mathbf{r}_0 \in Q$. Since the family Q_τ is continuous, the terminal point of $\mathbf{r}_0$ lies on the boundary of one of these bodies. Assume furthermore that τ satisfies this. Consider the outer normal $\mathbf{e}$ to the supporting plane P of Q_τ at $\mathbf{r}_0$. The set X splits into measurable parts: E_1 with $\mathbf{e}\,\mathbf{x}(t) < 0$; E_2 with $\mathbf{e}\,\mathbf{x}(t) = 0$; E_3 with $\mathbf{e}\,\mathbf{x}(t) > 0$. It is easy to verify that the common part of Q_τ and the supporting plane P is filled by the vectors of the form

$$\mathbf{r} = \int_{E_1} \mathbf{x}(t)\,d\mu + \int_{E_2} \mathbf{x}(t)\alpha(t)\,d\mu,$$

where $0 \le \alpha(t) \le 1$ is an arbitrary measurable function. In particular, the vector $\mathbf{r}_0$ can be expressed in this form for some function $\alpha_0(t)$. On the measurable parts E of E_2 there is some vector function of smaller dimension than n:

$$\psi(E) = \int_E \mathbf{x}(t)\,d\mu.$$

By the inductive assumption, for $\alpha_0(t)$ in E_2 there is a part E_2^0 such that

$$\int_{E_2} \mathbf{x}(t)\alpha_0(t)\,d\mu = \int_{E_2^0} \mathbf{x}(t)\,d\mu.$$

Therefore,

$$\mathbf{r}_0 = \int_{E_1} \mathbf{x}(t)\,d\mu + \int_{E_2^0} \mathbf{x}(t)\,d\mu = \varphi(E_1 + E_2^0) \in \Theta.$$

This proves the coincidence $Q = \Theta$ claimed in the lemma. □

4. The lemma proved above implies the results of Lyapunov [8] that the range of an atomless countably additive finite-dimensional vector function is always a centrally symmetric closed convex body containing the origin, and its intersection with each of its supporting $(n-1)$dimensional planes is also centrally symmetric. These results are strengthened in the next theorem.

Theorem 5 *In order for a body Q to be the range of a nonatomic countably additive finitedimensional vector function, it is necessary and sufficient that it be the indicatrix of the positive part of variation of some rectifiable curve.*

Necessity: Consider a function $\varphi(E) = \int_E \mathbf{x}(t)\,d\mu$ satisfying the hypotheses of the theorem. Its range Q has the support function

$$H_Q(\mathbf{e}) = \int_X \frac{|\mathbf{e}\,\mathbf{x}(t)| + \mathbf{e}\,\mathbf{x}(t)}{2}\,d\mu.$$

Subdivide the sphere Ω into parts of diameter less than δ so that X splits into measurable parts E_i, within each of which $\mathbf{x}(t)$ belongs to the same parts of Ω. Put $\mathbf{x}_i = \mathbf{x}(t_i)$, where $t_i \in E_i$. The parallelohedron Π_δ filled by the endpoints of the vectors

$$\mathbf{r}_i = \sum \lambda_i\,\mathbf{x}_i \mu(E_i) \quad (0 \le \lambda_i \le 1)$$

has the support function

$$H(\mathbf{e}) = \sum_i \frac{|\mathbf{e}\,\mathbf{x}_i| + \mathbf{e}\,\mathbf{x}_i}{2}\,\mu(E_i).$$

Then

$$\left|H_Q(\mathbf{e}) - H(\mathbf{e})\right| < \delta\mu(X).$$

As $\delta \to 0$, the zonotopes Π_δ converge to Q. In addition, Q contains the origin. Therefore, Theorem 2 implies that Q is the indicatrix of the positive part of variation of some rectifiable curve.

Sufficiency: Consider the indicatrix of variations Q of a curve $K = \mathbf{x}(s)$ of length l. The body Q is filled by the endpoints of the vectors of the form

$$\mathbf{r} = \int_0^l \mathbf{x}'(s)\alpha(s)\,ds \quad \big(0 \le \alpha(s) \le 1\big).$$

By the lemma proved above, in this case Q coincides with the range of the vector function

$$\varphi(E) = \int_E \mathbf{x}'(s)\,ds$$

defined on the measurable subsets of the segment $[0, l]$.

5. Chuĭkina in [2] introduced the concept of extendability of zonotopes, which we can state as follows: A zonotope Q_2 *extends* a zonotope Q_1 whenever there exist a broken line L_2 and a broken line L_1 inscribed into L_2 with $Q_2 = Q(L_2)$ and $Q_1 = Q(L_1)$. Here $Q(L)$ means the indicatrix of the positive part of variations of a curve L.

As established in [2], for Q to be the range of a continuous additive vector function it is necessary and sufficient that Q be a limit of zonotopes extending each other. As far as we know, Glivenko strengthened this result by showing that *every body serving as a limit of zonotopes and containing the origin is a limit of zonotopes extending each other*. Indeed, according to Theorem 2, this body Q is the indicatrix of the positive part of variation of some rectifiable curve K. Inscribe into K a sequence of broken lines L_i so that every successive line passes through all vertices of the previous line and L_i converge to K in length. The indicatrices $Q(L_i)$ obviously constitute a sequence of zonotopes extending each other and converging to Q. (In the proof by Glivenko the required curve K is constructed by a more complicated method than in Theorem 1 above.)

5. The Supporting Function of a Body That Is the Minkowski Sum of Straight Line Segments

1. It is known that the support function of a Minkowski sum of convex bodies is the corresponding linear combination of the support functions of the summands. It is natural to similarly obtain the support function for the Minkowski sum of infinitely many bodies (see [1, p. 28] for instance).

In order to avoid introducing extra regularity requirements, we use the Lebesgue–Stieltjes integral.

Theorem 6. 1) *If a body P is the indicatrix of variations of a rectifiable curve $K = \mathbf{x}(s)$ then the support function $H(\mathbf{e})$ of P is expressed by the integral*

$$H(\mathbf{e}) = \frac{1}{2}\int_{\Omega} |\mathbf{e}\,\mathbf{e}'| F(d\omega), \tag{8}$$

where $\mathbf{e}'$ is the unit vector going into the element $d\omega$, while $F(M)$ is the linear measure of the set of those values of the parameter s for which the tangent line $\mathbf{x}'(s)$ of K is directed into M.

2) *The even part of every countably additive function $F(M)$ defined on the Borel sets of the unit sphere Ω and satisfying*

$$H(\mathbf{e}) = \frac{1}{2}\int_{\Omega} |\mathbf{e}\,\mathbf{e}'| F(d\omega) \tag{9}$$

is uniquely determined by specifying $H(\mathbf{e})$.[9]

3) *For every nonnegative countably additive function $F(M)$ defined on the Borel sets of the sphere Ω there is a rectifiable curve $K = \mathbf{x}(s)$ for which $F(M)$ is the linear measure of the values of s for which $\mathbf{x}'(s) \in M$.*

Proof. 2. Consider the indicatrix of variations P of a rectifiable curve $K = \mathbf{x}(s)$. For Borel sets $M \subset \Omega$ the set E of values of s for which $\mathbf{x}'(s) \in M$ is measurable since K is rectifiable. Therefore, the function $F(M) = \mathrm{mes}E$ is defined. Verify that for this function F the support function $H(\mathbf{e})$ of P is expressed by (8). Indeed,

$$\frac{1}{2}\int_{\Omega} |\mathbf{e}\,\mathbf{e}'| F(d\omega) = \frac{1}{2}\lim\sum_i |\mathbf{e}\,\mathbf{e}_i| F(M_i) = \frac{1}{2}\lim\sum_i |\mathbf{e}\,\mathbf{e}_i| \mathrm{mes}E_i$$

$$= \frac{1}{2}\int_0^l \left|\mathbf{e}\frac{d\mathbf{x}}{ds}\right| ds = \frac{1}{2}V_{\mathbf{e}}(K) = H(\mathbf{e}).$$

The first the claim of Theorem 6 is established.

3.Proceed to the second claim of the theorem. Blaschke ([9, p. 155]) essentially proved it, but restricting himself only to the regular case. His proof uses the trick of expansion into spherical functions which was applied already by Minkowski [10].

Let us explain this trick in the form used by Aleksandrov ([11, pp. 1231–1233]) to obtain a similar result.

The eigenfunctions of the integral equation

$$Y(\mathbf{e}) = \lambda\int_{\Omega} |\mathbf{e}\,\mathbf{e}'| Y(\mathbf{e}')\, d\omega \tag{10}$$

[9] The even part of $F(M)$ is $\frac{1}{2}[F(M) + F(-M)]$, where $-M$ means the set symmetric to M with respect to the center of the sphere Ω. The odd part is $\frac{1}{2}[F(M) - F(-M)]$.

are the spherical functions $Y_{2k}(\mathbf{e})$ of even order. Multiply (9) by $Y_{2k}(\mathbf{e})$ and integrate over Ω. Switching the order of integrations on the right-hand side and considering that Y_{2k} satisfies (10) for some constant $\lambda_k \neq 0$, we obtain

$$\int_\Omega H(\mathbf{e})Y_{2k}(\mathbf{e}')\,d\omega = \frac{\lambda_k}{2}\int_\Omega Y_{2k}(\mathbf{e}')F(d\omega). \tag{11}$$

Consequently, as soon as $H(\mathbf{e})$ is defined, so are the integrals appearing on the right-hand side of (11). Verify that this defines the values of the even part of F on all Borel sets. It suffices to do this for the open sets lying in one hemisphere.

Take a set G of this form and the function $Z(\mathbf{e})$ equal to 1 on G and $-G$ and to 0 on the rest of the sphere. Then $Z(\mathbf{e})$ is a bounded even function. Since the system of spherical functions is complete, there is a sequence of linear combinations $Z_m(\mathbf{e})$ of spherical functions of even order converging to $Z(\mathbf{e})$. By (11), the integrals

$$\int_\Omega Z_m(\mathbf{e}')F(d\omega)$$

are determined by $H(\mathbf{e})$, and so is their limit,

$$\int_\Omega Z(\mathbf{e}')F(d\omega) = F(G) + F(-G),$$

as required.

4. Proceed to the third claim of Theorem 6.

Consider a countably additive nonnegative function $F(M)$ defined on the Bsets of the sphere Ω.

As in the proof of Theorem 1, fix a sequence of partitions $\xi_1, \xi_2, \dots$ of the unit sphere Ω into finite systems of enumerated disjoint Borel sets such that the maximal diameter δ_k of the sets M_i^k in ξ_k tends to zero with the growth of k, and every partition ξ_{k+1} results from ξ_k by subdivision.

Consider the segment $(0, l)$, where $l = F(\Omega)$, and construct a sequence $\eta_1, \eta_2, \dots$ of its partitions such that every set $M_i^k \in \xi_k$ corresponds to a set $N_i^k \in \eta_k$, while $\mathrm{mes}(N_i) = F(M_i)$, and the next partition η_{k+1} is constructed by subdividing the previous partition in complete accordance with the way ξ_{k+1} is obtained from ξ_k. (For simplicity we may assume that all sets N_i of nonzero measure are half-open intervals, while the measure zero sets are empty.)

On the segment $0 < s \le l$ define a sequence of vector functions $\mathbf{e}_k(s)$ by putting $\mathbf{e}_k(s)$ equal to the unit vector going into one of the points M_i^k, where M_i^k corresponds to the N_i^k containing s. It is easy to see that

$$\left|\mathbf{e}_k(s) - \mathbf{e}_{k+p}(s)\right| < \delta_k$$

holds for all k and p, which implies that the functions $\mathbf{e}_k(s)$ converge uniformly to a measurable unit vector function $\mathbf{e}(s)$ as $k \to \infty$. Verify that

$$\mathbf{x}(s) = \int_0^s \mathbf{e}(s)\,ds$$

is the required curve.

The tangent vector of this curve for almost all s is equal to $\mathbf{e}(s)$. It remains to verify that the measure of those s for which $\mathbf{e}(s) \in M$ coincides with $F(M)$ for all Borel sets $M \subset \Omega$. It suffices to establish this for open sets. Given an open set G, denote by A_k the union of all sets of the partition ξ_k included into G together with their closures $\bar{A}_k$, and by B_k, the union of the corresponding sets of the partition η_k.

Since A_k constitute an expanding sequence exhausting G, while the function F is countably additive, it follows that

$$F(G) = \lim_{k\to\infty} F(A_k) = \lim_{k\to\infty} \mathrm{mes} B_k = \mathrm{mes} \bigcup_{k=1}^{\infty} B_k. \tag{12}$$

It is easy to verify however that $\bigcup_{k=1}^{\infty} B_k = B$, where B is the set of values of s with $\mathbf{e}(s) \in G$. Indeed, for every $s \in B_k$ we have $\mathbf{e}_k(s) \in A_k$; hence, $\mathbf{e}_{k+p}(s) \in A_k$, and in the limit $\mathbf{e}(s) \in \bar{A}_k \subset G$. Therefore, $\bigcup_{k=1}^{\infty} B_k \subset B$. On the other hand, if $\mathbf{e}(s) \in G$ then, starting with a sufficiently large k, the vectors $\mathbf{e}_k$, converging to $\mathbf{e}(s)$, lie in G together with the sets M_i^k containing $\mathbf{e}_k(s)$, as well as their closures. For these k we have $s \in B_k$; thus, $B \subset \bigcup_{k=1}^{\infty} B_k$, which together with the previous inclusion yields $B = \bigcup_{k=1}^{\infty} B_k$.

Therefore, (12) implies that $F(G) = \mathrm{mes} B$; and the third claim of Theorem 6 is established.

Theorem 6 implies that the representability of the support function of some body in the integral form (8) with a nonnegative countably additive function F is a necessary and sufficient condition for this body to be the integral Minkowski sum of line elements. The definition of a body via the representation of its support function as (8) is often taken as the definition of the integral Minkowski sum of infinitely many straight line segments (see [1, p. 28]). Blaschke showed (see [9, pp. 147–155]) that we can express the support function of every centrally symmetric convex body (with center the origin) as (8) without requiring the nonnegativity of $F(M)$. (The converse claim fails, of course, since not every function $H(\mathbf{e})$ possessing a similar representation is a support function.)

5. Let us make the following remark regarding the third claim of Theorem 6. As $F(M)$ we can take, in particular, the usual measure on Ω. The theorem implies that every ball is a limit of parallelohedra, and *there exist curves possessing the same variation in all directions.*[10]

In the three-dimensional space we can obtain a curve K of this kind as follows. Subdivide the surface of the sphere into four equilateral spherical triangles Δ_i and form a broken line from the vectors

$$\mathbf{r}_i = \int_{\Delta_i} \mathbf{e}\, d\omega. \tag{13}$$

This is a closed broken line consisting of four straight line segments. Then divide every triangle into four triangles connecting the midpoints of its sides by the shortest paths on the sphere. Accordingly, replace every segment of the broken line by the fragments consisting of the vectors obtained using (13) for every triangle of the finer partition, and so on. Each successive broken line passes through the vertices of the previous broken line. These broken lines converge to the required curve K.

6. Note, without the proof, the following insignificant modification of Theorem 6:

(1) If a body Q is the indicatrix of the positive part of variation of a rectifiable curve $K = \mathbf{x}(s)$ then we can express its support function $H(\mathbf{e})$ as the integral

$$H(\mathbf{e}) = \int_{\frac{\Omega}{2}} (\mathbf{e}\,\mathbf{e}') F(d\omega), \tag{14}$$

[10] We can even show the existence of a smooth curve of this kind.

where $\mathbf{e}'$ is the unit vector going into the element $d\omega$; $F(M)$ is the linear measure of the values s for which $\mathbf{x}'(s) \in M$; $\frac{\Omega}{2}$ is the hemisphere within which $\mathbf{e}\,\mathbf{e}' > 0$.

(2) If the support function $H(\mathbf{e})$ of a convex body Q admits at least one representation (14), where $F(M)$ is a countably additive nonnegative function defined on the Borel sets of the unit sphere Ω, then the even part of $F(M)$ is uniquely determined by Q. We can vary the odd part of $F(M)$ (subject to the condition that the countable additivity and nonnegativity of F are preserved) by adding to $F(M)$ an arbitrary odd function $\Phi(M)$ satisfying

$$\int_\Omega \mathbf{e}'\Phi(d\omega) = 0;$$

moreover, (14) is preserved.

6. Criteria for the Bodies That Are Limits of Zonotopes

1. It is natural to pose the question of criteria distinguishing the bodies P that are limits of zonotopes. In the two-dimensional case a necessary and also sufficient condition is the central symmetry of the convex set P. In the general case Theorem 6 answers this question in some sense. From the support function of a centrally symmetric convex body we can uniquely reconstruct the even function $H(\mathbf{e})$ appearing in (8); according to Theorem 6, its nonnegativity turns out a necessary and sufficient condition.

For a regular surface the function $F(M)$ itself is an integral over the subset M of the unit sphere Ω of some function h defined at the points of Ω. As Blaschke showed (see [9, pp. 154–155]), in this case the function h is found from the support function $H(\mathbf{e})$ of a given body using quadratures.[11]

2. **Theorem 7.** *For a convex polyhedron to be a zonotope it is necessary and sufficient that its every two-dimensional face has center of symmetry.*

The necessity of the condition is obvious; let us prove its sufficiency. Consider a polyhedron P with centrally symmetric two-dimensional faces. Divide its edges (one-dimensional faces) into classes, putting in one class the edges that are equal and parallel. Take an arbitrary support plane R^{n-1} of P parallel to the edges of some class K. Let us prove (by induction on the dimension n of P) that it contains at least one edge of this class.

For $n = 2$ the claim holds. Assume that it holds for $n_0 = n-1$. In order to prove it for n, it suffices to verify that the distance δ from R^{n-1} to the nearest edge to R^{n-1} of the class K is equal to zero. Take an arbitrary edge $a \in K$. If it lies in R^{n-1} then $\delta = 0$. Otherwise, take the plane R_1^{n-1} parallel to R^{n-1} and passing through a. Consider all $(n-1)$dimensional faces passing through a. At least one of them contains points lying on the same side of R_1^{n-1} as R^{n-1}. Take the supporting plane to this face parallel to R^{n-1} passing on the same side of R_1^{n-1} as R^{n-1}. By the inductive assumption, this plane contains an edge of this class whose distance to R^{n-1} is less than the distances to the edge a. Since a is arbitrary and the number of edges is finite, this implies that $\delta = 0$.

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[11]Blaschke used the results of Funk [12] on reconstructing a function defined on the sphere from the values of its integrals over large circles. The latter results are developed and expanded in the $n$dimensional case in the student's diploma work by Yu. A. Volkov entitled "On the areas of flat sections of a centrally symmetric body" (Len. St. Univ., 1952).

Now choose one edge from every class and construct their Minkowski sum, the zonotope Q. Verify, again by induction on the dimension n of P, that P and Q can be matched by parallel translation. This will prove that P is a zonotope.

For $n = 2$ the claim holds. Assume that it is established for all polyhedra of dimension less than n. Take the supporting planes R_1^{n-1} and R_2^{n-1} to P and Q with the same outer normals in R^n. Select all classes of edges of P containing the edges parallel to R_1^{n-1}, if these classes exist at all. According to a previous result, the common part $P \cap R_1^{n-1}$ contains at least one edge from each of these classes. Select one edge in each of these classes. The common part $Q \cap R_2^{n-1}$ is the Minkowski sum of these edges.

By the inductive assumption, $P \cap R_1^{n-1}$ and $Q \cap R_2^{n-1}$ can be matched by a parallel translation. Therefore, every face (i.e., the common part with some supporting plane R^{n-1}) of the polyhedron P can be matched by parallel translation with the face of the parallelohedron Q selected out of Q by the supporting plane parallel to R^{n-1}. This implies that P and Q themselves can be matched by parallel translation. The proof of Theorem 7 is complete.

3. Note the following lemma.

Lemma. *The common part Q of a body P, that is a limit of parallelohedra in the space R^n and the supporting plane R^{n-1} to P is itself a limit of parallelohedra in R^{n-1}.*[12]

This implies not only the central symmetry of Q, but also the central symmetry of the faces of Q (if there are any), their faces, and so on. Essentially, the claim of this lemma is proved in [13, p. 469]. (A sequence of parallelohedra is constructed there contracting to the plane R^{n-1} and converging to Q. It suffices to project them onto R^{n-1} in order to obtain parallelohedra in R^{n-1} converging to Q.)

This lemma and Theorem 7 imply

Theorem 8. *A polyhedron which is a limit of parallelohedra is itself a parallelohedron.*

4. The property of a surface to bound a body which is a limit of parallelohedra is obviously preserved under affine transformations. The question of local characteristics of regular surfaces enjoying this property, arising in affine differential geometry ([13, p. 82]; [14, p. 250]), remains unsolved as far as we know.

7. On the Connection of Variations of Converging and Limit Curves

1. In subsection 3.5 we gave a definition of parallelohedra extending each other. Let us somewhat generalize the concept. Namely, say that a bounded body P extends a body Q whenever there exists two sequences of parallelohedra $P_k \to P$ and $Q_k \to Q$ such that every parallelohedron P_k extends the corresponding parallelohedron Q_k.

Lemma. *In order for P to extend Q, it is necessary and sufficient that both these bodies be the indicatrices of the positive part of variation of some curves, and*

[12] Since P is a limit of parallelohedra; P serves if we translate its center to the origin, as the indicatrix of variations of some curve $\mathbf{x}(s)$. It is not difficult, incidentally, to show that Q is filled by the endpoints of the vectors which can be expressed as

$$\mathbf{r} = \int_{E_1} \mu\,\mathbf{x}'\,ds + \frac{1}{2}\int_{E_2} \operatorname{sign}(\mathbf{e}\,\mathbf{x}')\cdot \mathbf{x}'\,ds,$$

where $\mathbf{e}$ is the outer normal to the plane R^{n-1}; E_1 is the set of values of s for which $\mathbf{e}\,\mathbf{x} = 0$; E_2 is the collection of the remaining values of s; $\mu(s)$ is an arbitrary measurable function with $-\frac{1}{2} \le \mu(s) \le \frac{1}{2}$.

that for every broken line L inscribed in an arbitrary generator K of Q there exists a generator K' of P passing through all vertices of L.[13]

Sufficiency: Take an arbitrary generator K of Q. Take a sequence of broken lines L with vertices accumulating on K. Approximate the generators of P passing through their vertices by broken lines passing through the vertices of L. We obtain two sequences of broken lines, the indicatrices of the positive parts of whose variations extend each other and converge respectively to P and Q.

Necessity: Suppose that P extends Q. Consider parallelohedra P_k and Q_k extending each other and converging to P and Q; broken lines L_k generating Q_k; broken lines K_k circumscribed about L_k generating P_k. Observe first of all that we can always transform the broken line K_k by rearranging its straight line segments so that it passes through an arbitrary predefined point of L_k. Indeed, if CD is a segment of L_k and a point E lies on it, then, replacing the fragment of the broken line K_k subtended by the chord CD by two fragments obtained from it by dilations with similarity coefficients $\frac{CE}{CD}$ and $\frac{ED}{CD}$, we obtain generators passing through E. Do the same for other points.

Take a broken line L inscribed in some generator K of Q. By rearranging the straight line segments, transform L_k into broken lines L'_k converging to K, which is always possible according to Remark 2 to Theorem 3. We may have to subdivide the straight line segments of L_k. By the argument above we may assume that K_k passes through the required points of the subdivision. Rearranging the fragments of L_k and L'_k, transform simultaneously K_k into K'_k. Mark on L'_k some points A_i converging to the vertices of L. We may assume that K'_k pass through these points. We may assume that on the fragments A_iA_{i+1} the straight line segments of K'_k are rearranged in accordance with Theorem 1 so that for some subsequence these fragments converge in length, forming the limit curve K'. Obviously, K' is the generator of P and passes through the vertices of L. The proof of the lemma is complete.

This lemma implies in particular that if P and Q are parallelohedra and L is a generator of Q which is a broken line then there exists a generator K of P passing through the vertices of L. Furthermore, we can also make K a broken line. (In order to prove this, we may consider the function $F(M)$ of these curves, which in case of a broken line reduces to finitely many "pointwise loadings.") Therefore, *if two parallelohedra extend each other in the sense of the definition in this subsection then they extend each other in the usual sense as well.*

2. If a sequence of rectifiable curves $K_1, K_2, \dots$ converges to a rectifiable curve K, while their indicatrices of the positive part of variation converge to a body P, then $P \subset Q$, where Q is the indicatrix of the positive part of variation of the limit curve. It was proved in [3, pp. 470–476] that in the planar case this exhausts the relation between P and Q in some sense, while for the spaces of greater dimension the situation is different. The next theorem presents the details of this relation.

Theorem 9. *In order for a bounded body P to be a limit of the indicatrices of the positive part of variation of curves K_m converging to a curve K with the indicatrix Q, it is necessary and sufficient that P extend Q.*

Necessity: Consider some curves K_m $(m = 1, 2, \dots)$ converging to a curve K generating the body Q, with their indicatrices of variations converging to P. Inscribe

[13] Here we refer as a generator to an arbitrary curve the indicatrix of the positive part of whose variation is this body.

into each of K_m a broken line L_m so that the difference between the lengths of L_m and K_m tends to zero with the growth of m. Into the limit curve K, inscribe broken lines M_p $(p = 1, 2, \dots)$ whose vertices accumulate without bound on K with the growth of p. Mark on the curves L_m some points converging to the vertices of M_p. Connecting them, we obtain broken lines M_p^m. Denote by m_p some integers such that the difference between the lengths of the broken lines $N_p = M_p^{m_p}$ and M_p and the distance between them are less than $\frac{1}{p}$. As $p \to \infty$, the broken lines $N_p \to K$ together with their lengths; thus, the parallelohedra $Q_p = Q(N_p)$ converge to Q, while the parallelohedra $P_p = Q(L_{m_p})$ converge to P. Since N_p are inscribed into L_{m_p}, it follows that P_p extends Q_p. The necessity of the condition is proved.

Sufficiency: Take a sequence of broken lines M_p inscribed into K with vertices accumulating on K. By the lemma in the beginning of this section, for each of them there is a generator K_p of P passing through all vertices of M_p. Replacing every arc of K_p subtended by a segment of M_p by an arc composed of m arcs, each of which is similar to the original arc and smaller by a factor of m, we obtain a curve K_p'. For sufficiently large m the distance between K_p' and M_p is less than $\frac{1}{p}$. It is obvious that the curves K_p' constitute the required sequence.

Observe that when P and Q are parallelohedra then the criterion established in Theorem 9 admits direct verification. Namely, if the vectors $\mathbf{x}_i$ $(i = 1, 2, \dots, m)$ constitute a broken line generating P, while the vectors $\mathbf{y}_j$ $(j = 1, 2, \dots, p)$, a broken line generating Q, then the question whether P extends Q reduces to the existence of nonnegative numbers λ_{ij} satisfying

$$\sum_{i=1}^{m} \lambda_{ij}\, \mathbf{x}_i = \mathbf{y}_j; \quad \sum_{j=1}^{p} \lambda_{ij} = 1.$$

If these $m+p$ equations with mp unknowns have a solution then the latter depends linearly on a number of parameters. It remains to verify the consistency of the linear inequalities $\lambda_{ij} \geq 0$.

3. Finally, let us give a refinement of another result of [3]. It was proved there that the convergence of curves in variation in a set of directions which is everywhere dense on the sphere Ω implies the convergence of curves in length. The question is: How much can we reduce the set of directions so that the convergence in variation in these directions will always imply the convergence of curves in length?

Theorem 10. *In order for the convergence of curves in variation in the direction of the vectors $\mathbf{e} \in M$ to always imply their convergence in length, it is necessary and sufficient that for all unit vectors $\mathbf{e}_1 \neq \mathbf{e}_2$ there is a vector $\mathbf{e} \in M$ whose orthogonal plane strictly separates $\mathbf{e}_1$ and $\mathbf{e}_2$.*

The proof rests on the following result of Tonelli ([15, p. 331]).

Consider a function $F(\mathbf{x}, \alpha)$ of vectors $\mathbf{x}$ and α such that

(1) $F(\mathbf{x}, \alpha) \geq 0$;

(2) $F(\mathbf{x}, \lambda\alpha) = \lambda F(\mathbf{x}, \alpha)$ for $\lambda > 0$;

(3) $F(\mathbf{x}, \alpha + \beta) \leq F(\mathbf{x}, \alpha) + F(\mathbf{x}, \beta)$; moreover, equality holds if and only if $\alpha = \lambda\beta$ and $\lambda > 0$.

If the curves

$$K_m \to K$$

and

$$\int_0^{s(K_m)} F\big(\mathbf{x}_m(s), \mathbf{x}_m'(s)\big)\, ds \to \int_0^{s(K)} F\big(\mathbf{x}(s), \mathbf{x}'(s)\big)\, ds$$

then

$$s(K_m) \to s(K),$$

where $s(K)$ stands for the length of K.

Basing on this, we prove Theorem 10.

Necessity: Suppose that some unit vectors $\mathbf{e}_1 \neq \mathbf{e}_2$ are not separated by any plane with the normal $\mathbf{e} \in M$: for every $\mathbf{e} \in M$ the products $\mathbf{e}_1\,\mathbf{e}$ and $\mathbf{e}_2\,\mathbf{e}$ are of the same sign. Construct a broken line L_m with $2m$ straight line segments alternately equal to $\frac{\mathbf{e}_1}{m}$ and $\frac{\mathbf{e}_2}{m}$. As $m \to \infty$, the broken lines L_m converge to the segment $L = \mathbf{e}_1 + \mathbf{e}_2$. Obviously, $\nu_{\mathbf{e}}(L_m) \to \nu_{\mathbf{e}}(L)$ for $\mathbf{e} \in M$, but the lengths $s(L_m)$ fail to converge to $s(L)$.

Sufficiency: Consider a countable set of directions $\mathbf{e}_k \in M$ $(k = 1, 2, \dots)$ everywhere dense in M. Put

$$F(\alpha) = \sum_{k=1}^{\infty} \frac{|\,\mathbf{e}_k \alpha|}{2^k}.$$

It is obvious that $F(\alpha) \geq 0$; $F(\lambda\alpha) = |\lambda| F(\alpha)$ for every λ. If $\alpha \neq \lambda\beta$ for no $\lambda > 0$ then there is $\mathbf{e}_p$ such that $\alpha\,\mathbf{e}_p$ and $\beta\,\mathbf{e}_p$ are of opposite signs. Hence,

$$F(\alpha+\beta) = \sum_k \frac{\big|\,\mathbf{e}_k(\alpha+\beta)\big|}{2^k} < \sum_{k \neq p} \frac{\big|\,\mathbf{e}_k(\alpha+\beta)\big|}{2^k} + \frac{|\,\mathbf{e}_p\alpha| + |\,\mathbf{e}_p\beta|}{2^p} \leq F(\alpha) + F(\beta);$$

thus,

$$F(\alpha+\beta) < F(\alpha) + F(\beta).$$

If we verify that for all curves $K_m \to K$ together with variations in the directions $\mathbf{e} \in M$ we have

$$\int_0^{s(K_m)} F(\,\mathbf{x}'_m)\,ds \to \int_0^{s(K)} F(\,\mathbf{x}')\,ds,$$

then by the result of Tonelli this would imply that

$$s(K_m) \to s(K).$$

The set M obviously contains n linearly independent directions. Therefore, the lengths of K_m are jointly bounded by a number A. We have

$$\int_0^{s(K_m)} F(\,\mathbf{x}'_m)\,ds = \sum_{k=1}^{\infty} \frac{1}{2^k} \nu_{\,\mathbf{e}_k}(K_m),$$

whence

$$\left| \int_0^{s(K_m)} F(\,\mathbf{x}'_m)\,ds - \sum_{k=1}^{N} \frac{1}{2^k} \nu_{\,\mathbf{e}_k}(K_m) \right| < \frac{A}{2N}$$

and similarly for K. Since

$$\sum_{k=1}^{N} \frac{1}{2^k} \nu_{\,\mathbf{e}_k}(K_m) \to \sum_{k=1}^{N} \frac{1}{2^k} \nu_{\,\mathbf{e}_k}(K),$$

while the number N is arbitrary, it follows that

$$\int_0^{s(K_m)} F(\,\mathbf{x}'_m)\,ds \to \int_0^{s(K)} F(\,\mathbf{x}')\,ds,$$

as required. □

Theorem 10 implies, for instance, that in the three-dimensional space the convergence of curves in variation in all directions lying in three mutually orthogonal planes implies the convergence in length. If we restrict to all directions lying, for instance, only in two of these planes then the hypotheses of the theorem are not fulfilled and there are sequences of curves not converging to a certain curve in length, but converging to it in variation in all directions mentioned.

References

[1] *T. Bonnesen and W. Fenchel.* Theorie der konvexen Körper, Berlin (1934). (The short survey of results and references on the theory of convex bodies until 1934.) Zbl 0008.07708

[2] *K. I. Chuĭkina.* On additive set-functions. Dokl. Akad. Nauk SSSR, LXXVI, No. 6 (1951), p. 801–804. A full presentation see in Uchen. Zap. Moskovsk. Gorn. Pedagogich. Inst. im. Potemkina, vol. XVI, No. 3 (1951), p. 97–126.

[3] *V. A. Zalgaller.* The variation of curves along a fixed direction. Izv. Akad. Nauk SSSR Ser. Mat., No. 5, 15 (1951), p. 463–476. MR0043875

[4] *G. H. Hardy, D. E. Littlewood, and G. Pólya.* Inequalities. Moscow (1948). MR0083530

[5] *P. R. Halmos.* Measure Theory. Moscow (1953). MR0062816

[6] *S. Saks.* Theory of the Integral. Moscow (1949).

[7] *A. D. Alexandrov.* Additive set-functions in abstract spaces. Mat. Sb., 13 (55): 2–3 (1943), p. 169–238. MR0012207

[8] *A. A. Lyapunov.* On countably additive set-functions. Izv. Akad. Nauk SSSR Ser. Mat., No. 6, 4, (1940), p. 465–478; part II, Izv. Akad. Nauk SSSR Ser. Mat., No. 3, 10 (1946), p. 277–279.

[9] *W. Blaschke.* Kreis und Kugel, Leipzig (1916). JFM 46.1109.01

[10] *H. Minkowski.* On bodies of constant width. Mat. Sb., 25 (1905), No. 3, p. 505–508. JFM 36.0526.01

[11] *A. D. Alexandrov.* To the theory of mixed volumes of convex bodies. II. Mat. Sb., 2 (1937), No. 6, p. 1205–1238.

[12] *P. Funk.* Über Flächen mit lauter geschlossenen geodätischen Linien. Math. Ann., 74 (1913), p. 278–300. MR1511763

[13] *W. Blaschke and K. Reidemeister.* Über die Entwicklung der Affingeometrie, Jahresbericht der Dtsch Mathematikervereinigung, 31 (1922), p. 63–82. JFM 48.0804.01

[14] *W. Blaschke.* Vorlesungen über Differentialgeometrie II, Berlin (1923). JFM 49.0499.01

[15] *L. Tonelli.* Fondamenti di calcolo delle variazioni I, Bologna (1921).

Comments on the Article by V. A. Zalgaller and Yu. G. Reshetnyak

A few words about the history of this article are in order. In the beginning of 1953 Aleksandr Danilovich Alexandrov was asked to review the article by E. V. Glivenko which contained some characterization of the range of a countably additive set function, i. e a measure in modern parlance, which takes values in $\mathbb{R}^n$. This set is convex by the now classical theorem of Alekseĭ Andreevich Lyapunov.

Ms. Glivenko was a student of Luyapunov. In the article sent to A.D. there was proved that each set of the sort is the limit of a sequence of some polyhedra of a special type. A.D. was well known in the mathematical community as a specialist in the theory of convex bodies and measure theory. It was natural that the Editorial Board of *Matematicheskiĭ Sbornik* chose him as a reviewer of the article.

Alexandrov asked me to read the article and make my comments. (I was then a second-year postgraduate at Leningrad State University.) I had heard nothing count the Lyuapunov work on additive set functions and the topic attracted me strongly. I noticed that the convex sets, arising in the theorem, coincided with the convex set that had already been distinguished by Zalgaller in studying rectifiable curves in $\mathbb{R}^n$. So I proposed Viktor Abramovich that we wrote a joint paper on the matter. He agreed.

We had finished the paper shortly and submitted it the editorial office of the mathematical series of *Herald of Leningrad State University.* Some of out colleagues wondered why we had submitted the paper to the journal rather than to some central Soviet mathematical periodicals. It is difficult to recall the reasons of our choice now, as some 60 years elapsed since then. Unfortunately, the article remained unnoticed by the intended readership.

I think there were two reasons behind this. The first is that the article was published in the journal that was not listed as leading in the mathematical community. Vising the USA, I saw that only few university libraries have the issues of the 1954 volume of *Herald of Leningrad State University.* The second reason is concealed in the structure of the article itself, for it was focused on some problems of the theory of curves. Therefore, the problems of the theory of set functions seemed to be drifted slightly out of the main light.

In the 1960s the theory of set functions had again become topical In particular this related to the Lyapunov Theorem. The new term *zonoid* was suggested for the range of any measure in question. I did not find out the authorship of the term. In our article we used the bulkier term, a body this is a mixing of straight line segments.

Some of the results of our article ([ZR] for the sake of brevity in the sequel) are widely used, as I can discern, in the papers dealing with various aspects of the theory of zonoids. As sources these papers usually indicated the papers than were published after 1969. This shows only some shortcomings of information available to the authors, since since some of this results were proved in [ZR] and published in 1954.

Below I list the results of the articles that I view as fundamental for the theory of zonoids. In the Western mathematical literature the statements and proofs of these results were published not before 1979. I will use the numeration that differs from that in [ZR]. To avoid confusion, I put the letter C before each successive numeral (i. e., C1, C2, etc.).

So, let $(X, \mathcal{B})$ be a measure space; i. e., X is a set equipped with some σ-algebra $\mathcal{B}$ of subsets of X.

Lemma C1. *Each measure $\varphi : \mathcal{B} \to \mathbb{R}^n$ admits the representation*

$$\varphi(E) = \int_E \mathbf{x}(t)\, d\mu(t),$$

Here μ is a positive measure, the variation of the vector-measure φ, while $\mathbf{x}(t)$ is a function to $\mathbb{R}^n$ define on the whole of X and such that $|\mathbf{x}(t)| = 1$ for all $t \in X$.

In [ZR] this is Theorem 4. I think that the title of theorem is undeserved by this proposition, since this is a trivial corollary of the Radon–Nikodym, and the proof in [ZR] may be slightly shortened.

Assume given a vector measure $\varphi : \mathcal{B} \to \mathbb{R}^n$. Denote by $Q(\varphi)$ the set of all vectors of $\mathbb{R}^n$ that are presentable as

$$\mathbf{z} = \int_X \alpha(t)\mathbf{x}(t) d\mu(t),$$

where $\mathbf{x}(t)$ and μ have the same meaning as above, and $\alpha(t)$ is $\mathcal{B}$-measurable functions such that $0 \le \alpha(t) \le 1$ for all $t \in X$.

The next proposition of [ZR] bore neither the title of theorem nor at least the title of lemma. Its statement and full proof are given in the first part of subsection 3 of §4.

Theorem C1. *If Let $\varphi : \mathcal{B} \to \mathbb{R}^n$ be a vector measure. Then $Q(\varphi)$ is a bounded closed convex subset of $\mathbb{R}^n$. The support function $H(\nu)$ of $Q(\varphi)$ admits the representation*

$$H(\mathbf{e}) = \int_X \frac{\left|\langle \mathbf{e}, \mathbf{x}(t)\rangle\right| + \langle \mathbf{e}, \mathbf{x}(t)\rangle}{2}\, d\mu.$$

The boundedness and convexity of $Q(\varphi)$ are straightforward by definition. The proof of sufficiency requires application of the properties of weak convergence of a sequence of measures.

Theorem C2.*If φ is a atomless (i. e., continuous in the sense of Lyapunov), then $Q(\varphi)$ coincides with the range of φ.*

This is the Lemma of §3 of [ZR].

Theorem C3. The range of a atomless vector measure acting to $\mathbb{R}^n$ is a bounded closed convex set in $\mathbb{R}^n$.

Theorem C2 is the Lemma of §3 in [ZR]. Theorem C3 is an obvious corollary of Theorem C2. Theorem C3 is the Lyapunov Convexity Theorem. We thus have a proof of the theorem which was new in 1954. Note the claim of Theorem '1 has applications in optimization theory.

The following proposition belongs to the basic facts of the theory of zonoids.

Theorem C4. *For a compact convex set $\mathbf{x}$ to be a zonoid it is necessary and sufficient that its support function $H(\mathbf{x})$ admit the representation:*

$$H(\mathbf{x}) = \int_{\Omega_{n-1}} \langle \mathbf{x}, \xi\rangle^+ \, d\mu(\xi),$$

where Ω_{n-1} is the $(n-1)$-dimensional sphere in $\mathbb{R}^n$.

There is no exactly this formulation in [ZR]. But [ZR] has the following Theorem 5:

Theorem 5 of [ZR]. *For a body Q to be the range of a atomless vector measure acting to $\mathbb{R}^n$ it is necessary and sufficient that Q be the indicatrix of the positive part od the variation of some rectifiable curve.*

I will not state Theorem 6 of [ZR] since its statement is bulky. This theorem implies in particular that the support function of the indicatrix of a rectifiable curve admits the representation

$$H(\mathbf{x}) = \int_{\Omega_{n-1}} \langle \mathbf{x}, \xi\rangle^+ \, d\mu(\xi),$$

By Theorem 5 of [ZR] this yields Theorem C4.

Finally, Lemma 6 states that if a measure μ on the sphere Ω_{n-1} is even then μ is uniquely determined from $H(\mathbf{x})$ in accord with the equation

$$H(\mathbf{x}) = \int_{\Omega_{n-1}} \langle \mathbf{x}, \xi\rangle^+ \, d\mu(\xi)$$

The proof of this fact was based on using spherical functions and followed the ideas of W. Blaschke and A. D. Alexandrov.

The *zonotopes*, i. e. the polyhedra presenting the sums of finitely many line segments, were called *parrallelohedra* in [ZR]. Here we followed the terminology of Glivenko and Tyapkina, students of A. A. Lyapunov.

I also want to mention the next result of [ZR]:

Theorem C5. *For a convex polyhedron in* $\mathbb{R}^n$ *to be a zonotope it is necessary and sufficient that its every two-dimensional face have a center of symmetry.*

This is Theorem 6 of [ZR]. The theorem is accompanied by the following

Theorem C6. *If a convex polyhedron* P *in* $\mathbb{R}^n$ *is a zonoid, then* P *is a zonotope, i. e., the sum of finitely many line segments.*

In closing the author expresses his sincere gratitude to Professor Semën Samsonovich Kutateladze for attention to my ancient paper written with my friend Viktor Abramovich Zalgaller more than half a century ago. I also thank S. S. Kutateladze for organizing the translation of our article into Latin of today, i. e., English.

Yu. G. Reshetnyak